Extended mimetic gravity:

Hamiltonian analysis and gradient instabilities

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Based on

- **KT**, H. Motohashi, T. Suyama, and T. Kobayashi
  Phys. Rev. D **95**, 084053 (2017), "General invertible transformation and physical degrees of freedom"
- **KT** and T. Kobayashi
  JCAP **1711**, 038 (2017), "Extended mimetic gravity: Hamiltonian analysis and gradient instabilities"

Theorem of Ostrogradsky

- Many scalar-tensor theories with 2nd-order EOMs
  - Brans-Dicke theory

$$S_{\mathrm{BD}} = \int d^4x\sqrt{-g}\left(\phi\mathcal{R} - \frac{\omega_{\mathrm{BD}}}{\phi}X\right)$$

  - $f(\mathcal{R})$ gravity (as scalar-tensor theory)

$$S_f = \int d^4x\sqrt{-g}f(\mathcal{R}) \leftrightarrow S_f' = \int d^4x\sqrt{-g}[f(\phi) + f'(\phi)(\mathcal{R} - \phi)]$$

No extra DOF
(2 tensor + 1 scalar)

- No go for higher-order EOMs: Theorem of Ostrogradsky

Given a Lagrangian $L(q^i, \dot{q}^i, \ddot{q}^i)$. If the kinetic matrix $k_{ij} \equiv \partial^2 L/\partial\ddot{q}^i\partial\ddot{q}^j$ is nondegenerate, then the Hamiltonian of the system is unbounded

$\det k_{ij} \neq 0$

extra DOF associated w/ higher-order EOMs
= "Ostrogradsky ghost"

- A theory without Ostrogradsky ghost ≡ "healthy"

→ Healthy higher-order theories must have degenerate Lagrangian ($\det k_{ij} = 0$)

- The most general ST theory having 2nd-order EL eqs. ⋯ Horndeski theory (Horndeski 1974)
(sufficient condition for the healthiness)

➢ Horndeski theory

- Horndeski/generalized Galileons (Horndeski 1974, Deffayet+ 2011, Kobayashi+ 2011)

$$S_{\rm H} = \int d^4x\sqrt{-g}(L_2 + L_3 + L_4 + L_5),$$

where

$$L_2 = G_2(\phi, X)$$
$$L_3 = G_3(\phi, X)\Box\phi$$
$$L_4 = G_4(\phi, X)\mathcal{R} - 2G_{4X}\left[(\Box\phi)^2 - \phi_\mu^\nu\phi_\nu^\mu\right]$$
$$L_5 = G_5(\phi, X)\mathcal{G}^{\mu\nu}\phi_{\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)\phi_\mu^\nu\phi_\nu^\mu + 2\phi_\mu^\nu\phi_\nu^\lambda\phi_\lambda^\mu\right]$$

$\mathcal{R}$: 4D Ricci scalar
$\mathcal{G}_{\mu\nu}$: 4D Einstein tensor
$X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$
$\phi_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\phi$
$G_{iX} \equiv \partial_X G_i$

- Characterized by 4 arbitrary functions G_2, G_3, G_4, G_5 of (ϕ, X)

- The most general single-field scalar-tensor theory in 4D that yields at most 2nd-order Euler-Lagrange eqs. both for the metric and the scalar field
… trivially evades Ostrogradsky ghost (2 tensor + 1 scalar modes)

- Existing theories are reproduced by suitable choice of G_i's

➢ Diversity of the Horndeski theory

- GR + canonical scalar

$$S_{\rm GR} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}\mathcal{R} - \frac{X}{2} - V(\phi)\right]$$

$X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

$$\Rightarrow \quad G_2 = -\frac{X}{2} - V(\phi), \qquad G_4 = \frac{M_{\rm Pl}^2}{2}, \qquad G_3 = G_5 = 0$$

- $f(\mathcal{R})$ gravity

$$S_f = \int d^4x\sqrt{-g}f(\mathcal{R}) \leftrightarrow S_f' = \int d^4x\sqrt{-g}[f(\phi) + f'(\phi)(\mathcal{R} - \phi)]$$

$$\Rightarrow \quad G_2 = f(\phi) - \phi f'(\phi), \qquad G_4 = f'(\phi), \qquad G_3 = G_5 = 0$$

