Gravitational Wave in Modified Gravities

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S. Capozziello, M. De Laurentis, S. Nojiri and S. D. Odintsov,
"Evolution of gravitons in accelerating cosmologies: The case of extended gravity,"
Phys. Rev. D **95** (2017) no.8, 083524
doi:10.1103/PhysRevD.95.083524
arXiv:1702.05517 [gr-qc]

S. Nojiri and S. D. Odintsov,
"Cosmological Bound from the Neutron Star Merger GW170817 in scalar-tensor and $F(R)$ gravity theories,"
Phys. Lett. B **779** (2018) 425
doi:10.1016/j.physletb.2018.01.078
arXiv:1711.00492 [astro-ph.CO].

K. Bamba, S. Nojiri and S. D. Odintsov,
"Propagation of gravitational waves in strong magnetic fields,"
Phys. Rev. D **98** (2018) no.2, 024002
doi:10.1103/PhysRevD.98.024002
arXiv:1804.02275 [gr-qc].

Introduction

Gravitational Waves
$\Leftarrow$ linearizing ($g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$) the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu}$$

by choosing the transverse and traceless gauge,

$$\nabla^\mu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0$$

$\Rightarrow$

$$\frac{1}{2}\left[-\nabla^2 h_{\mu\nu} - 2R^\lambda{}_\nu{}^\rho{}_\mu h_{\lambda\rho} + R^\rho{}_\mu h_{\rho\nu} + R^\rho{}_\nu h_{\rho\mu} - h_{\mu\nu} R + g_{\mu\nu} R^{\rho\lambda} h_{\rho\lambda} \right] = \kappa^2 \delta T_{\mu\nu} \,.$$

$T_{\mu\nu}$ depends on the metric.
The dependence carries the informations on the mechanism of the expansion of the universe.
$\delta T_{\mu\nu}$ can be different in models even if the expansion history of the universe is identical.

Example: Scalar Tensor Theory

$$S_\phi = \int d^4x \sqrt{-g}\mathcal{L}_\phi\,, \quad \mathcal{L}_\phi = -\frac{1}{2}\omega(\phi) g^{\mu\nu} \partial_\mu\phi\partial_\nu\phi - V(\phi)\,,$$

$$\Rightarrow \ T_{\mu\nu} = -\omega(\phi)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L}_\phi\,,$$

$$\Rightarrow \ \delta T_{\mu\nu} = h_{\mu\nu}\mathcal{L}_\phi + \frac{1}{2}g_{\mu\nu}\omega(\phi)\partial^\rho\phi\partial^\lambda\phi h_{\rho\lambda}\,,$$

Assuming a FRW spatially flat metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} \left(dx^i\right)^2 ,$$

and $\phi = \phi(t)$, we may choose $\phi = t$
$\left(\phi = \phi\left(\tilde\phi\right) \Rightarrow \omega(\phi)\partial_\mu\phi\partial^\mu = \tilde\omega\left(\tilde\phi\right)\partial_\mu\tilde\phi\partial^\mu\tilde\phi,\ \tilde\omega\left(\tilde\phi\right) \equiv \omega\left(\phi\left(\tilde\phi\right)\right)\phi'\left(\tilde\phi\right)^2\right)$,
the FRW equations $\left(H \equiv \frac{\dot a}{a}\right)$

$$\frac{3}{\kappa^2}H^2 = \frac{\omega}{2} + V , \quad -\frac{1}{\kappa^2}\left(2\dot H + 3H^2\right) = \frac{\omega}{2} - V ,$$
$$\Rightarrow\ \omega = -\frac{2}{\kappa^2}\dot H , \quad V = \frac{1}{\kappa^2}\left(\dot H + 3H^2\right) .$$

Then

$$a(t) = \left(\frac{t}{t_0}\right)^\alpha \ \Leftrightarrow\ \omega(\phi) = \frac{2\alpha}{\kappa^2 t_0^2 \phi^2} , \quad V(\phi) = \frac{3\alpha^2 - \alpha}{\kappa^2 t_0^2 \phi^2} .$$

t_0, α: real constants

$$\Leftrightarrow \quad \alpha = \frac{2}{3\,(1+w)}\,.$$

w: equation of state (EoS) parameter
(when Universe is filled with perfect fluid).

