RGE AND GRAVITATIONAL COUPLING CONSTANTS

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Introduction :
Inflation and late-time acceleration of the universe

2 important acceleration of the universe

Inflation : Accelerated expansion at the early universe

Late-time acceleration : Accelerated expansion at the current universe

The most simple candidate for these acceleration is the Cosmological constant

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \Lambda\right] + S_{\mathrm{matter}}$$

and it cause the de-Sitter expansion which realize the inflation and late-time acceleration

Connect the Inflation and Late-time Acceleration by the Running Gravitational Coupling Constant

The value of cosmological constant of Inflation and late-time acceleration are very different

We construct a kind of RGE of the cosmological constant model and assume that there are two fixed point (UV and IR). Then we connect inflation and late-time acceleration by RG flow.

Low — Energy scale — High

RGE flow

IR fixed point
||
Late-time acceleration

UV fixed point
||
Inflation

Construction of the Model for the Running Gravitational Coupling Constant

We construct the model for RGE of the cosmological constant

$$S = \int d^4x\sqrt{-g}\left[\underbrace{\lambda_\alpha R - \lambda_\Lambda}_{\text{Einstein gravity + Cosmological constant}} + \sum_{i=\Lambda,\alpha}\left(\partial_\mu\lambda_{(i)}\partial^\mu\varphi_{(i)} + \lambda_{(i)}f_{(i)}(\lambda_{(j)})\varphi_{(i)}\right)\right] + S_{\mathrm{matter}}$$

Einstein gravity
+ Cosmological constant

$\lambda_{(i)}$: Gravitational couplingconstant

$f_{(i)}$: Functions of $\lambda_{(i)}$

$\varphi_{(i)}$: Scalar fields

Assumption
- FRW background $ds^2 = -dt^2 + a(t)^2\sum_{i=1}^{3}\left(dx^i\right)^2$
- Scalar fields only depend on time

E.o.M for $\lambda_{(i)}$ under the FRW back ground

$$\frac{d^2\lambda_{(i)}}{dt^2} + 3H\frac{d\lambda_{(i)}}{dt} = \lambda_{(i)}f_{(i)}\left(\lambda_{(j)}\right)$$

→ $\lambda_{(i)}$ are functions of scale factor $\boxed{\lambda_{(i)} = \lambda_{(i)}(a(t))}$

Construct RGE Flow to Connect IR Fixed Point and UV Fixed Point

We regard scale factor as the energy scale of the universe

Small $a(t)$ = High energy

Large $a(t)$ = Low energy

We can construct : $\boxed{\dfrac{d\lambda_{(i)}}{d\tau} = g_i\left(\lambda_{(j)}\right)}$ likewise RGE $a \equiv \mathrm{e}^{\tau}$ g_i : beta function

The important point is that

E.o.M $\dfrac{d^2\lambda_{(i)}}{dt^2} + 3H\dfrac{d\lambda_{(i)}}{dt} = \lambda_{(i)} f_{(i)}\left(\lambda_{(j)}\right)$ can be written by $g_{(i)}(\lambda_{(j)})$ and $f_i\left(\lambda_{(j)}\right)$

$$\xrightarrow{a \equiv \mathrm{e}^{\tau}} \quad H^2\left\{\sum_k \frac{\partial g_i\left(\lambda_{(j)}\right)}{\partial \lambda_{(k)}} g_k\left(\lambda_{(j)}\right) + 3g_i\left(\lambda_{(j)}\right)\right\} - \lambda_{(i)} f_i\left(\lambda_{(j)}\right) = 0$$

Our aim We assume that g_i has UV (IR) fixed point and choose proper $f_i\left(\lambda_{(j)}\right)$ to connect IR fixed point and UV fixed point as an RG flow

$$\lambda_{(i)\mathrm{IR}} \xrightarrow[f_i\left(\lambda_{(j)}\right)]{\text{RGE flow}} \lambda_{(i)\mathrm{UV}}$$

IR fixed point — UV fixed point

Method for the Construction of RGE Flow

Constraint for constructing of the RG flow

① UV (IR) fixed point is obtained from the UV (IR) limit

$$\lambda_{(i)} \begin{cases} \nearrow \ \lambda_{(i)\mathrm{UV}} & \text{: UV limit (early Universe) } a(t) \to 0 \\ \searrow \ \lambda_{(i)\mathrm{IR}} & \text{: IR limit (late-time Universe) } a(t) \to \infty \end{cases}$$