- nonminimal coupling to the Gauss-Bonnet scalar (Kobayashi+ 2011)

$$S_{\rm GB} = \int d^4x\sqrt{-g}\xi(\phi)\left(\mathcal{R}^{\mu\nu\lambda\sigma}\mathcal{R}_{\mu\nu\lambda\sigma} - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^2\right)$$

$$\Rightarrow \quad G_2 = 2\xi^{(4)}X^2(3 - \ln|X|), \qquad G_3 = 2\xi^{(3)}X(7 - 3\ln|X|),$$

$$G_4 = -2\xi^{(2)}X^2(2 - \ln|X|), \qquad G_5 = -4\xi^{(1)}\ln|X|$$

$\xi^{(n)} \equiv \frac{d^n\xi}{d\phi^n}$

Degenerate scalar-tensor theories

- Degenerate scalar-tensor theories w/ 3 DOFs
  - Horndeski/generalized Galileons (Horndeski 1974, Deffayet+ 2011, Kobayashi+ 2011)
  - GLPV theories (Gleyzes, Langlois, Piazza, and Vernizzi 2015) [also called "beyond Horndeski"]
  - quadratic/cubic DHOST (Langlois+ 2016, Crisostomi+ 2016, Ben Achour+ 2016)

    (Degenerate Higher-Order Scalar-Tensor theories) [also called "extended ST"]

→ Specify all the degenerate theories up to cubic order in $\nabla_\mu \nabla_\nu \phi$

constructed from ϕ and ϕ_μ

$F_0 + F_1 \Box\phi$ could further be included

$$S_{\mathrm{q/c}} = \int d^4x \sqrt{-g} \Big[\underbrace{f_2 \mathcal{R} + a^{\mu\nu\lambda\sigma} \phi_{\mu\nu} \phi_{\lambda\sigma}}_{\text{quadratic}} + \underbrace{f_3 \mathcal{G}^{\mu\nu} \phi_{\mu\nu} + b^{\mu\nu\lambda\sigma\alpha\beta} \phi_{\mu\nu} \phi_{\lambda\sigma} \phi_{\alpha\beta}}_{\text{cubic}} \Big]$$

Chosen so that the Lagrangian is degenerate

$\mathcal{R}$: 4D Ricci scalar

$\mathcal{G}_{\mu\nu}$: 4D Einstein tensor

$X \equiv g^{\mu\nu} \phi_\mu \phi_\nu$

- Some specific case with only 2 DOFs (on a cosmological b.g.)

→ ϕ behaves as an auxiliary field = "cuscuton" (Afshordi+ 2006)

A general class of such theories can be specified ⋯ "extended cuscuton"

(KT, Iyonaga, Kobayashi, *in prep.*)

See Aya Iyonaga's talk!

Viability of DHOST theories

- The absence of Ostrogradsky ghost is not a sufficient condition for a theory to be fully viable.

✓ Stability of...

- cosmological solutions → We focused on the stability of flat FLRW background
- BH solutions

✓ The almost simultaneous detection of GW170817 and GRB170817A implies

$$|c_{GW} - 1| < 10^{-15}$$

→ A trivial class of DHOST theories where $c_{GW} = 1$ (Langlois+ 2017)

$$L^{\mathrm{DHOST}}_{c_{GW}=1} = F(\phi,X)\mathcal{R} + P(\phi,X) + Q(\phi,X)\Box\phi + A_3(\phi,X)L_3 + A_4(\phi,X)L_4 + A_5(\phi,X)L_5,$$

