Theory of Quantum Hall Effect in Monolayer and Bilayer Graphene

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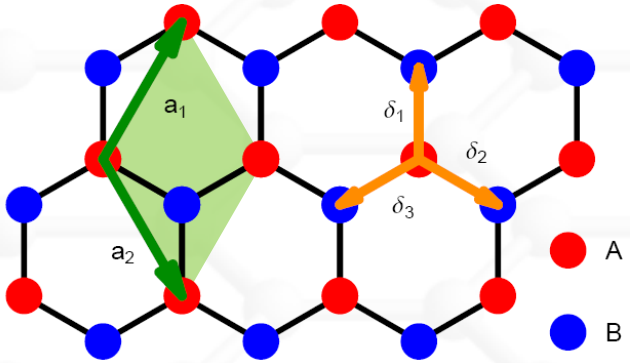
Lattice in coordinate & reciprocal space

- Translation vectors

$$\mathbf{a}_1 = a\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \mathbf{a}_2 = a\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

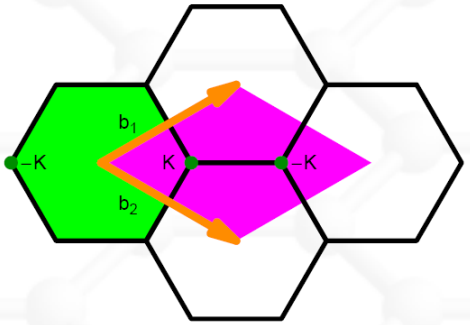
where a is the lattice constant



- Two carbon atoms per primitive cell

- Reciprocal lattice vectors

$$\mathbf{b}_1 = 2\pi/a(1, 1/\sqrt{3}), \; \mathbf{b}_2 = 2\pi/a(1, -1/\sqrt{3})$$



Tight binding model

- Strong covalent sigma-bonds between nearest neighbors (carbon atoms)
- Hamiltonian

$$H = -t \sum_{\mathbf{n},\boldsymbol{\delta}_i,s} \left[a^{\dagger}_{\mathbf{n},s} \exp\left(\frac{ie}{\hbar c} \boldsymbol{\delta}_i \mathbf{A} \right) b_{\mathbf{n}+\boldsymbol{\delta},s} + \text{h.c.} \right]$$

where $a_{\mathbf{n},s}$ and $b_{\mathbf{n}+\delta,s}$ are the annihilation operators of electrons with spin $s=\uparrow,\downarrow$

- The nearest neighbor vectors are

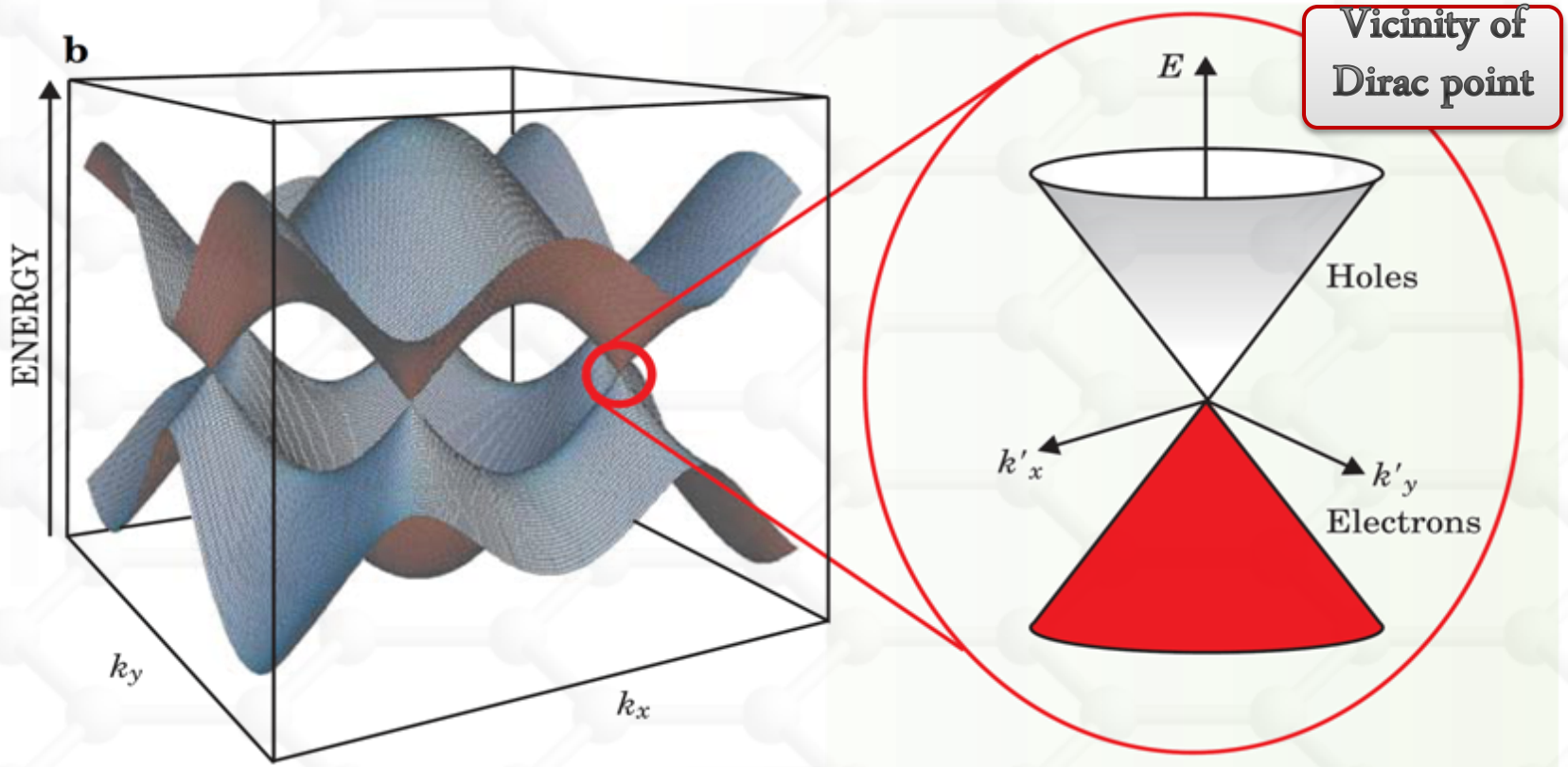
$$\boldsymbol{\delta}_1 = (\mathbf{a}_1 - \mathbf{a}_2)/3\,, \quad \boldsymbol{\delta}_2 = \mathbf{a}_1/3 + 2\mathbf{a}_2/3\,,$$
$$\boldsymbol{\delta}_3 = -\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2 = -2\mathbf{a}_1/3 - \mathbf{a}_2/3$$

Low energy Dirac fermions

$$\mathcal{L} = \sum_{s=\pm 1} \bar{\Psi}_s(t,\mathbf{r})\left[i\gamma^0(\hbar\partial_t - i\mu_s) + i\hbar v_F \gamma^1 D_x + i\hbar v_F \gamma^2 D_y\right]\Psi_s(t,\mathbf{r})$$

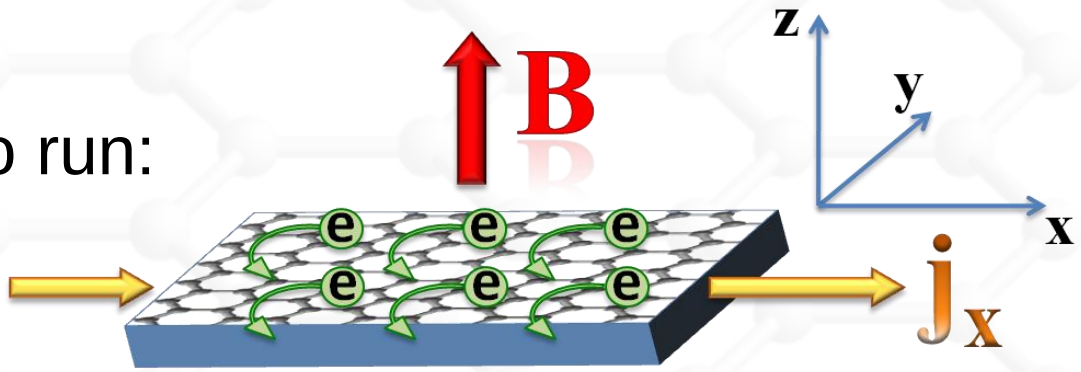
$$\Psi_s^T = (\psi_{KAs}, \psi_{KBs}, \psi_{K'Bs}, \psi_{K'As})$$

P. R. Wallace, Phys. Rev. **71,** 622 (1947)
G.W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984)

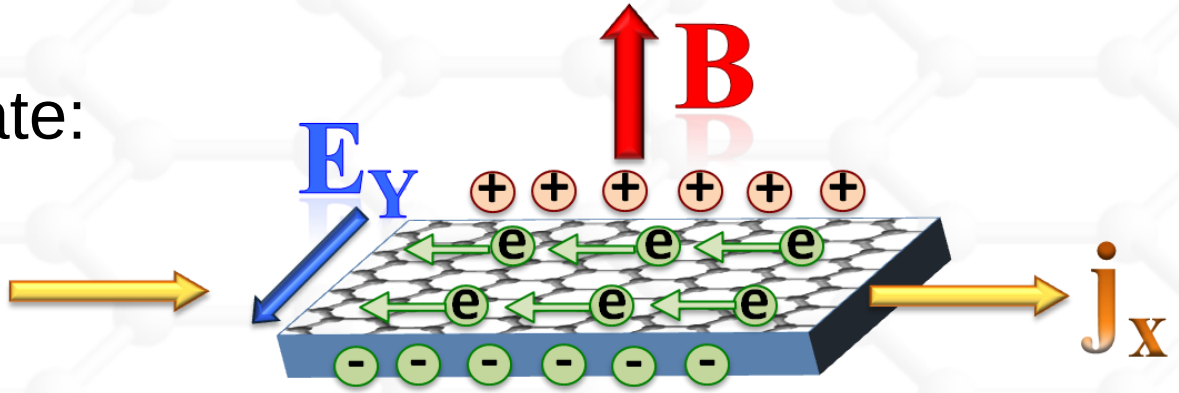


Quantum Hall effect

- General setup
  - Current starts to run:



  - Steady state:



Classical HE: $j_x = \sigma_{xx} E_x + \sigma_{xy} E_y, \quad j_y = \sigma_{yy} E_y + \sigma_{yx} E_x = 0$

Quantum HE: $j_x = \sigma_{xy} E_y, \quad E_x = 0, \quad \sigma_{xx} = \sigma_{yy} = 0, \quad j_y = 0,$

$\sigma_{yx} = -\sigma_{xy}, \quad \rho_{xx} = \rho_{yy} = 0$

Quantum Hall effect in graphene

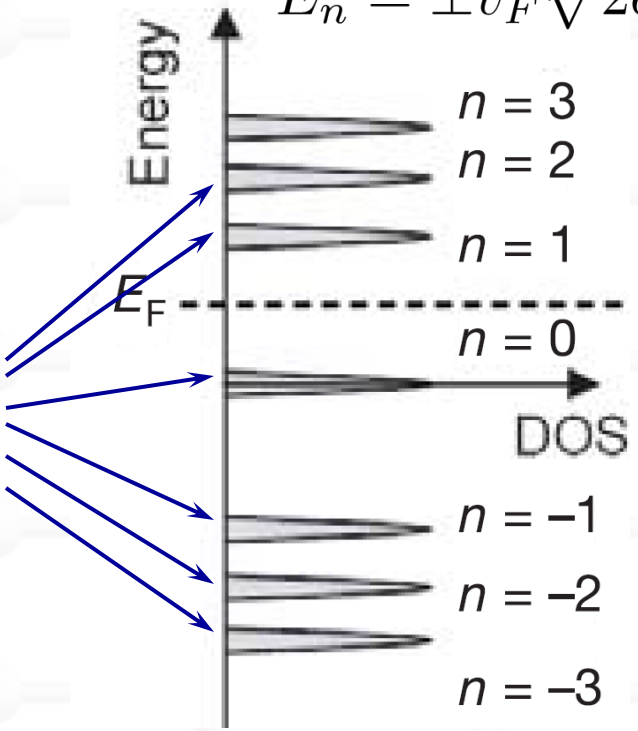
[1] Zheng & Ando, PRB **65**, 245420 (2002)
[2] Gusynin & Sharapov, PRL **95**, 146801 (2005)
[3] Peres, Guinea, & Castro Neto, PRB **73**, 125411 (2006)
[4] Novoselov et al., Nature **438**, 197 (2005)
[5] Zhang et al., Nature **438**, 201 (2005)

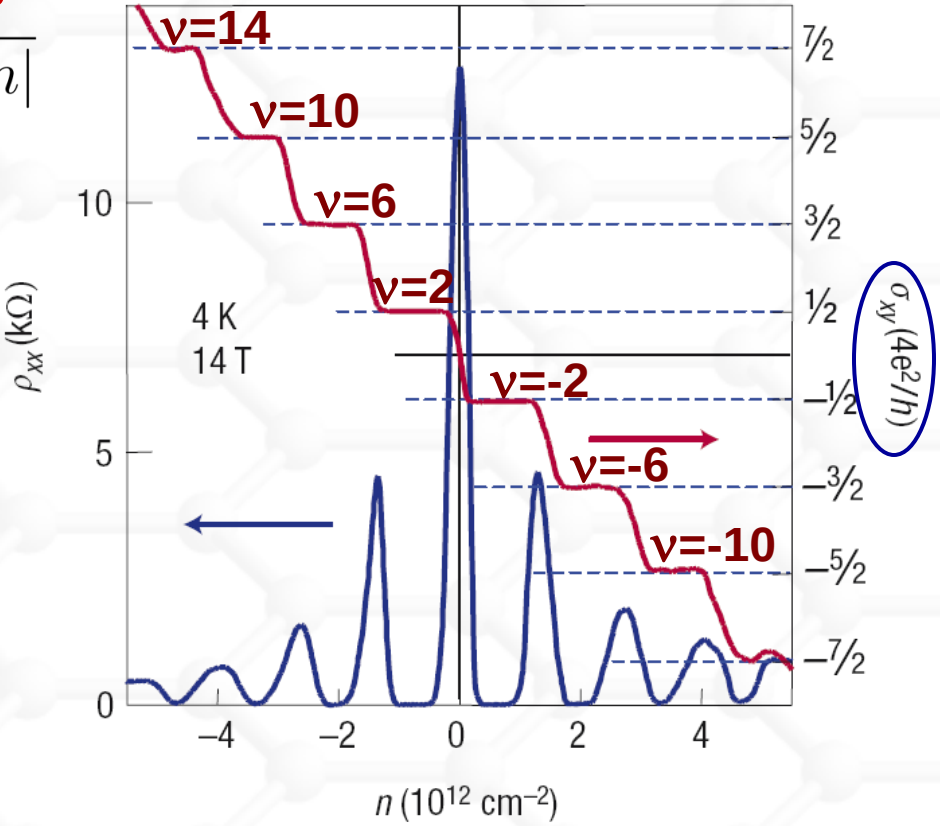
Experiment

$$\sigma_{xy} = \frac{\nu e^2}{h} = \frac{4e^2}{h}\left(n + \frac{1}{2}\right)$$

4–fold degenerate levels

$$E_n = \pm v_F \sqrt{2e\hbar B|n|}$$





Anomalous QHE

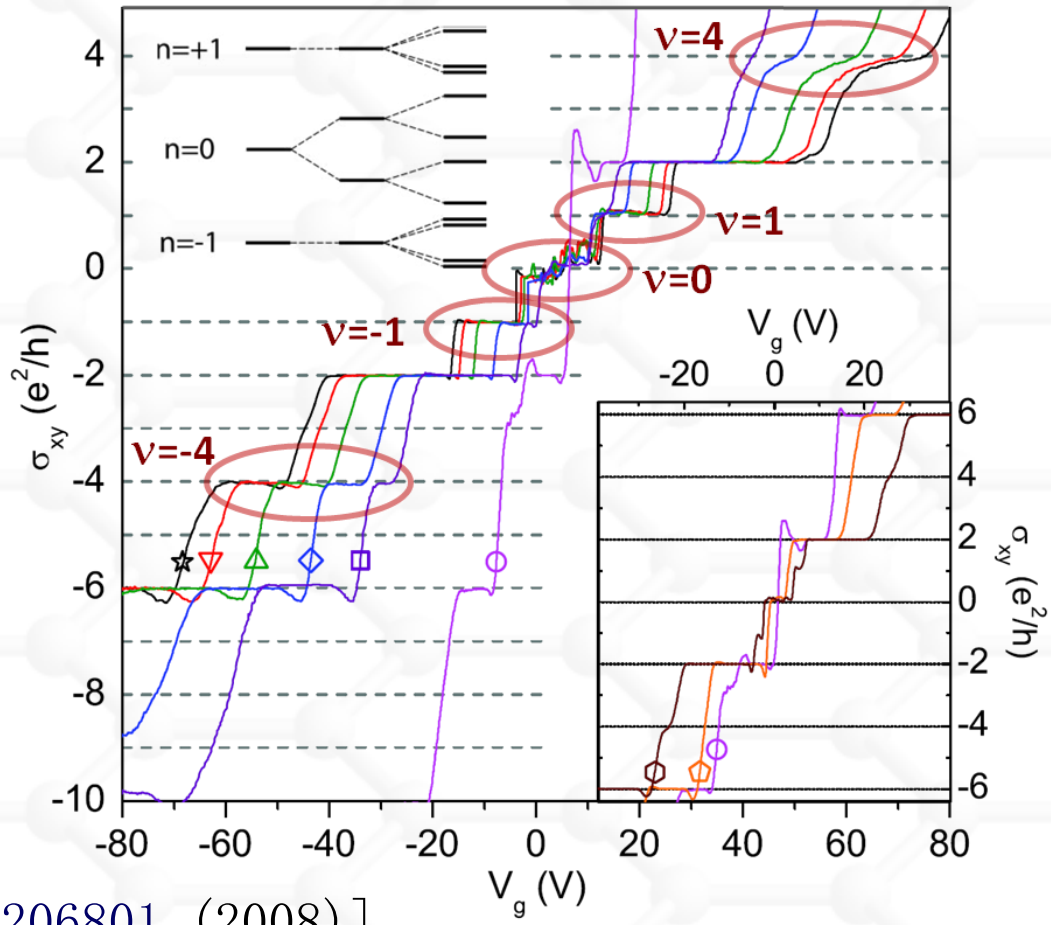
- New plateaus are observed at

$$\nu = 0$$
$$\nu = \pm 1$$
$$\nu = \pm 3$$
$$\nu = \pm 4$$

[Novoselov et al., Science **315**, 1379 (2007)]

[Abanin et al., PRL **98**, 196806 (2007)]

Zhang et al., PRL **96,** 136806 (2006)



[Checkelsky et al., PRL 100, 206801 (2008)]

[Xu Du et al., Nature **462**, 192 (2009)]

Magnetic Catalysis

- Electron wave functions in magnetic field are localized [Gusynin, V.M., Shovkovy, PRL **73** (1994) 3499]
- Effective dimensional reduction