$w = 0 \Leftrightarrow$ dust $\sim$ cold dark matter (CDM)

$$\omega(\phi) = \frac{4}{3\kappa^2 t_0^2 \phi^2}\,, \quad V(\phi) = \frac{2}{3\kappa^2 t_0^2 \phi^2}\,.$$

$w = \frac{1}{3} \Leftrightarrow$ radiation

$$\omega(\phi) = \frac{1}{\kappa^2 t_0^2 \phi^2}\,, \quad V(\phi) = \frac{1}{4\kappa^2 t_0^2 \phi^2}\,.$$

Example: Quantum Thermodynamical Scalar Field

Free real scalar field ϕ with mass M

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left(-\frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi - \frac{1}{2} M^2 \phi^2 \right) .$$

Estimation in finite temperature T and chemical potential μ in the flat background,

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_T = \frac{1}{12\pi^2} \delta_{ij} \delta^{kl} \int_0^\infty dk \frac{k^4}{\sqrt{k^2 + M^2}} \frac{\mathrm{e}^{-\beta \left(k^2 + M^2\right)^{\frac{1}{2}} - i\mu}}{1 - \mathrm{e}^{-\beta (k^2 + M^2)^{\frac{1}{2}} - i\mu}} .$$

Tensor structures:

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_T \propto \delta_{ij}\delta^{kl} \ \Leftrightarrow \ \left.\frac{\partial T_{ij}}{\partial g_{kl}}\right|_{\text{Scalar Tensor Theory}} \propto \frac{1}{2}\left(\delta_i{}^k\delta_j{}^l + \delta_i{}^l\delta_j{}^k\right)$$

$\Leftarrow$

$$\frac{\partial T_{ij}}{\partial g_{kl}} = \frac{1}{4}\left(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k\right)\left(\pi^2 - \sum_{n=1,2,3}(\partial_n\phi)^2 - M^2\phi^2\right) + \frac{1}{2}\delta_{ij}\partial^k\phi\partial^l\phi\,.$$

In case of thermal quanta,
1st term$=0$ by on-shell condition $\left(E^2 - k^2 - M^2 = 0\right)$,
2nd term $\sim \left\langle k^k k^l \right\rangle \propto \delta^{kl}$.

In case of scalar tensor theory $\left(M^2\phi^2 \Rightarrow V(\phi)\right)$,
$\phi = \phi(t) \Rightarrow$ 2nd term$=0$.

When the number N of the particles is fixed $N = N_0$ and $T \to 0$ ($\Leftrightarrow$ Cold Dark Matter (CDM))

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_{T=0, N=N_0} = \left. \frac{\partial T_{ij}}{\partial g_{kl}} \right|_{\mathrm{Scalar\ Tensor\ Theory}} = 0\,,$$

but in general,

$$\left\langle : \frac{\partial T_{ij}}{\partial g_{kl}} : \right\rangle_{T} \neq \left. \frac{\partial T_{ij}}{\partial g_{kl}} \right|_{\mathrm{Scalar\ Tensor\ Theory}}\,,$$

for example, $w = \frac{1}{3}$ (radiation).

B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations],
"GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral,"
Phys. Rev. Lett. **119** (2017) no.16, 161101
arXiv:1710.05832 [gr-qc]

Gravitational Wave from Neutron Star Merger

$$\left|\frac{c_{\mathrm{GW}}^2}{c^2} - 1\right| < 6 \times 10^{-15}\,.$$

c: propagating speed of the light
c_{GW}: the propagating speed of the gravitational wave