where

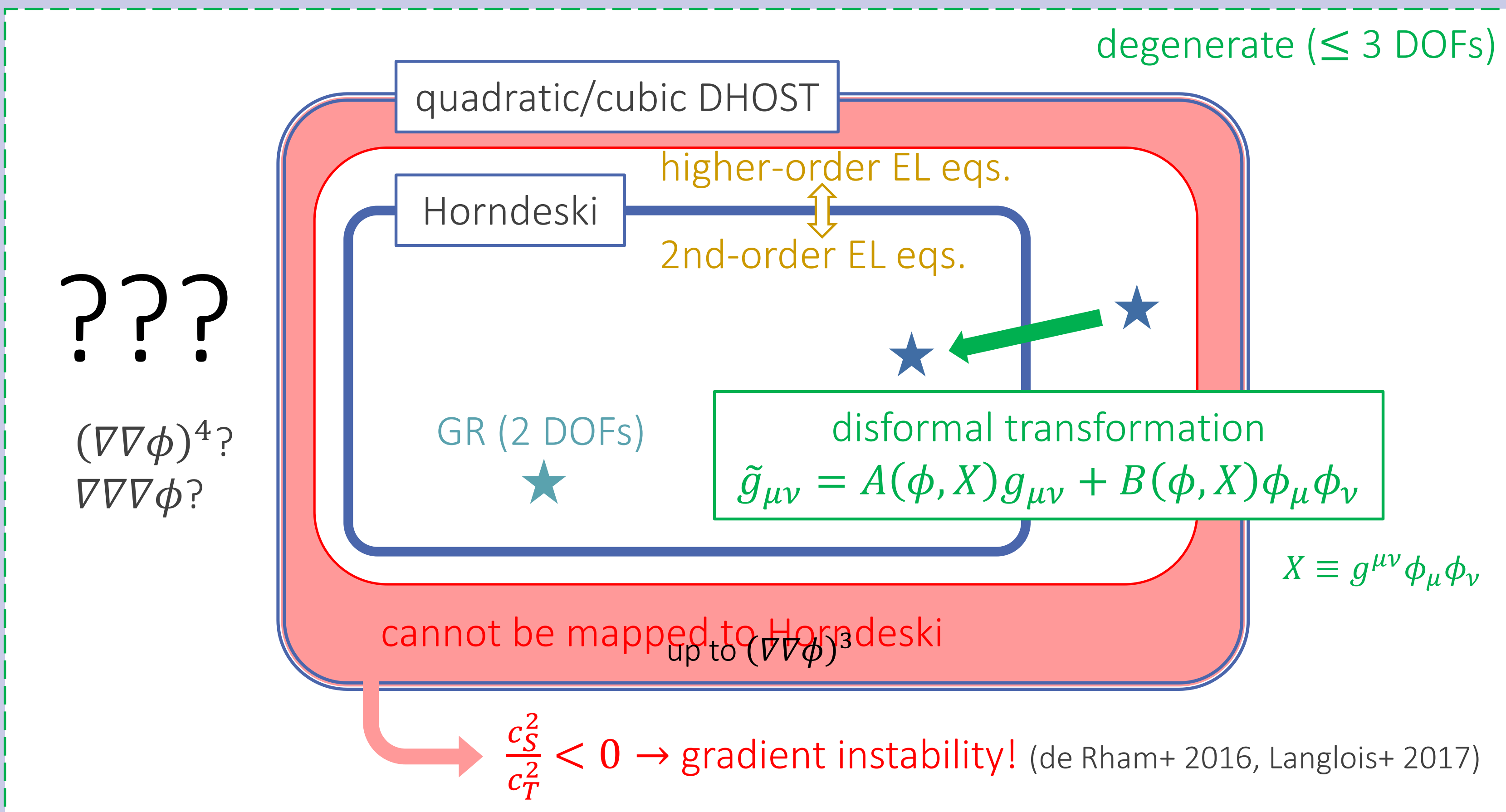
$$L_3 = (\Box\phi)\phi^\mu\phi^\nu\phi_{\mu\nu}, \qquad L_4 = \phi^\mu\phi^\nu_\mu\phi^\lambda_\nu\phi_\lambda, \qquad L_5 = \left(\phi^\mu\phi^\nu\phi_{\mu\nu}\right)^2,$$

$$A_4 = \frac{1}{8F}\left[48F_X^2 - 8(F - XF_X)A_3 - X^2A_3^2\right], \qquad A_5 = \frac{1}{2F}(4F_X + XA_3)A_3.$$

However, note that the constraint on c_{GW} applies only to low-z universe ($z < 0.01$).