$a(t) \to \infty$ $\qquad$ $a(t)$ $\qquad$ $a(t) \to 0$

$\longleftrightarrow$

IR fixed point $\qquad$ UV fixed point

② At the UV (IR) limit, it realize de-Sitter expansion which correspond to the value of Hubble constant of Inflation (late-time acceleration)

$$\begin{cases} a(t) \to 0 \\ a(t) \to \infty \end{cases} \Rightarrow \begin{cases} H \to H_{\mathrm{UV}} = \mathrm{constant} \\ H \to H_{\mathrm{IR}} = \mathrm{constant} \end{cases}$$

Determine the Behavior of the beta function in the neighborhood of UV (IR) fixed point

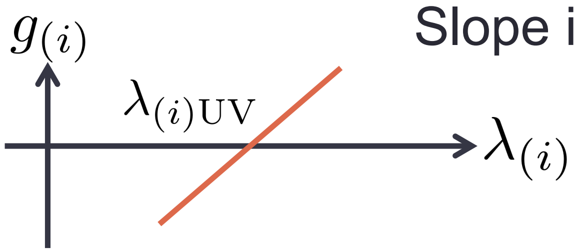
Constraint ① UV (IR) fixed point is obtained from the UV (IR) limit of the scale factor

We determine the relation between beta function and $\lambda_{(i)}$ in the neighborhood of the UV (IR) fixed point (approximation)

Neighborhood of the UV fixed point

$$\boxed{\frac{dg_{(i)}}{d\lambda_{(i)}} = k_{(i)\mathrm{UV}}(\lambda_{(j)}) > 0}$$

Slope is positive



$$\lambda_{(i)} \sim \lambda_{(i)\mathrm{UV}} + a(t)^{k_{(i)\mathrm{UV}}(\lambda_{(j)\mathrm{UV}})}$$

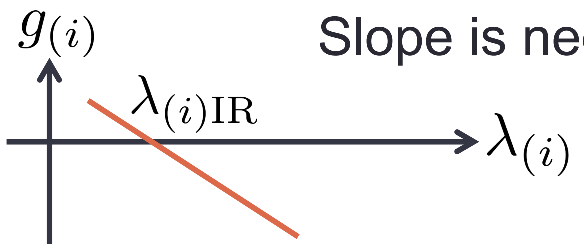
UV limit $a(t) \to 0$

$$\lambda_{(i)} = \lambda_{(i)\mathrm{UV}}$$

Neighborhood of the IR fixed point

$$\boxed{\frac{dg_{(i)}}{d\lambda_{(i)}} = -k_{(i)\mathrm{IR}}(\lambda_{(j)}) < 0}$$

Slope is negative



$$\lambda_{(i)} \sim \lambda_{(i)\mathrm{IR}} + a(t)^{-k_{(i)\mathrm{IR}}(\lambda_{(j)\mathrm{IR}})}$$

IR limit $a(t) \to \infty$

$$\lambda_{(i)} = \lambda_{(i)\mathrm{IR}}$$

Construct RGE Flow

$f_i\left(\lambda_{(j)}\right)$ is determined in the neighborhood of UV (IR) fixed point

$$f_i\left(\lambda_{(j)}\right) \sim \frac{H^2}{\lambda_{(i)\mathrm{UV}}}\left(k_{(i)\mathrm{UV}}(\lambda_{(j)\mathrm{UV}})+3\right) k_{(i)\mathrm{UV}}(\lambda_{(j)\mathrm{UV}})\,\underline{\left(\lambda_{(i)}-\lambda_{(i)\mathrm{UV}}\right)}$$

In the neighborhood of UV fixed point

$+$

$$f_i\left(\lambda_{(j)}\right) \sim \frac{H^2}{\lambda_{(i)\mathrm{IR}}}\left(k_{(i)\mathrm{IR}}(\lambda_{(j)\mathrm{IR}})-3\right) k_{(i)\mathrm{IR}}(\lambda_{(j)\mathrm{IR}})\,\underline{\left(\lambda_{(i)}-\lambda_{(i)\mathrm{IR}}\right)}$$