In case of covariant Galileon model
C. Deffayet, G. Esposito-Farese and A. Vikman, "Covariant Galileon,", Phys. Rev. D **79** (2009) 084003, doi:10.1103/PhysRevD.79.084003, [arXiv:0901.1314 [hep-th]]

$$\begin{aligned}\mathcal{L} =& X + G_4(X)R + G_{4,X}\left(\left(\nabla^2\phi\right)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right)\,, \\ X =& -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\,, \\ G_4(X) =& \frac{M_{\mathrm{Pl}}^2}{2} + \frac{2c_0}{M_{\mathrm{Pl}}}\phi + \frac{2c_4}{\Lambda_4^6}X^2\,,\end{aligned}$$

c_4 term induces the modification of the effective metric for the gravitational wave,

$$\begin{aligned} g_{\mu\nu} \to& g_{\mu\nu} + C\partial_\mu\phi\partial_\nu\phi\,, \\ \Rightarrow \left|\frac{c_{\mathrm{GW}}^2}{c^2} - 1\right| =& \left|\frac{4c_4x^2}{1-3c_4x^2}\right|\,, \quad x = \frac{\dot\phi}{HM_{\mathrm{Pl}}}\,.\end{aligned}$$

J. Sakstein and B. Jain, "Implications of the Neutron Star Merger GW170817 for Cosmological Scalar-Tensor Theories," arXiv:1710.05893.

Propagation of Light

$$0 = \nabla^\mu F_\mu{}^\nu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}\right) = \nabla^2 A^\nu - \nabla^\nu\nabla^\mu A_\mu + R^{\mu\nu}A_\mu\,,$$

$$\Rightarrow$$

$$0 = \sum_{i=1,2,3}\partial_i\left(\partial_i A_t - \partial_t A_i\right)\,,$$

$$0 = \left(\partial_t + H\right)\left(\partial_i A_t - \partial_t A_i\right) + a^{-2}\left(\triangle A_i - \partial_i\sum_{j=1,2,3}\partial_j A_j\right)\,,$$

by assuming a FRW spatially flat metric

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1,2,3}\left(dx^i\right)^2\,,$$

Landau gauge:

$0 = \nabla^\mu A_\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} A_\nu \right) = -\partial_t A_t + 3HA_t + a^{-2} \sum_{i=1,2,3} \partial_i A_i$

$$\Rightarrow \ 0 = \nabla^2 A^\nu + R^{\mu\nu} A_\mu \, .$$

Assume $0 = A_t = \sum_{i=1,2,3} \partial_i A_i$

$$0 = - \left(\partial_t^2 + H \partial_t \right) A_i + a^{-2} \triangle A_i \, .$$

de Sitter space-time $H = H_0$, $a = \mathrm{e}^{H_0 t}$.
Assume $A_i \propto \mathrm{e}^{i\mathbf{k}} \times$ (the part only depending on t)
$\triangle$ by $-k^2 \equiv -\mathbf{k} \cdot \mathbf{k}$.
$s \equiv \mathrm{e}^{-H_0 t}$

$$\Rightarrow \quad 0 = \left(\frac{d^2}{ds^2} + \frac{k^2}{H_0^2} \right) A_i \,,$$
$$\Rightarrow \quad A_i = A_{i0} \cos \left(\frac{k}{H_0} s + \theta_0 \right) \,.$$

Propagation in Scalar-Tensor Theory by GW170817

Gravitational wave in de Sitter space-time by cosmological constant,

$$u = \frac{k}{H_0} s \,, \quad h_{ij} = s^{-\frac{1}{2}} l_{ij} \,.$$

$\Rightarrow$

$$0 = \left(\frac{d^2}{du^2} + \frac{1}{u} + 1 - \frac{\left(\frac{5}{2}\right)^2}{u^2} \right) l_{ij} \,,$$

Bessel's differential equation $\Rightarrow$ Bessel functions $J_{\pm\frac{5}{2}}(u)$.

Black hole/neutron star merger $s \equiv \mathrm{e}^{-H_0 t} \sim 1$. $\frac{k}{H_0} \gg 1$.

$$h_{ij} \sim \frac{1}{s} \cos\left(\frac{k}{H_0} s + \frac{\pm 5 + 1}{4} \pi \right) \,.$$

$\Rightarrow c = c_{\mathrm{GW}}$.