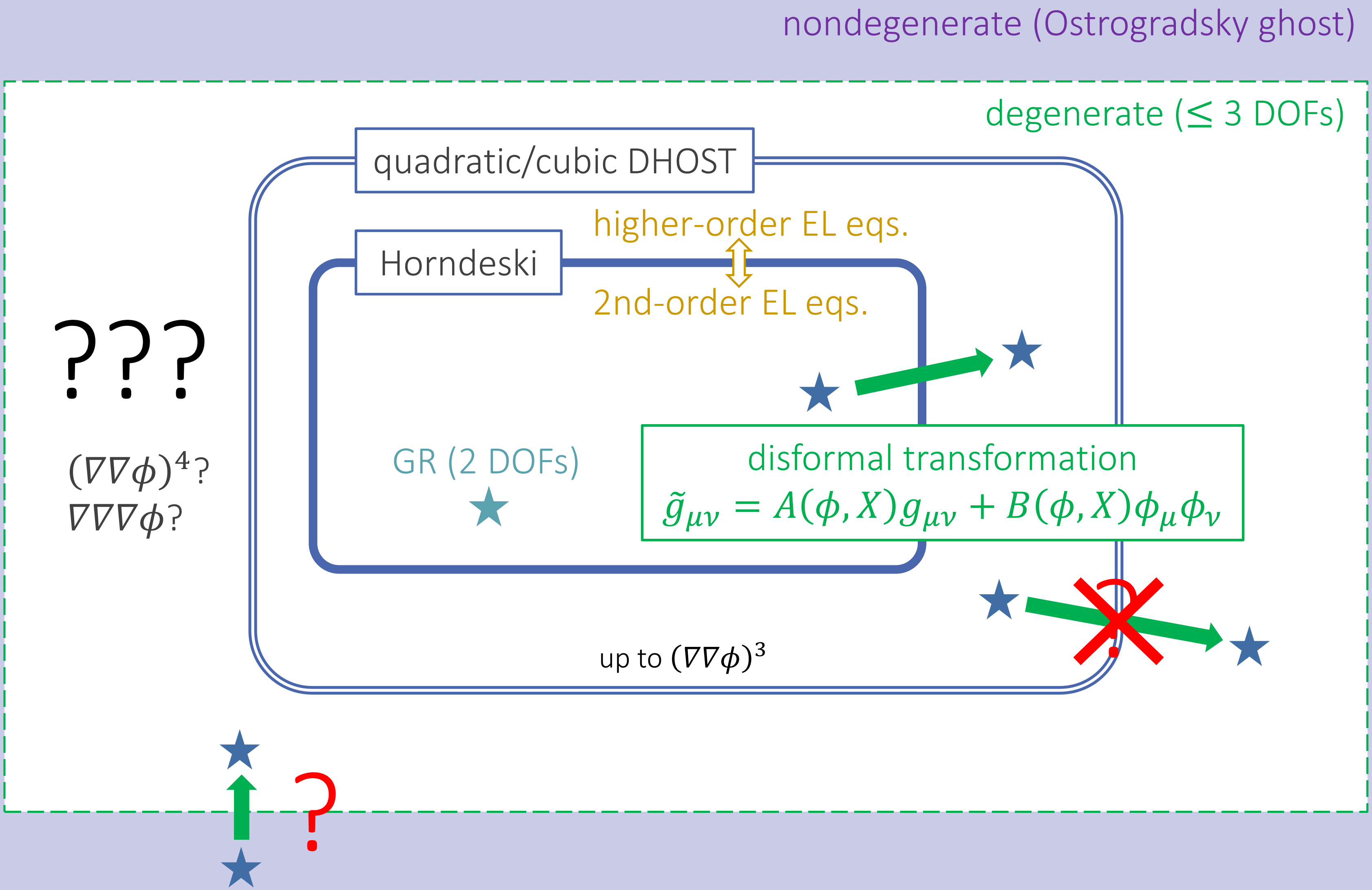
➤ Problem in beyond Horndeski theories?

nondegenerate (Ostrogradsky ghost)



(except for those w/ nondynamical tensor modes)

Possible extension



Way out

- How can we go outside the quadratic/cubic DHOST class?

  ➡ Let's start from nondegenerate theories!

invertible: $\tilde{g}_{\mu\nu} \leftrightarrow g_{\mu\nu}$ one-to-one

- If the transformation is invertible, the resultant theory is also nondegenerate

  (∵ DOFs are invariant under invertible trnsf.) KT, Motohashi, Suyama, and Kobayashi 2017

  ➡ Let's consider noninvertible (singular) transformations!