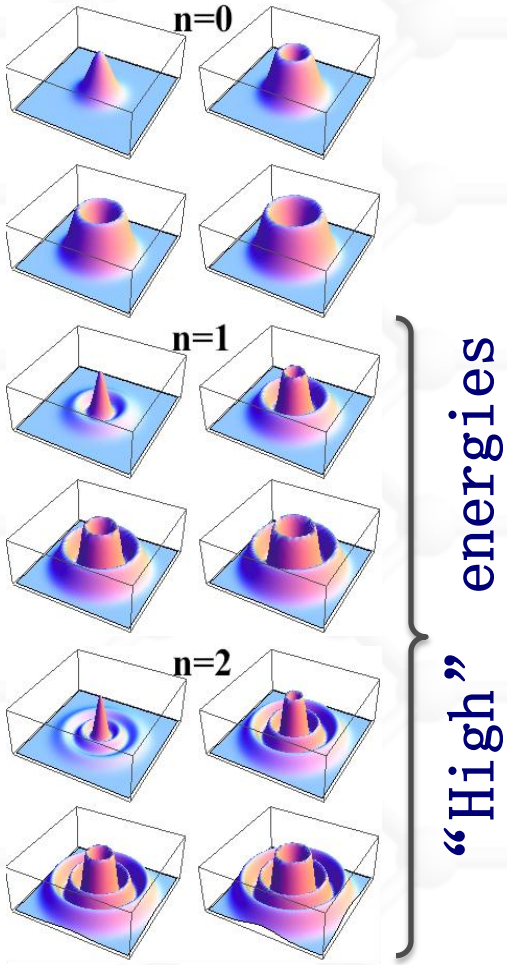
  D space directions ➡ ***D-2*** space directions
- Quantized energy levels:

$$E_n = \pm\sqrt{m^2c^4 + 2n\,\hbar c^2|eB|}$$

$$\simeq \pm\sqrt{2n\,\hbar c^2|eB|} \quad \text{for} \quad m \simeq 0$$

n* = *0: zero energy

n* ≥ *1 states have “high” energies



Spontaneous Chiral Symmetry Breaking

- At $m_0=0$, a dynamical Dirac mass m_{dyn} is generated

  2D: $m_{\rm dyn} \propto \alpha\sqrt{|eB|}$

  3D: $m_{\rm dyn} \propto \sqrt{|eB|}\; e^{\pi/[\alpha \ln(\alpha N_f)]}$

- This happens even at the *weakest* interaction ("catalysis")

- The phenomenon is *universal* (specific details of interaction are mostly irrelevant)

- Dimensional *reduction* is the key ingredient

Magnetic Catalysis in Graphene

- Charge carriers are Dirac fermions with $m=0$
- Theoretically, $m_{dyn} \neq 0$ must be generated in a sufficiently strong magnetic field

  [Gorbar, Gusynin, V.M., Shovkovy, PRB **66** (2002) 045108]
- Possible complications:
  - several types of Dirac masses may exist in **2D**
  - competition with quantum Hall ferromagnetism
  - nonzero electron/hole density
  - impurities, lattice defects, ripples, etc.

General Approach

Model Hamiltonian

[Gorbar, Gusynin, V. M., Shovkovy, Phys. Rev. B **78** (2008) 085437]

$$H = H_0 + H_C + \int d^2\mathbf{r}\,[\underbrace{\mu_B B \Psi^\dagger \sigma^3 \Psi}_{\textbf{Zeeman term}} - \mu_0 \Psi^\dagger \Psi]$$

where

$$H_0 = v_F \int d^2\mathbf{r}\,\overline{\Psi}\left(\gamma^1 \pi_x + \gamma^2 \pi_y\right)\Psi,$$

is the Dirac Hamiltonian, and

$$H_C = \frac{1}{2}\int d^2\mathbf{r} d^2\mathbf{r}' \Psi^\dagger(\mathbf{r})\Psi(\mathbf{r})U_C(\mathbf{r}-\mathbf{r}')\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}')$$

is the Coulomb interaction term.

Note that $\Psi^T_{(s)} = (\psi_{KAs}, \psi_{KBs}, \psi_{K'Bs}, \psi_{K'As}), \quad s = \pm$

Spin index

$v_F \approx 10^6$ m/s

Symmetry

- The Hamiltonian $H = H_0 + H_C$ possesses "flavor" *U(4)* symmetry (no Zeeman term)
- 16 generators read (*spin* $\otimes$ *valley*)

$$\frac{\sigma^\alpha}{2} \otimes I_4, \ \frac{\sigma^\alpha}{2i} \otimes \gamma_3, \ \frac{\sigma^\alpha}{2} \otimes \gamma_5 \text{ and } \frac{\sigma^\alpha}{2} \otimes \gamma_3\gamma_5$$

$$\text{where } \gamma^3\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} - \text{diagonal valley matrix}$$

- A ferromagnetic like order parameter breaks spin degeneracy.
  Thus, *U(4)* breaks down to $U^{(+)}(2)_v \times U^{(-)}(2)_v$
- QHF order parameter and/or Dirac mass breaks *U(4)* down to $U^{(K)}(2)_s \times U^{(K')}(2)_s$

Full propagator

- One can use the following general ansatz:

$$iG_s = \left[(i\hbar\partial_t + \mu_s + \tilde{\mu}_s\gamma^3\gamma^5)\gamma^0 - v_F(\boldsymbol{\pi}\cdot\boldsymbol{\gamma}) - \tilde{\Delta}_s + \Delta_s\gamma^3\gamma^5\right]^{-1}$$

Electron chemical potential

"Pseudospin" chemical potential

Dirac mass

T-odd mass

- Physical meaning of the order parameters

$$\mu_3: \quad \langle\Psi^\dagger\sigma^3\Psi\rangle = \sum_{\kappa=K,K'}\sum_{a=A,B}\langle\psi^\dagger_{\kappa a+}\psi_{\kappa a+} - \psi^\dagger_{\kappa a-}\psi_{\kappa a-}\rangle$$

$$\tilde{\mu}_s: \quad \langle\Psi^\dagger\gamma^3\gamma^5 P_s\Psi\rangle = \langle\psi^\dagger_{KAs}\psi_{KAs} - \psi^\dagger_{K'As}\psi_{K'As} + \psi^\dagger_{KBs}\psi_{KBs} - \psi^\dagger_{K'Bs}\psi_{K'Bs}\rangle$$

$$\Delta_s: \quad \langle\bar{\Psi}\gamma^3\gamma^5 P_s\Psi\rangle = \langle\psi^\dagger_{KAs}\psi_{KAs} - \psi^\dagger_{KBs}\psi_{KBs} - (\psi^\dagger_{K'As}\psi_{K'As} - \psi^\dagger_{K'Bs}\psi_{K'Bs})\rangle$$

$$\tilde{\Delta}_s: \quad \langle\bar{\Psi}P_s\Psi\rangle = \langle\psi^\dagger_{KAs}\psi_{KAs} + \psi^\dagger_{K'As}\psi_{K'As} - \psi^\dagger_{KBs}\psi_{KBs} - \psi^\dagger_{K'Bs}\psi_{K'Bs}\rangle$$

$$P_\pm = \frac{1\pm\sigma^3}{2}$$

Energy scales in graphene

- Large Landau energy scale

$$\epsilon_B \equiv \sqrt{2\hbar|eB_\perp|v_F^2/c} \simeq 424\sqrt{|B_\perp[\mathrm{T}]|}\ \mathrm{K}$$

- Small Zeeman energy

$$Z \simeq \mu_B B = 0.67 B[\mathrm{T}]\ \mathrm{K}$$

- Intermediate dynamical scales

$$Z \ll A \leq M \ll \epsilon_B\,; \quad A \to \mu_s, \tilde{\mu}_s\,; \quad M \to \Delta_s, \tilde{\Delta}_s$$

$$A \leqslant M \sim 10^{-2}\epsilon_B$$

- In a model calculation [Phys. Rev. B **78** (2008) 085437]

$$M = 4.84 \times 10^{-2}\epsilon_B \text{ and } A = 3.90 \times 10^{-2}\epsilon_B$$

Dispersion relations

The dispersion relations for LLs with $n \geq 1$ are

$$\omega_{ns}^{(\sigma)} = -\mu_s + \sigma\tilde{\mu}_s \pm \sqrt{n\epsilon_B^2 + (\tilde{\Delta}_s + \sigma\Delta_s)^2}$$

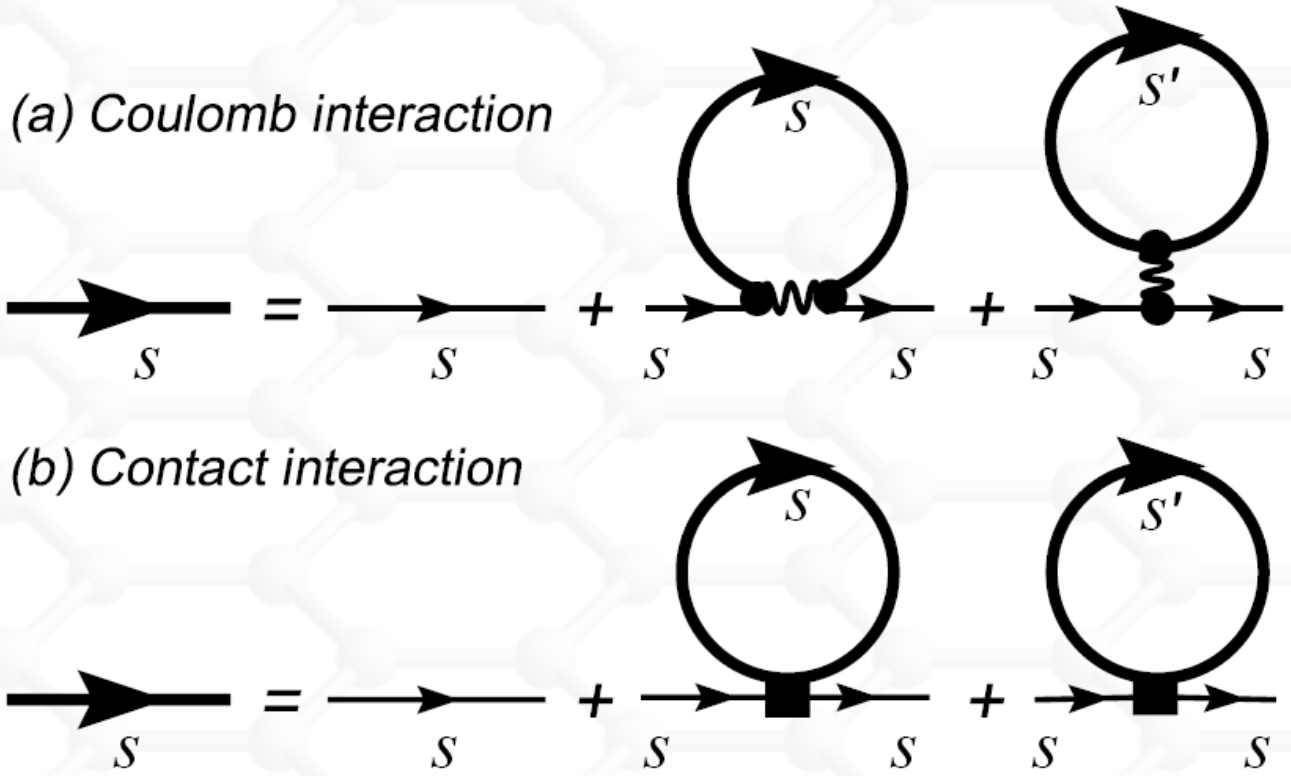
where $\sigma = \pm 1$ is connected with the eigenvalues of the diagonal pseudospin matrix $\gamma^3\gamma^5$.