Power-law expansion $a(t) = \left(\frac{t}{t_0}\right)^{\alpha}$ in Scalar-Tensor Theory.

$$\omega(\phi) = \frac{2\alpha}{\kappa^2 t_0^2 \phi^2}, \quad V(\phi) = \frac{3\alpha^2 - \alpha}{\kappa^2 t_0^2 \phi^2}.$$

$\sim$ perfect fluid with a constant equation of state prameter w, $\alpha = \frac{2}{3(1+w)}$.

$$H = \frac{\alpha}{t}, \quad \dot{H} = -\frac{\alpha}{t^2}.$$

Black hole/neutron star merger $\Rightarrow H \sim$ a constant, $H \sim H_0$.
$H^2 \sim \dot{H} \Rightarrow \dot{H} \sim$ a constant, $\dot{H} = H_1$

$$0 = \left(2\dot{H} + 6H^2 + H\partial_t - \partial_t^2 + \frac{\triangle}{a^2} \right) h_{ij}$$

$\Rightarrow$

$$0 = \left(\frac{d^2}{du^2} + \frac{1}{u} + 1 - \frac{\left(\frac{5}{2}\right)^2 - \frac{2H_1}{H_0^2}}{u^2} \right) l_{ij},$$

Solution $J_{\pm\frac{5}{2}\sqrt{1-\frac{4H_1}{25H_0^2}}}(u)$.

$$h_{ij} \sim \frac{1}{s}\cos\left(\frac{k}{H_0}s + \frac{\pm 5\sqrt{1+\beta}+1}{4}\pi\right), \quad \beta \equiv -\frac{4H_1}{25H_0^2},$$

The propagation of the light is not changed.
The propagation of the gravitational wave is not changed, either.
The difference is in phase,

$$\beta = -\frac{4}{25\alpha} = -\frac{6(1+w)}{25},$$

$$S_{F(R)} = \int d^4x\sqrt{-g}F(R)\,.$$

$F(R)$ gravity $\Leftrightarrow$ scalar-tensor theory, under the scale transformation

$$\widetilde{g}_{\mu\nu} = \mathrm{e}^{2\Phi} g_{\mu\nu}\,,$$

Because $\widetilde{h}_i^{\ j} = \widetilde{g}^{lj}\delta\widetilde{g}_{il} = \mathrm{e}^{-2\Phi} g^{lj}\mathrm{e}^{2\Phi}\delta g_{il} = h_i^{\ j}$, $h_i^{\ j}$ results scale invariant in this sense.

Transverse and traceless gauge,

$$\nabla^\mu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0$$

Scale transformation

$$\widetilde{\nabla}^\mu \widetilde{h}_\mu^{\ \nu} = \mathrm{e}^{-\Phi}\nabla^\mu h_{\mu\nu} + D\mathrm{e}^{-\Phi} g^{\mu\sigma} g^{\nu\rho}\Phi_{,\sigma} h_{\mu\rho} - \mathrm{e}^{-\Phi} g^{\nu\rho}\Phi_{,\rho} g^{\mu\sigma} h_{\mu\sigma}\,,$$

$$\Rightarrow\ \widetilde{\nabla}^\mu \widetilde{h}_\mu^{\ \nu} = D\mathrm{e}^{-\Phi} g^{\mu\sigma} g^{\nu\rho}\Phi_{,\sigma} h_{\mu\rho}\,.$$

D: the dimensions of space-time.

$$\widetilde{\nabla}^\mu \widetilde{h}_\mu^{\ \nu} = D\mathrm{e}^{-\Phi}\Phi_{,\mu} h^{\mu\nu} \, .$$

We assume that the background metric and therefore Φ only depend on the cosmological time t and also $g_{ti} = 0$.
$\Rightarrow$ when the perturbation with $h_{t\mu} = 0$,

$$\widetilde{\nabla}^\mu \widetilde{h}_\mu^{\ \nu} = \widetilde{g}^{\mu\nu} \widetilde{h}_{\mu\nu} = 0 \, .$$

The gauge conditions for the graviton are not changed by the scale transformation.