In the neighborhood of IR fixed point

To satisfy the above condition, we choose $f_i\left(\lambda_{(j)}\right)$ as

$$f_{(i)}\left(\lambda_{(j)}\right) = C_{(i)}\left(\lambda_{(j)}\right)\underline{\left(\lambda_{(i)}-\lambda_{(i)\mathrm{UV}}\right)}\,\underline{\left(\lambda_{(i)}-\lambda_{(i)\mathrm{IR}}\right)} \qquad C(\lambda_{(j)}) > 0$$

Comparing these equations, We determine the region of $C(\lambda_{(j)})$which realize RG flow between UV and IR fixed point from the constraint

Constraint for the Function of the RGE Flow

In the neighborhood of UV fixed point, the relation between $C(\lambda_{(j)})$ and $k_{(i)\mathrm{UV}}$ is

$$\frac{H_{\mathrm{UV}}^2}{\lambda_{(\lambda)\mathrm{UV}}}\left(k_{(i)\mathrm{UV}}(\lambda_{(j)\mathrm{UV}})+3\right)k_{(i)\mathrm{UV}}(\lambda_{(j)\mathrm{UV}})=C_{(i)}\left(\lambda_{(j)\mathrm{UV}}\right)\left(\lambda_{(i)\mathrm{UV}}-\lambda_{(i)\mathrm{IR}}\right)$$

H_{UV} : Hubble constant at the UV fixed point

Solve for $k_{(i)\mathrm{UV}}$

$$\frac{dg_{(i)}}{d\lambda_{(i)}}=k_{(i)\mathrm{UV}}(\lambda_{(j)})>0$$

$$k_{(i)\mathrm{UV}}=-\frac{3}{2}+\underbrace{\frac{1}{2}\sqrt{9+\frac{4\lambda_{(i)\mathrm{UV}}C_{(i)}\left(\lambda_{(j)\mathrm{UV}}\right)}{H_{\mathrm{UV}}^2}\left(\lambda_{(i)\mathrm{UV}}-\lambda_{(i)\mathrm{IR}}\right)}>0}_{\geqq \frac{3}{2}}$$

No constraint for $C(\lambda_{(j)})$

In the neighborhood of IR fixed point, the relation between $C(\lambda_{(j)})$ and $k_{(i)\mathrm{IR}}$ is

$$\frac{H_{\mathrm{IR}}^2}{\lambda_{(\lambda)\mathrm{IR}}}\left(k_{(i)\mathrm{IR}}(\lambda_{(j)\mathrm{IR}})-3\right)k_{(i)\mathrm{IR}}(\lambda_{(j)\mathrm{IR}})=-C_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right)\left(\lambda_{(i)\mathrm{UV}}-\lambda_{(i)\mathrm{IR}}\right)$$

H_{IR} : Hubble constant at the IR fixed point

Solve for $k_{(i)\mathrm{IR}}$

$$\frac{dg_{(i)}}{d\lambda_{(i)}}=-k_{(i)\mathrm{IR}}(\lambda_{(j)})<0$$

$$k_{(i)\mathrm{IR}}=\frac{3}{2}\pm\frac{1}{2}\sqrt{\underbrace{9-\frac{4\lambda_{(i)\mathrm{IR}}C_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right)}{H_{\mathrm{IR}}^2}\left(\lambda_{(i)\mathrm{UV}}-\lambda_{(i)\mathrm{IR}}\right)}_{\geqq 0}}$$

Constraint for $C(\lambda_{(j)})$

$$\boxed{C_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right)\leqq\frac{9H_{\mathrm{IR}}^2}{4\lambda_{(i)\mathrm{IR}}\left(\lambda_{(i)\mathrm{UV}}-\lambda_{(i)\mathrm{IR}}\right)}}$$

As long as $C(\lambda_{(j)})$ satisfy the above constraint, this model surely connect the IR fixed point with the UV fixed point by the RG flow $f_i\left(\lambda_{(j)}\right)$

Obtain Hubble constant by the de-Sitter Solution

We can obtain Hubble constant at the UV (IR) limit by considering de-Sitter solution

$$S=\int d^4x\sqrt{-g}\left[\lambda_\alpha R-\lambda_\Lambda+\sum_{i=\Lambda,\alpha}\left(\partial_\mu\lambda_{(i)}\partial^\mu\varphi_{(i)}+\lambda_{(i)}f_{(i)}(\lambda_{(j)})\varphi_{(i)}\right)\right]+S_{\mathrm{matter}}$$