For the LLL, the dispersion relations read

$$\omega_s^{(\sigma)} = -\mu_s + \sigma\left(\tilde{\mu}_s \,\mathrm{sign}(eB_\perp) + \tilde{\Delta}_s\right) + \Delta_s \,\mathrm{sign}(eB_\perp).$$

Schwinger-Dyson (gap) equation

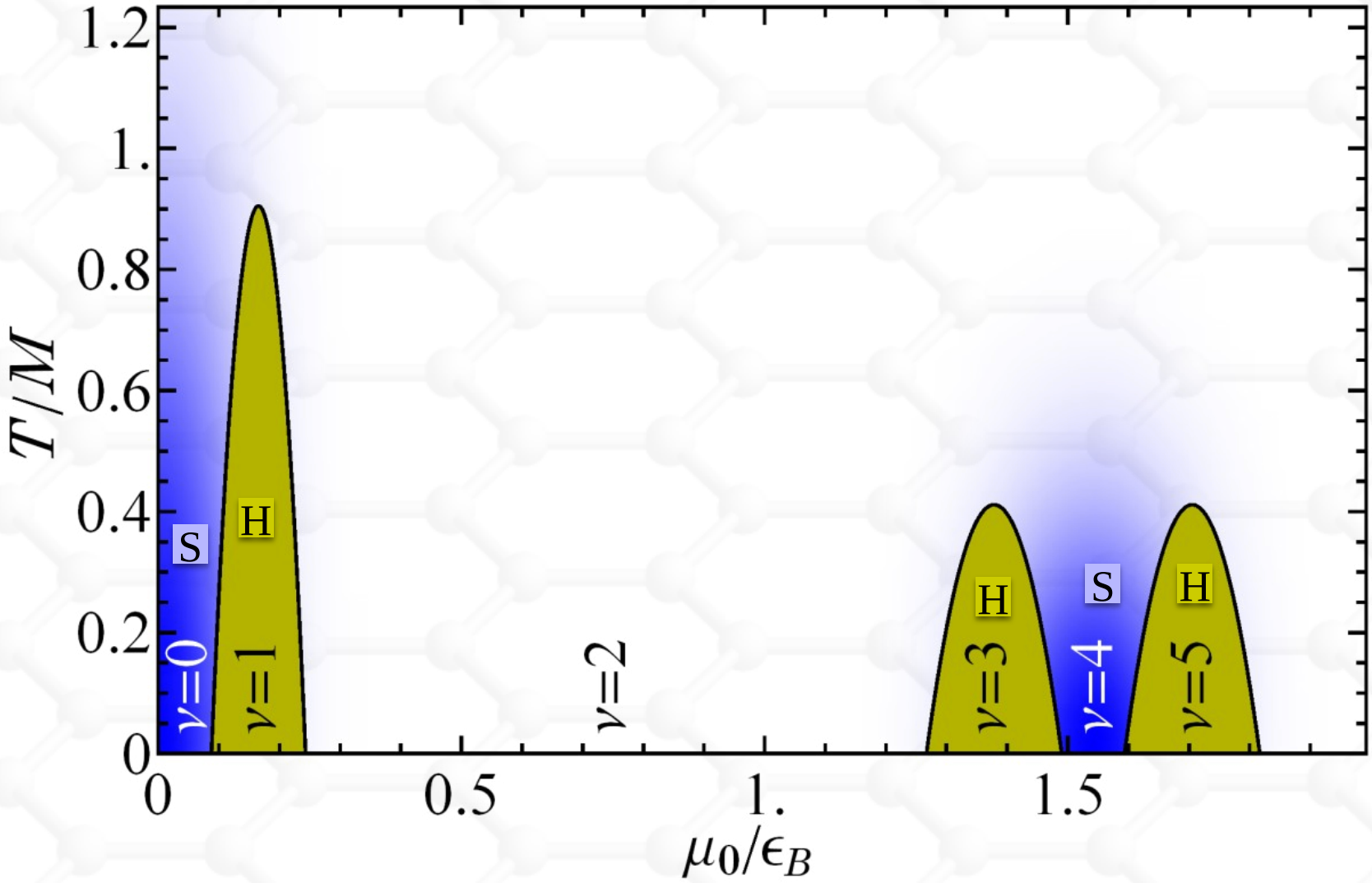
- Hartree-Fock (mean field) approximation:



Three types of solutions

- $\boldsymbol{S}$ (*singlet* with respect to $U^{(+)}(2)_{\mathrm{v}} \times U^{(-)}(2)_{\mathrm{v}}$); $v=0$
  - Order parameters: μ_{s} and Δ_{s}
  - Symmetry: $U^{(+)}(2)_{\mathrm{v}} \times U^{(-)}(2)_{\mathrm{v}}$
- $\boldsymbol{T}$ (*triplet* with respect to $U^{(\pm)}(2)_{\mathrm{v}}$); $v=0$
  - Order parameters: $\tilde{\mu}_{\mathrm{s}}$ and/or $\tilde{\Delta}_{\mathrm{s}}$
  - Symmetry: $U^{(K)}(2)_{\mathrm{s}} \times U^{(K')}(2)_{\mathrm{s}}$
- $\boldsymbol{H}$ (*hybrid*, i.e., singlet + triplet); $v=\pm 1$
  - Order parameters: mixture of $\boldsymbol{S}$ and $\boldsymbol{T}$ types
  - Symmetry: $U^{(+)}(2)_{\mathrm{v}} \times U^{(-)}(1)_{K} \times U^{(-)}(1)_{K'}$ or $U^{(+)}(1)_{K} \times U^{(+)}(1)_{K'} \times U^{(-)}(2)_{\mathrm{v}}$

Phase diagram



Bilayer graphene

The effective low energy Hamiltonian [McCann & Falko, PRL, 96, 086805 (2006)]

Free Hamiltonian:

$$H_0 = -\frac{1}{2m}\int d^2x \Psi^+_{Vs}(x) \begin{pmatrix} 0 & (\pi^\dagger)^2 \\ \pi^2 & 0 \end{pmatrix} \Psi_{Vs}(x)$$

$$m \sim 10^{-2} m_e \sim 10^8 \mathrm{K/c}^2, \pi = \hat{p}x_1 + i\hat{p}x_2\,, \hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla} + e\mathbf{A}/c\,.$$

Bernal (A_2-B_1) stacking: $\Psi^T_{Ks} = (\psi_{A1}, \psi_{B2})_{Ks}$,whereas $\Psi^T_{K's} = (\psi_{B2}, \psi_{A1})_{K's}$.

$E = \pm\frac{p^2}{2m}$ without magnetic field B and $E_n = \pm\hbar\omega_c\sqrt{n(n-1)}$, $\omega_c = |eB|/mc$ with magnetic field B.

Interaction Hamiltonian:

$$H_{int} = \mu_B B\int d^2x \Psi^+(x)\sigma^3\Psi(x) + \frac{e^2}{2\kappa}\int d^3x d^3x' \frac{n(\mathbf{x})n(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + \tilde{\Delta}_0\int d^2x \Psi^+(x)\xi\tau_3\Psi(x)\,,$$

where $n(\mathbf{x}) = \delta(z-\frac{d}{2})\rho_1(x) + \delta(z+\frac{d}{2})\rho_2(x)$ is the three dimensional charge density ($d \simeq 0.35$ nm is the distance between the layers). The Pauli matrix τ^3 in the voltage imbalance term acts on layer components, and $\xi = \pm 1$ for the valleys *K* and *K'*, respectively.

Quantum Hall effect in bilayer graphene

FIRST EXPERIMENTS:

Novoselov et al., Nature Phys. **2**, 177 (2006)

Henriksen et al., PRL **100**, 087403 (2008)

- Quantum Hall states with the filling factor $\nu = \pm 4n,\ n = 1, 2...$ predicted in the framework of the one electron problem were revealed.