Power law case,
$F(R) \sim R^m \Rightarrow a(t) = \left(\frac{t}{t_0}\right)^\alpha \ \left(\alpha = -\frac{(m-1)(2m-1)}{m-2}\right)$
Scale transformation $F(R)$ gravity $\Rightarrow$ Scalar-Tensor Theory.

$$\tilde{\alpha} = 3\frac{(m-1)^2}{(m-2)^2}, \quad 1 + \tilde{w} = \frac{2}{3\tilde{\alpha}} = \frac{2(m-2)^2}{9(m-1)^2}.$$

Speed of the propagation in the gravitational wave could not be changed by the scale transformation but there is a change of the phase

$$\beta = -\frac{4}{25\tilde{\alpha}} = \frac{12(m-2)^2}{25(m-1)^2} \sim \frac{243\,(1+w)^2}{25},$$

(by assuming $w \sim -1$)
which is different from the case of the scalar-tensor theory.

$$\beta|_{\mathrm{Scalar\ Tensor\ Theory}} = -\frac{6(1+w)}{25},$$

Scalar Tensor Theory

Equation for Gravitational Wave

$$0 = \left(2\dot{H} + 6H^2 + H\partial_t - \partial_t^2 + \frac{\triangle}{a^2} \right) h_{ij} \, .$$

Assuming

$$h_{ij}\left(\mathbf{x}, t\right) = \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} a(t)^2 \hat{h}_{ij}(t) \, ,$$

and defining new time coordinate τ by $d\tau = a(t)^{-3} dt$,

$$0 = \frac{d^2 \hat{h}_{ij}}{d\tau^2} + 4a^6 H^2 \hat{h}_{ij} + k^2 a^4 \hat{h}_{ij} \, , \quad \left(k^2 \equiv \mathbf{k} \cdot \mathbf{k} \right) \, .$$

We consider

$$a(t) = \left(\frac{t}{t_0}\right)^{\alpha} .$$

t_0, α: real constants.
Accelerating expansion: $\alpha \geq 1$,
Decelerating expansion: $0 < \alpha < 1$,
Phantom: $\alpha < 0$.

1. When $t \to 0$,
   1. $\alpha > 1$. In this case, $t \to 0$ corresponds to $\tau \to -\infty$. The solution oscillates $\hat{h}_{ij} \propto \mathrm{e}^{\pm i \left(\frac{1-3\alpha}{1-\alpha}\right) k \left(\frac{t}{t_0}\right)^{1-\alpha}}$ and the absolute value of $\hat{h}_{ij}$ is finite.
   2. $\frac{1}{3} < \alpha < 1$. Even in this case, $t \to 0$ corresponds to $\tau \to -\infty$. The solution diverges as $\hat{h}_{ij} \sim \left(\frac{t}{t_0}\right)^{\eta(1-3\alpha)}$. Here $\eta = \frac{1 \pm \sqrt{1 - \frac{16\alpha^2}{(1-3\alpha)^2}}}{2}$ and the real part of η is positive.
   3. $0 < \alpha < \frac{1}{3}$ case. $\hat{h}_{ij}$ goes to a finite value $\hat{h}_{ij} \propto \mathrm{e}^{-\frac{(1-3\alpha)^2 k^2}{2(1-\alpha)(1+\alpha)} \left(\frac{t}{t_0}\right)^{2(1-\alpha)}}$.
2. When $t \to +\infty$,
   1. $\alpha > 1$. The solution is given by $\hat{h}_{ij} \sim \left(\frac{t}{t_0}\right)^{\eta(1-3\alpha)}$, which decreases for large t because the real part of η is positive.
   2. $0 < \alpha < 1$. The solution oscillates as $\hat{h}_{ij} \propto \mathrm{e}^{\pm i \left(\frac{1-3\alpha}{1-\alpha}\right) k \left(\frac{t}{t_0}\right)^{1-\alpha}}$ and the absolute value of $\hat{h}_{ij}$ is finite.

$F(R)$ gravity

$$F(R) \sim R^m \;\Rightarrow\; a(t) = \left(\frac{t}{t_0}\right)^{\alpha}, \quad \left(\alpha = -\frac{(m-1)(2m-1)}{m-2}\right)$$

When $t \to 0$,

1. $m < \frac{1-\sqrt{3}}{2}$ $(\alpha > 1)$: The solution $\hat{h}_{ij}$ oscillates but its absolute value is finite.
2. $\frac{1-\sqrt{3}}{2} < m < \frac{1}{2}$ $(0 < \alpha < 1)$: The solution diverges.
3. $\frac{1}{2} < m < \frac{5}{4}$ $\left(-5 + 2\sqrt{6} \le \alpha < \frac{1}{3}\right)$: The solution $\hat{h}_{ij}$ goes to a finite value.
4. $\frac{5}{4} < m < \frac{1+\sqrt{3}}{2}$ $\left(\frac{1}{3} < \alpha < 1\right)$: The solution diverges.
5. $\frac{1+\sqrt{3}}{2} < m < 2$ $(\alpha > 1)$: The solution $\hat{h}_{ij}$ oscillates but its absolute value is finite.
6. $m > 2$ $\left(\alpha \le -5 - 2\sqrt{6}\right)$: We note that this case corresponds to $t_E \to \infty$. Then the solution decreases for small t.