Neglecting the matter contribution, (0,0)-component of the E.o.M for $\delta g^{\mu\nu}$ is

$$H^2=\frac{1}{6\lambda_{(\alpha)}}\left\{\lambda_{(\Lambda)}-\sum_{i=\Lambda,\alpha}\left(\dot{\lambda}_{(i)}\dot{\varphi}_{(i)}-\lambda_{(i)}f_i(\lambda_j)\varphi_{(i)}\right)\right\}$$

Neighborhood of UV fixed point, E.o.M becomes,

$$H^2\sim\frac{1}{6\lambda_{(\alpha)}}\left(\lambda_{(\Lambda)\mathrm{UV}}+a(t)^{k_\Lambda(\lambda_{(j)})}+\sum_{i=\Lambda,\alpha}k_{(i)}(\lambda_{(j)})Ha(t)^{k_{(i)}(\lambda_{(j)})}\dot{\varphi}_{(i)}\right)$$
$$+\frac{1}{6\lambda_{(\alpha)}}\sum_{i=\Lambda,\alpha}C_{(i)}(\lambda_{(j)})a(t)^{k_{(i)}(\lambda_{(j)})}(\lambda_{(i)\mathrm{UV}}+a(t)^{k_{(i)}(\lambda_{(j)})}-\lambda_{(i)\mathrm{IR}})$$

$\downarrow$ UV limit $a(t)\to 0$

$$\boxed{H=H_{\mathrm{UV}}=\sqrt{\frac{\lambda_{(\Lambda)\mathrm{UV}}}{6\lambda_{(\alpha)\mathrm{UV}}}}=\mathrm{const.}}$$

de-Sitter solution

Obtain Hubble constant by the de-Sitter Solution

Similarly, in the neighborhood of IR fixed point, E.o.M becomes,

$$H^2 \sim \frac{1}{6\lambda_{(\alpha)}}\left(\lambda_{(\Lambda)\mathrm{IR}} + a(t)^{-k_\Lambda(\lambda_{(j)})} - \sum_{i=\Lambda,\alpha} k_{(i)}(\lambda_{(j)}) H a(t)^{-k_{(i)}(\lambda_{(j)})}\dot{\varphi}_{(i)}\right)$$
$$+\frac{1}{6\lambda_{(\alpha)}}\sum_{i=\Lambda,\alpha} C_{(i)}(\lambda_{(j)}) a(t)^{-k_{(i)}(\lambda_{(j)})}(\lambda_{(i)\mathrm{UV}} + a(t)^{-k_{(i)}(\lambda_{(j)})} - \lambda_{(i)\mathrm{IR}})$$

$\downarrow$ IR limit $a(t) \to \infty$

$$\boxed{H = H_{\mathrm{IR}} = \sqrt{\frac{\lambda_{(\Lambda)\mathrm{IR}}}{6\lambda_{(\alpha)\mathrm{IR}}}} = \mathrm{const}}$$

Summary

Two de-Sitter expansion (Inflation and late-time acceleration) is connected by the flow of

$$f_{(i)}\left(\lambda_{(j)}\right) = C_{(i)}\left(\lambda_{(j)}\right)\left(\lambda_{(i)} - \lambda_{(i)\mathrm{UV}}\right)\left(\lambda_{(i)} - \lambda_{(i)\mathrm{IR}}\right)$$

with the constraint

$$C_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right) \leqq \frac{9H_{\mathrm{IR}}^2}{4\lambda_{(i)\mathrm{IR}}\left(\lambda_{(i)\mathrm{UV}} - \lambda_{(i)\mathrm{IR}}\right)} \quad \Rightarrow \quad C_{(i)}\left(\lambda_{(j)\mathrm{IR}}\right) \leqq \frac{4\lambda_{\Lambda)\mathrm{IR}}}{8\lambda_{\alpha)\mathrm{IR}}\lambda_{(i)\mathrm{IR}}\left(\lambda_{(i)\mathrm{UV}} - \lambda_{(i)\mathrm{IR}}\right)}$$