RECENT EXPERIMENTS:

Suspended graphene:

Feldman, Martin, Yacoby, Nature Phys. **5**, 889 (2009)

Graphene on SiO_2/Si substrates:

Zhao, Cadden-Zimansky, Jiang, Kim, PRL **104**, 066801 (2010)

- Complete lifting the eightfold degeneracy in the LLL: $\nu = 0, \pm 1, \pm 2, \pm 3.$
- The $\nu = 0$ state is insulating.
- Suspended graphene: $\Delta E \sim 3.5 - 10.5 B_\perp[\mathrm{T}]\mathrm{K},\ B_\perp \lesssim 10\mathrm{T}$;

QHE in bilayer graphene: theory

Barlas, Cote, Nomura, MacDonald, PRL, **101**, 097601 (2008);

Gorbar, Gusynin, V. M., JETP Lett. **91**, 314 (2010); PRB **81**, 155451 (2010);

Gorbar, Gusynin, Junji Jia, V. M., arXiv: 1108.0650 [cond-mat];

Nandkishore and Levitov, arXiv:0907.5395, arXiv:1002.1966;

Tőke, Fal'ko, PRB **91**, 115455 (2011);

Symmetries

H_{int} can be rewritten as

$$H_{int} = \mu_B B \int d^2x \Psi^+(x)\sigma^3\Psi(x) + \tilde{\Delta}_0 \int d^2x \Psi^+(x)\xi\tau_3\Psi(x)$$
$$+\frac{1}{2}\int d^2x d^2x' \left[V(x-x')\left(\rho_1(x)\rho_1(x') + \rho_2(x)\rho_2(x')\right) + 2V_{12}(x-x')\rho_1(x)\rho_2(x')\right] .$$

Intralayer potential *V(x)* $\xrightarrow{\text{FT}} \tilde{V}(k) = 2\pi e^2/\kappa k$

interlayer potential $V_{12}(x)$ $\xrightarrow{\text{FT}} \tilde{V}_{12}(k) = (2\pi e^2/\kappa)(e^{-kd}/k).$

The two-dimensional charge densities $\rho_1(x)$ and $\rho_2(x)$ are:

$$\rho_1(x) = \Psi^+(x)P_1\Psi(x)\,, \quad \rho_2(x) = \Psi^+(x)P_2\Psi(x)\,,$$

where $P_1 = \frac{1+\xi\tau^3}{2}$ and $P_2 = \frac{1-\xi\tau^3}{2}$ are projectors on states in the layers 1 and 2, respectively.

If both the Zeeman and $\tilde{\Delta}_0$ terms are ignored, the bilayer symmetry is

$$G_2 = U^{(K)}(2)_S \times U^{(K')}(2)_S \times Z_{2V}^{(+)} \times Z_{2V}^{(-)} \quad ,$$

where $Z_{2V}^{(s)}$ describes the valley transformation $\xi \to -\xi$ for a fixed spin $s = \pm$.

G_2 is much lower than $G_1 = U(4)$ in monolayer graphene.

Order parameters

Although the G_1 and G_2 symmetries are quite different, it is noticeable that their spontaneous breakdowns can be described by the same QHF and MC order parameters. G_1 and G_2 define the same four conserved commuting currents whose charge densities (and four corresponding chemical potentials) span the QHF order parameters:

$$\mu_3: \quad \langle \Psi^\dagger \sigma^3 \Psi \rangle = \sum_{\kappa=K,K'} \sum_{a=A_1,B_2} \left\langle \psi^\dagger_{\kappa a+} \psi_{\kappa a+} - \psi^\dagger_{\kappa a-} \psi_{\kappa a-} \right\rangle ,$$

$$\tilde{\mu}_s: \quad \langle \Psi^\dagger_s \xi \Psi_s \rangle = \left\langle \psi^\dagger_{KA_1 s} \psi_{KA_1 s} - \psi^\dagger_{K'A_1 s} \psi_{K'A_1 s} + \psi^\dagger_{KB_2 s} \psi_{KB_2 s} - \psi^\dagger_{K'B_2 s} \psi_{K'B_2 s} \right\rangle .$$

While the first order parameter describes $SU^{(K)}(2)_s \times SU^{(K')}(2)_s$ spin symmetry breakdown, the second one breaks the discrete subgroup $Z^{(s)}_{2V}$. Their MC cousins are

$$\Delta_s: \quad \langle \Psi^\dagger_s \tau_3 \Psi_s \rangle = \left\langle \psi^\dagger_{KA_1 s} \psi_{KA_1 s} - \psi^\dagger_{KB_2 s} \psi_{KB_2 s} - \left(\psi^\dagger_{K'A_1 s} \psi_{K'A_1 s} - \psi^\dagger_{K'B_2 s} \psi_{K'B_2 s} \right) \right\rangle ,$$

$$\tilde{\Delta}_s: \quad \langle \Psi^\dagger_s \xi \tau_3 \Psi_s \rangle = \left\langle \psi^\dagger_{KA_1 s} \psi_{KA_1 s} + \psi^\dagger_{K'A_1 s} \psi_{K'A_1 s} - \psi^\dagger_{KB_2 s} \psi_{KB_2 s} - \psi^\dagger_{K'B_2 s} \psi_{K'B_2 s} \right\rangle .$$

LLL quasiparticle propagator

Bare LLL propagator

$$\tilde{S}_{\xi s}(\mathbf{r};\omega) = \frac{1}{2\pi l^2}\exp\left(-\frac{\mathbf{r}^2}{4l^2}\right)\left[L_0\left(\frac{\mathbf{r}^2}{2l^2}\right)+L_1\left(\frac{\mathbf{r}^2}{2l^2}\right)\right]S_{\xi s}(\omega)P_-,$$

where

$$S_{\xi s}(\omega) = \frac{1}{\omega+\bar{\mu}_s+\xi\tilde{\Delta}_0+i\delta\,\mathrm{sgn}\,\omega}, \quad \bar{\mu}=\mu_0-sZ, \quad P_\pm=\frac{1\pm\tau_3}{2}, \quad l=\sqrt{\frac{\hbar c}{|eB|}}.$$

Full LLL propagator

$$\tilde{G}_{\xi s}(\mathbf{r};\omega) = \frac{1}{2\pi l^2}\exp\left(-\frac{\mathbf{r}^2}{4l^2}\right)\left[G_{\xi s0}(\omega)L_0\left(\frac{\mathbf{r}^2}{2l^2}\right)+G_{\xi s1}(\omega)L_1\left(\frac{\mathbf{r}^2}{2l^2}\right)\right]P_-,$$

where

$$G_{\xi sn}(\omega) = \frac{1}{\omega-E_{\xi ns}+i\delta\mathrm{sgn}\omega}, \text{ and } E_{\xi ns} = -(\mu_s(n)+\Delta_s(n))+\xi(\tilde{\mu}_s(n)-\tilde{\Delta}_s(n)), \quad n=0,1,$$

are the energies of the LLL states depending on the order parameters $\mu_s(n), \tilde{\mu}_s(n), \Delta_s(n), \tilde{\Delta}_s(n)$.

Exchange (FOCK) interactions:

$$\tilde{V}_{eff}(\omega,k)=\frac{2\pi e^2}{\kappa}\frac{1}{k+\frac{4\pi e^2}{\kappa}\Pi(\omega,\mathbf{k}^2)}$$, κ -- dielectric constant

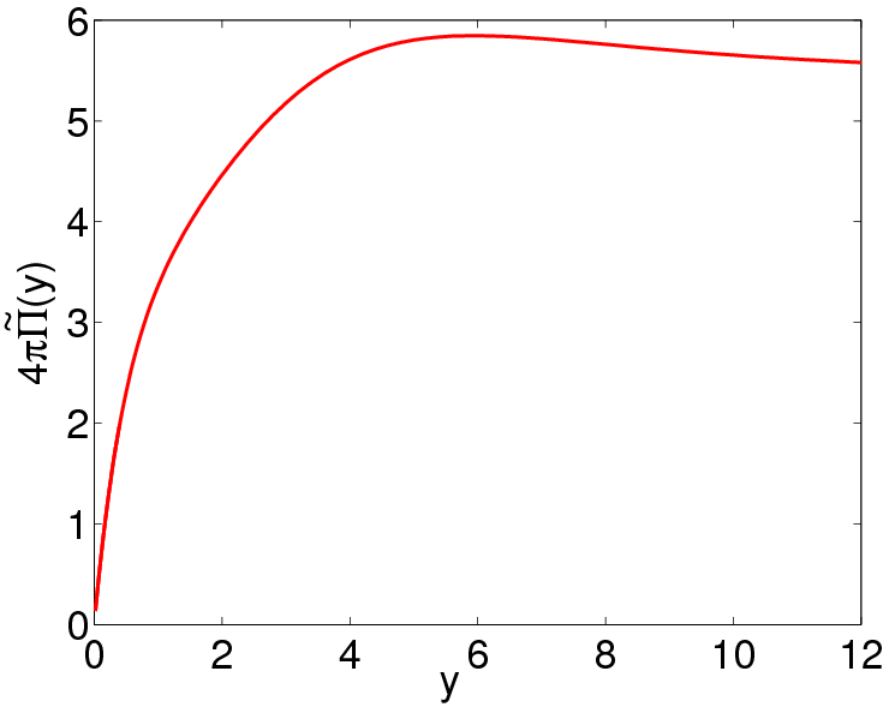
The static polarization function $\Pi(0,\mathbf{k}^2)\equiv\frac{m}{\hbar^2}\tilde{\Pi}(y),\quad y\equiv\frac{\mathbf{k}^2l^2}{2}$.