When $t \to +\infty$,

1. $m < \frac{1-\sqrt{3}}{2}$ $(\alpha > 1)$: The solution decreases for large t.
2. $\frac{1-\sqrt{3}}{2} < m < \frac{1}{2}$ $(0 < \alpha < 1)$: The solution oscillates and the absolute value of $\hat{h}_{ij}$ is finite.
3. $\frac{1}{2} < m < \frac{5}{4}$ $\left(-5 + 2\sqrt{6} \leq \alpha < \frac{1}{3}\right)$: The solution oscillates and the absolute value of $\hat{h}_{ij}$ is finite.
4. $\frac{5}{4} < m < \frac{1+\sqrt{3}}{2}$ $\left(\frac{1}{3} < \alpha < 1\right)$: The solution oscillates and the absolute value of $\hat{h}_{ij}$ is finite.
5. $\frac{1+\sqrt{3}}{2} < m < 2$ $(\alpha > 1)$: The solution decreases for large t.
6. $m > 2$ $\left(\alpha \leq -5 - 2\sqrt{6}\right)$: The solution oscillates but the absolute value of $\hat{h}_{ij}$ is finite.

Some differences,

For $m > 2$ or $\frac{1}{2} < m < 1$, phantom evolution, which does not appear in case of scalar tensor theory.
$t \to \infty$ corresponds to the infinite past
$t \to 0$ corresponds to the Big Rip singularity.
Near the Big Rip singularity, in case $m > 2$, $\hat{h}_{ij}$ decreases; for $\frac{1}{2} < m < 1$, $\hat{h}_{ij}$ is finite and not oscillating near the Big Rip singularity for $t \to 0$.

Even in quintessence evolution, there is a difference in the exponent.

Scalar tensor: $\hat{h}_{ij} \sim \left(\frac{t}{t_0}\right)^{\eta(1-3\alpha)}$, $F(R)$ gravity: $\hat{h}_{ij} \sim= \left(\frac{t}{t_0}\right)^{\tilde{\eta}(1-3\tilde{\alpha})(-m+2)}$

$\tilde{\alpha} \equiv 3\frac{(m-1)^2}{(m-2)^2}$ $\tilde{\eta} = \tilde{\eta}_\pm \equiv \frac{1\pm\sqrt{1-\frac{16\tilde{\alpha}^2}{(1-3\tilde{\alpha})^2}}}{2}$.

For example, when $m \to -\infty$ ($\alpha \to -m \to +\infty$ and $\tilde{\alpha} \to 3$)

$$\tilde{\eta}\,(1-3\tilde{\alpha})\,(-m+2) \to -\left(4 \pm 2i\sqrt{5}\right)\alpha\,, \quad \eta\,(1-3\alpha) \sim -\left(\frac{3}{2} \pm i\frac{\sqrt{7}}{2}\right)\alpha\,.$$

The decreasing exponent in the $F(R)$ gravity is much larger than the corresponding exponent in the scalar-tensor theory, the ratio is $8/3$.
If one is capable of fixing the exponent by observations, we can distinguish the $F(R)$ gravity from the scalar-tensor theory.

Summary

- We discussed the evolution of cosmological gravitational wave showing how the cosmological background affects their dynamics.
- The detection of cosmological gravitational wave could constitute an extremely important signature to discriminate among different cosmological models.
- We especially considered the cases of scalar-tensor gravity and $F(R)$ gravity where it is demonstrated the amplification of graviton amplitude changes if compared with General Relativity.
- We also show the speed of the gravitational wave by the modified gravity does not change different from the scalar tensor theory without higher derivative couplings.