Problems of negative norm and BRS structure of the model

Problem : This model may includes negative norm in QFT

$$S = \int d^4x\sqrt{-g}\left[\lambda_\alpha R - \lambda_\Lambda + \sum_{i=\Lambda,\alpha}\left(\underline{\partial_\mu\lambda_{(i)}\partial^\mu\varphi_{(i)}} + \lambda_{(i)}f_{(i)}(\lambda_{(j)})\varphi_{(i)}\right)\right] + S_{\text{matter}}$$

Indefinite metric → generate negative norm

To remove the negative norm we use BRS quantization, we introduce FP ghosts $b_{(i)}$ and $c_{(i)}$

$$S = \int d^4x\sqrt{-g}\left[\underline{\sum_{i=\Lambda,\alpha}\left(\lambda_{(i)}\mathcal{O}_{(i)} + \partial_\mu\lambda_{(i)}\partial^\mu\varphi_{(i)} + \lambda_{(i)}f_i\left(\lambda_{(j)}\right)\varphi_{(i)} - \partial_\mu b_{(i)}\partial^\mu c_{(i)}\right) + f_i\left(\lambda_{(j)}\right)b_{(i)}c_{(i)}}\right]$$
$$+ S_{\text{matter}} \qquad \equiv \mathcal{L}$$

$$\mathcal{O}_{(\Lambda)} = 1, \quad \mathcal{O}_{(\alpha)} = R$$

This Lagrangian in invariant under the following BRS transformation

$$\boxed{\delta\lambda_{(i)} = \delta c_{(i)} = 0\,, \quad \delta\varphi_{(i)} = \epsilon c\,, \quad \delta b_{(i)} = \epsilon\lambda_{(i)}} \qquad \epsilon\ :\ \text{Fermionic parameter}$$

Negative norm can be consistently removed!

Ref : T. Kugo and I. Ojima, Phys. Lett. B 73 (1978) 459.doi10.1016/0370-2693(78)90765-7

BRS Structure → Topological Structure

Furthermore, this model can be regarded as a kind of topological field theory

E. Witten, Topological Quantum Field Theory," Colmmun. Math. Phys. 117 (1988) 353. doi:10.1007/BF01223371

Topological field theory

Lagrangian

$$\mathcal{L} = \sum_{i=\Lambda,\alpha} \left(\lambda_{(i)}\mathcal{O}_{(i)} + \partial_\mu \lambda_{(i)}\partial^\mu \varphi_{(i)} + \lambda_{(i)} f_i \left(\lambda_{(j)}\right)\varphi_{(i)} - \partial_\mu b_{(i)}\partial^\mu c_{(i)} + f_i\left(\lambda_{(j)}\right) b_{(i)} c_{(i)}\right)$$

is BRS exact = No physical parameter

Lagrangian is obtained from the BRS transformation of

$$\sum_{i=\Lambda,\alpha} \left(-b_{(i)}\left(\mathcal{O}_i + \nabla_\mu \partial^\mu \varphi_{(i)} + f_i\left(\lambda_{(j)}\right)\varphi_{(i)}\right)\right)$$

$$\boxed{\delta\lambda_{(i)} = \delta c_{(i)} = 0\,, \quad \delta\varphi_{(i)} = \epsilon c\,, \quad \delta b_{(i)} = \epsilon\lambda_{(i)}}$$

$$\delta\left(\sum_{i=\Lambda,\alpha} \left(-b_{(i)}\left(\mathcal{O}_i + \nabla_\mu \partial^\mu \varphi_{(i)} + f_i\left(\lambda_{(j)}\right)\varphi_{(i)}\right)\right)\right) = \epsilon\left(\mathcal{L} + (\text{total derivative terms})\right)$$

We can regard the Lagrangian as a kind of topological field theory !

Summary and discussion

Summary

- Since the cosmological constant depends on the scale factor in our model, we can consider RGE and assume that there are IR and UV fixed point
- We determine that UV fixed point corresponds to the Inflation and IR fixed point corresponds to the late-time acceleration
- We construct the model to connect the IR fixed point and UV fixed point by RG flow
- Since this model is BRS invariant, negative norm can be consistently removed
- Furthermore, this model can be regarded as a kind of topological fields theory

Discussion

- Since the scalar field $\lambda_{(i)}$ should have physical value, BRS symmetry

$$\delta\lambda_{(i)} = \delta c_{(i)} = 0\,, \quad \delta\varphi_{(i)} = \epsilon c\,, \quad \delta b_{(i)} = \epsilon\lambda_{(i)}$$

must be broken