Because

$$m\sim 10^{-2}m_e\sim 10^8\mathrm{K}/c^2\gg\hbar^2/e^2l,$$

the polarization function dominates in $\tilde{V}_{eff}$:

$$\tilde{V}_{eff}(k)=C(y)\hbar^2/ml^2k^2\,.$$



HARTREE interactions:

$$\tilde{V}_H=-\frac{2\pi e^2 d}{\kappa}$$

$\nu = 0$ QH state

Two competing solutions of the gap equation at the neutrality point $\nu = 0$

(I) spin polarized solution

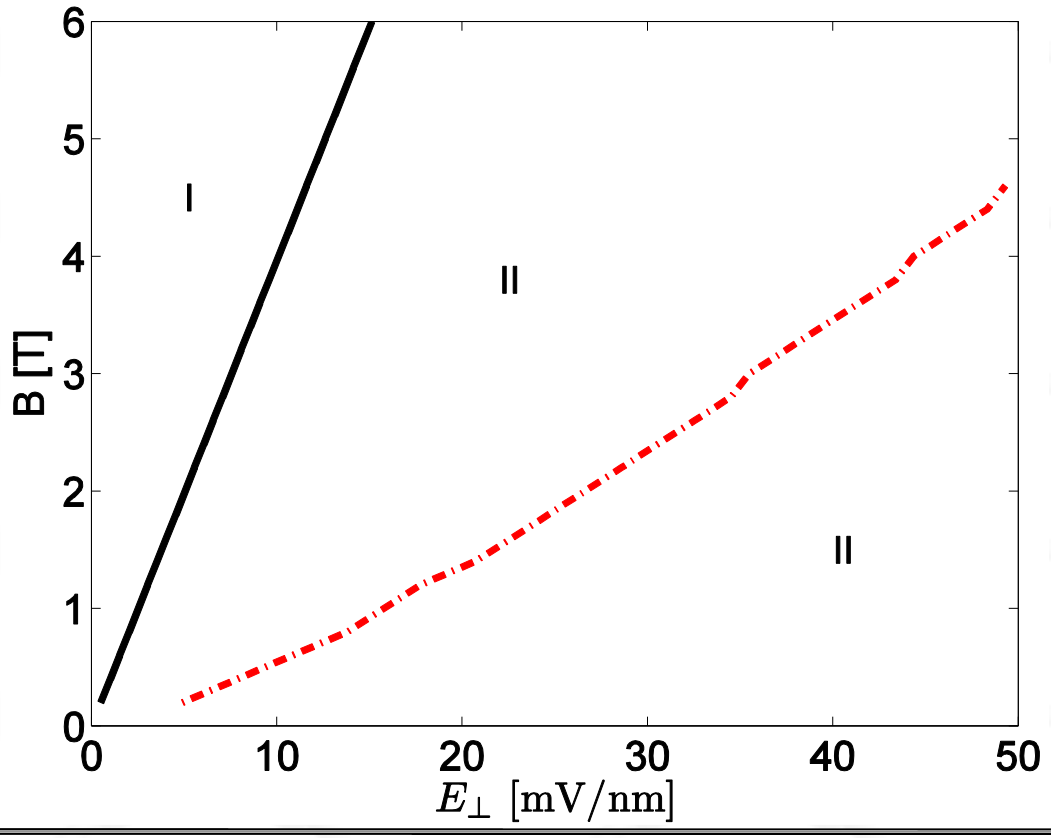
$$\begin{aligned}\Delta E^{(I)} &= (E^{(I)}_{\xi 1+} - E^{(I)}_{\xi 1-})/2 \\ &\simeq 14.4 B[\mathrm{T}]\mathrm{K}\end{aligned}$$

(II) layer polarized solution

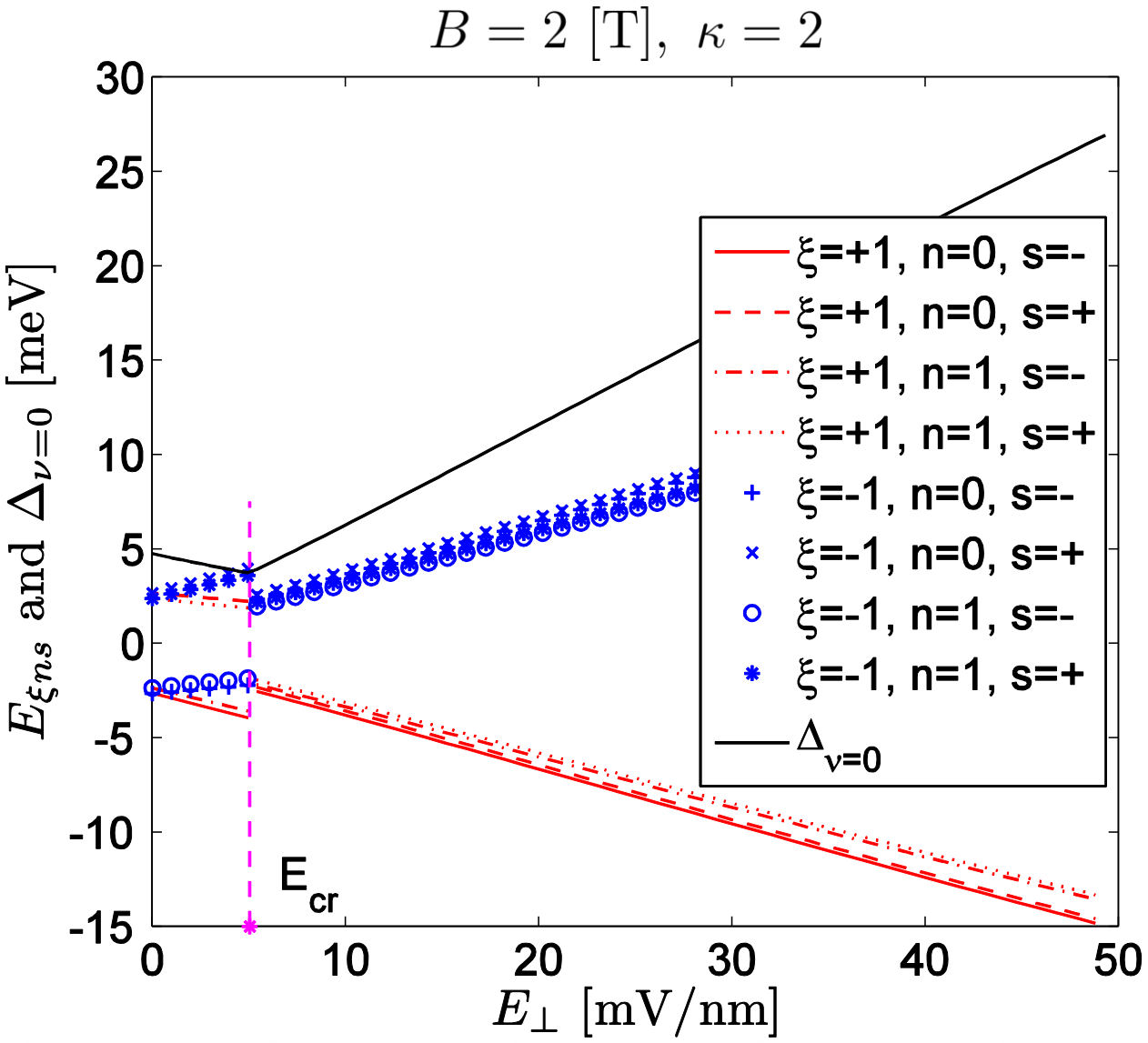
$$\begin{aligned}\Delta E^{(II)} &= (E^{(II)}_{-1-} - E^{(II)}_{+1+})/2 \\ &\simeq 9.3 B[\mathrm{T}]\mathrm{K} + \tilde{\Delta}_0\end{aligned}$$

$$\tilde{\Delta}_0 = eE_\perp d/2$$

$$E^{cr}_\perp \left[\frac{\mathrm{mV}}{\mathrm{nm}}\right] = CB\,[\mathrm{T}], \quad C \lesssim 5$$



Energy spectrum and gap of the $\nu = 0$ QH state



The values of the external field $E_\perp$ where the conductance is not quantized correspond to the minima of the gaps.

Latest Experiments

Weitz, Allen, Feldman, Martin, Yacoby, Science, **330**, 812 (2010)

Martin, Feldman, Weitz, Allen, Yacoby, PRL, **105**, 256806 (2010)

Kim, Lee, Tutuc, PRL. **107**, 016803 (2011)

Freitag, Trbovic, Weiss, Schonenberger, arXiv:1104.3816[cond-mat.mes-hall].

The confirmation of the phase transition between the ferromagnetic (spin polarized) and layer asymmetric (layer polarized) QH states

$$E_{\perp}^{cr}\left[\frac{\mathrm{mV}}{\mathrm{nm}}\right] \sim 10B\,[\mathrm{T}]$$

Higher, $\nu = \pm 1,\ \pm 2$ and ± 3 LLL plateaus

Main experimental results:

a) For $\nu = \pm 1,$ there are two phases separated by approximately the same critical line as that in the $\nu = 0$ QH state.

b) There is only one phase for $\nu = \pm 2$ and $\nu = \pm 3$ QH states.

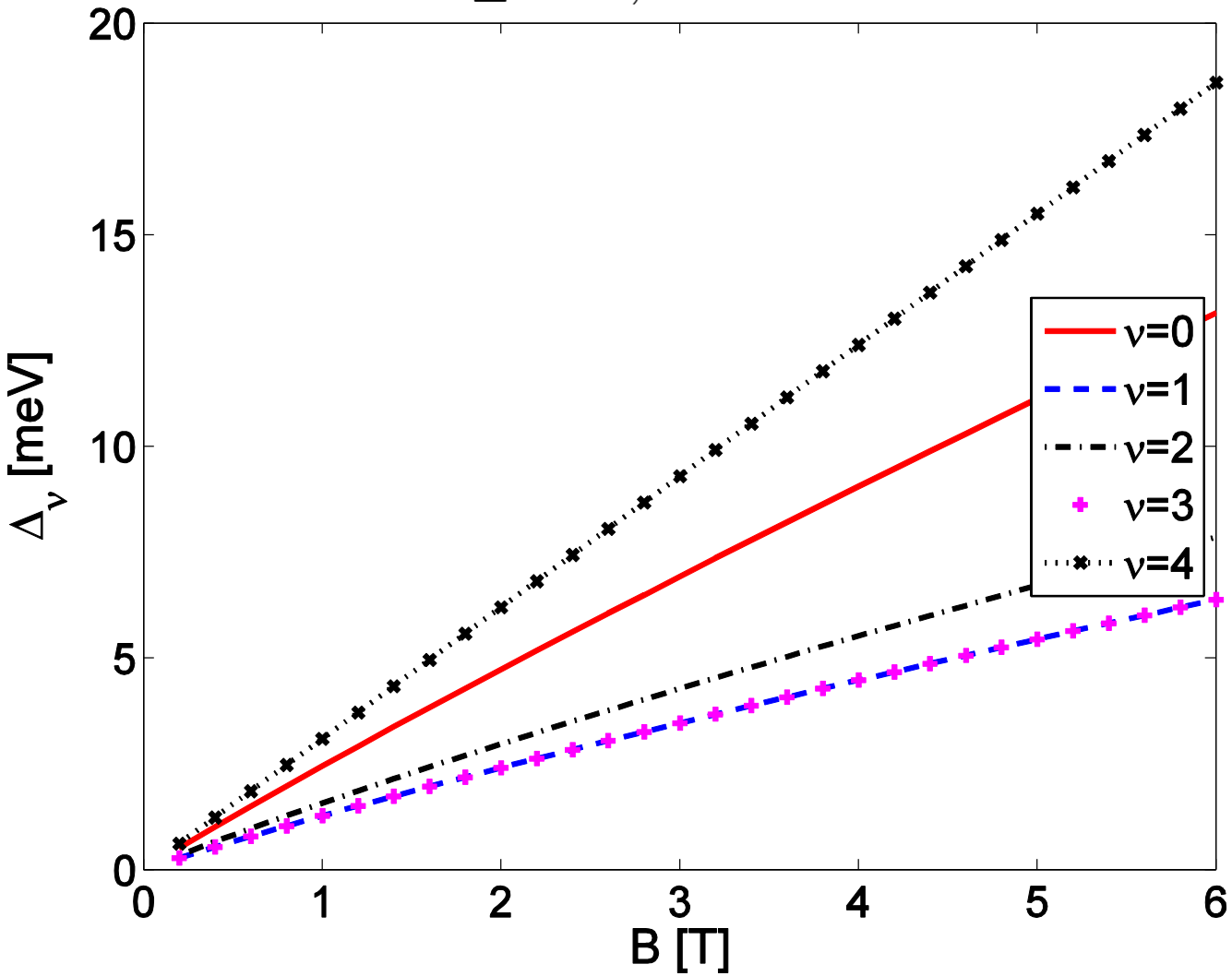
c) The $\nu = 0$ gap is approximately 30-40% larger than the $\nu = \pm 2$ one and significantly (by factor 10) exceeds the $\nu = \pm 1$ gap.

a) For $\nu = 0,\ \pm 1,\ \pm 2$ and $\nu = \pm 3$, the conductance is quantized except at particular values of the electric field $E_{\perp}$.

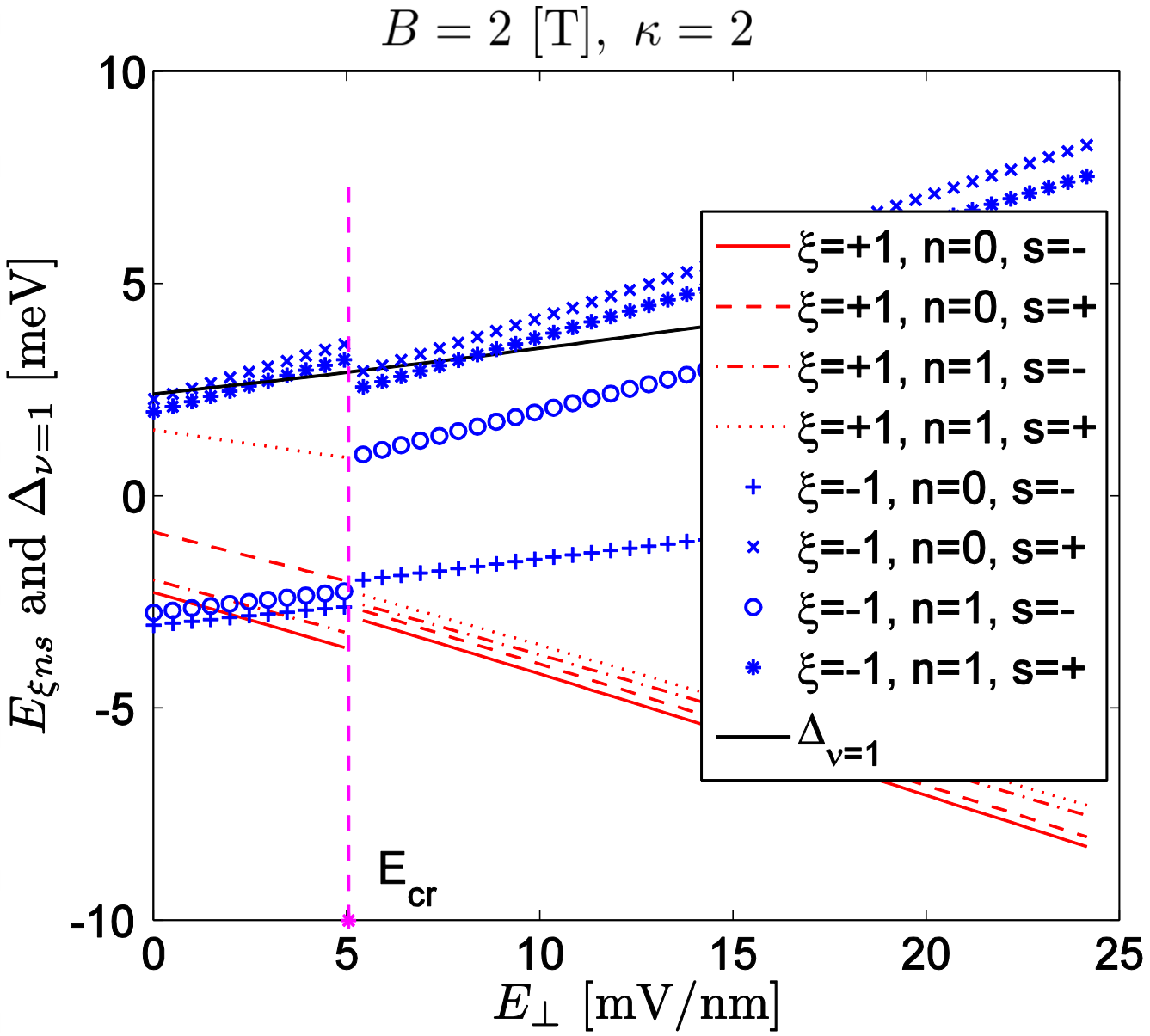
Theory (Gorbar, Gusynin, Junji Jia, V.M., arXiv: 1108.0650 [cond-mat])

The energy gaps as functions of magnetic field at zero electric field for different filling factors:

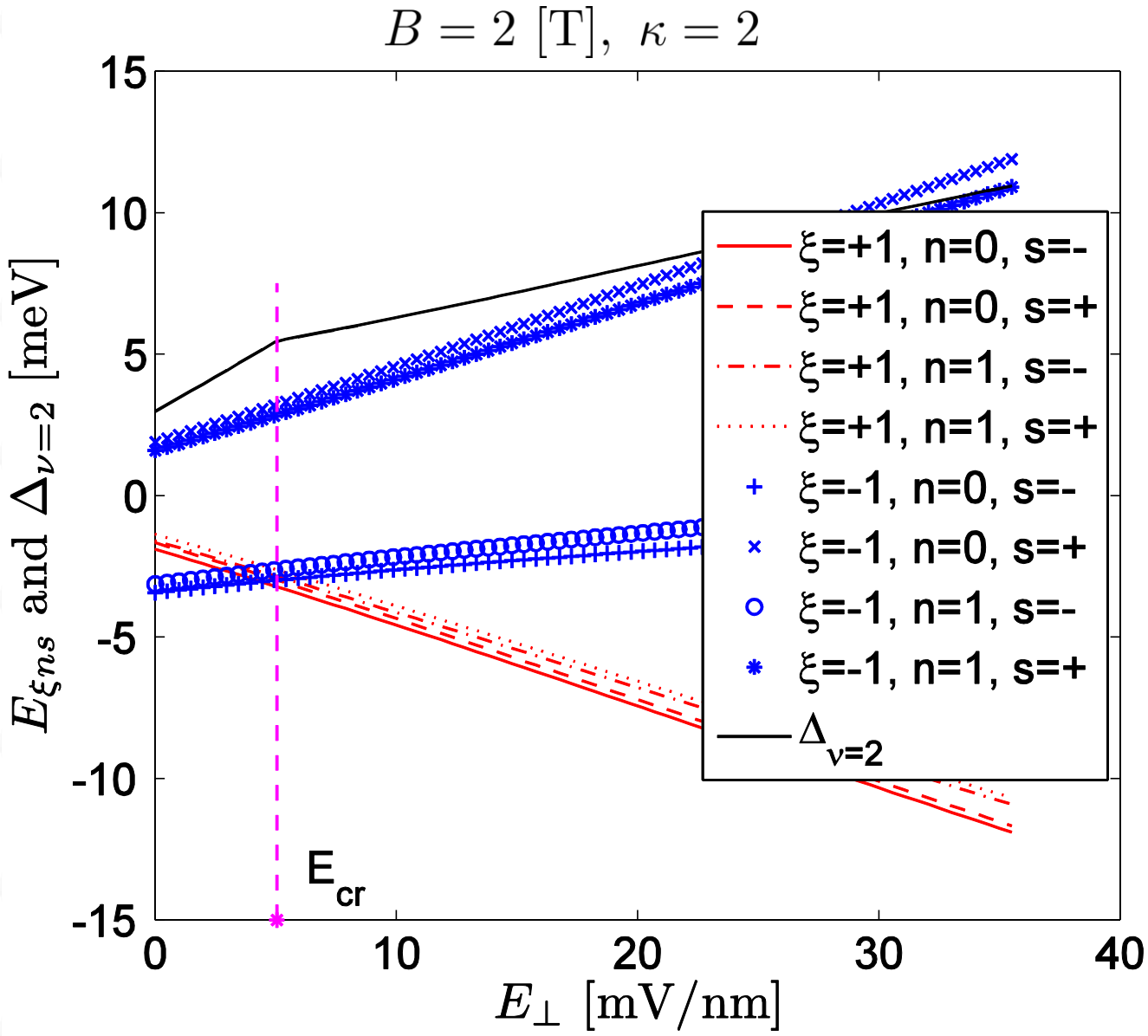
$$E_\perp = 0,\ \kappa = 2$$



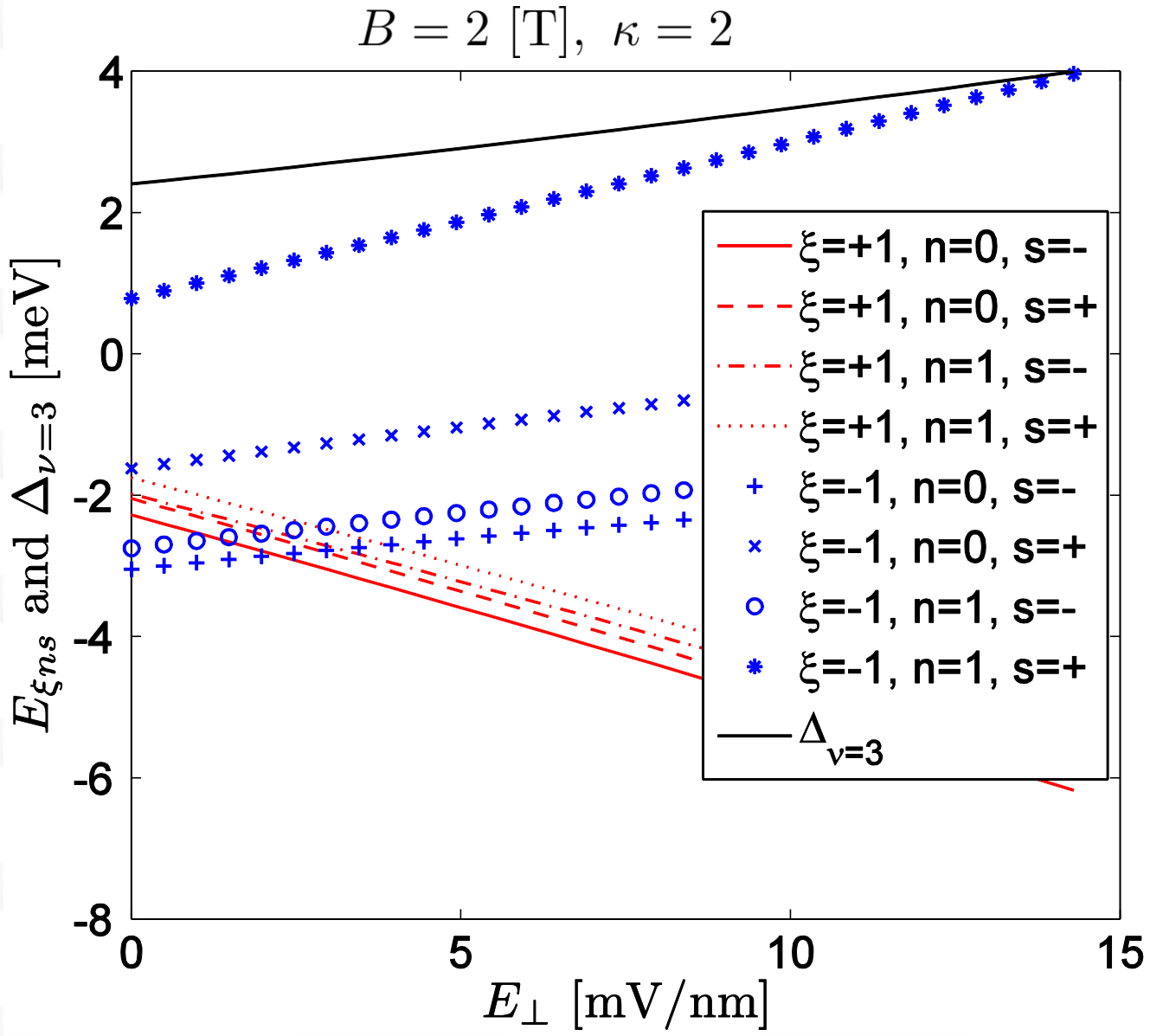
Energy spectrum and gap of the $\nu = 1$ QH state



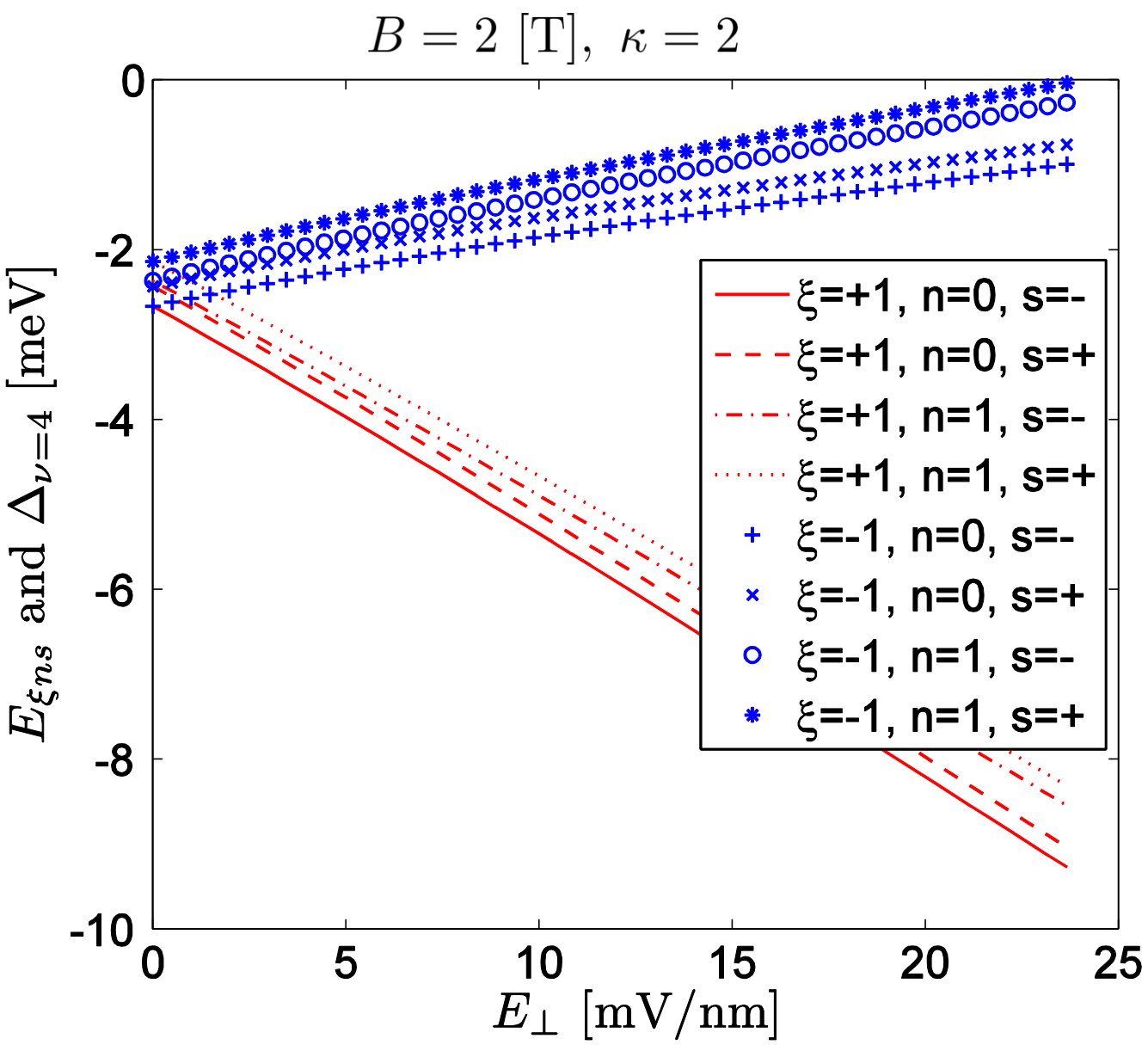
Energy spectrum and gap of the $\nu = 2$ QH state



Energy spectrum and gap of the $\nu = 3$ QH state



Energy spectrum and gap of the $\nu = 4$ QH state



Outlook

(i) The present ansatz with the sixteen order parameters is the minimal one for describing the breakdown of the $U^{(K)}(2)_S \times U^{(K')}(2)_S \times Z_{2V}^{(+)} \times Z_{2V}^{(-)}$ symmetry in bilayer graphene.
It could be extended in order to look for other solutions of the gap equation. A natural extension would be to include order parameters that mix the *n=0* and *n=1* LLL states.

(ii) It would be important to analyze the gap equation with a non-static polarization function.

(iii) It would be interesting to describe explicitly the dynamics around the threshold value B_{thr}, when the crossover between the regimes with the nonrelativistic-like scaling $\Delta E \sim |eB|$ and the relativistic-like one $\Delta E \sim \sqrt{|eB|}$ should take place.

Conclusion

- It seems that a dynamical Dirac mass (masses) is (are) necessarily produced in graphene in a strong magnetic field
- The set of order parameters which describes the QHE in graphene is quite large
- Feedback of QHE in graphene for particle physics: dynamics in dense Quantum Chromodynamics in a strong magnetic field [Gorbar, V.M., Shovkovy, PRD **83**, 085003 (2011)]

  In 3+1 dimensions, the analog of the Haldane mass term, $\Delta\bar{\Psi}\gamma^3\gamma^5\Psi,$ describes an axial-vector current density, rather than a mass.