- Noninvertible conformal transformation

  N.B. Any noninvertible disformal trnsf can be recast in this form

$$\tilde{g}_{\mu\nu} = -X g_{\mu\nu}$$

$X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

$X \to \Omega^{-2}X$ under $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$

$\Rightarrow$ $\tilde{g}_{\mu\nu}$: invariant

- Mimetic gravity

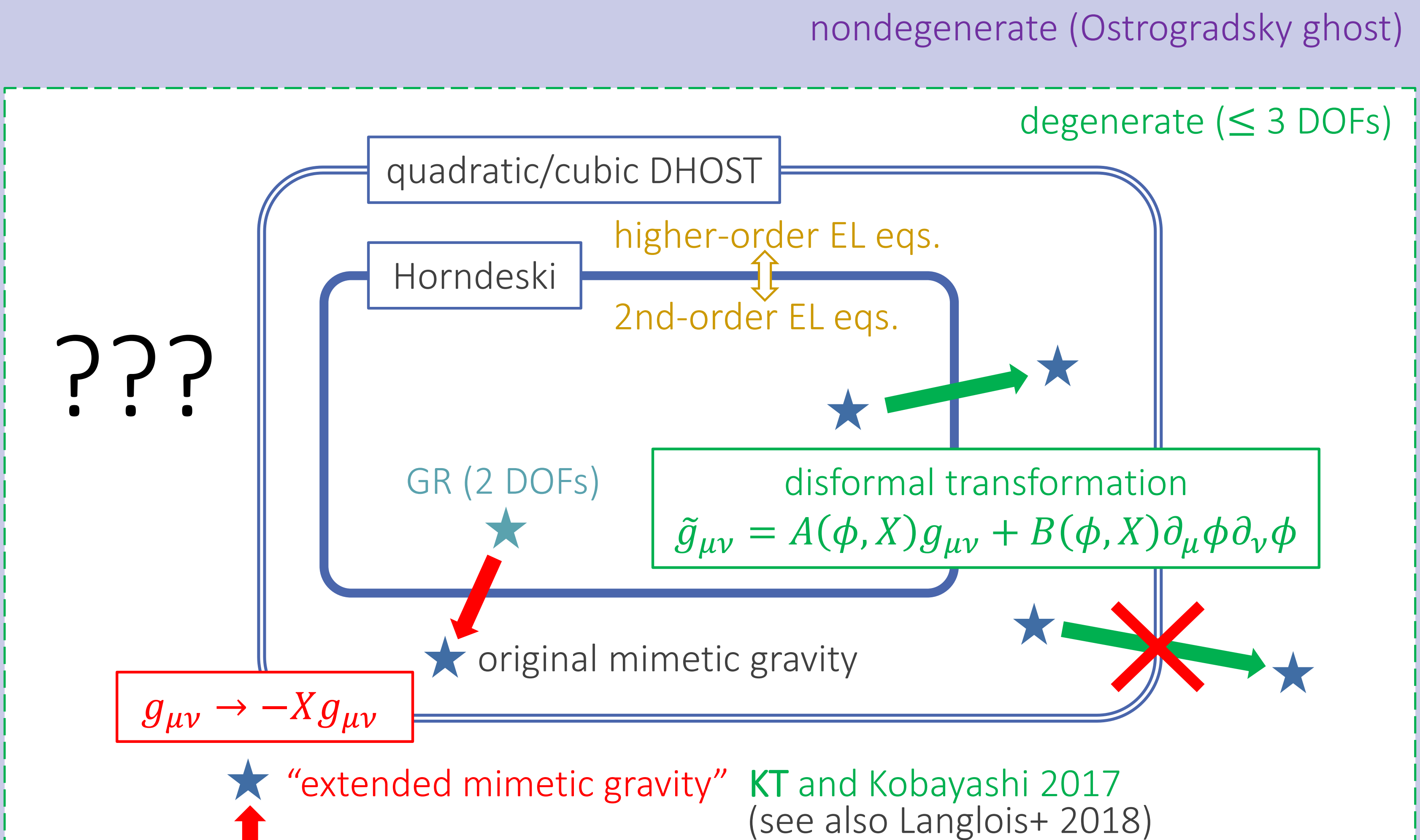
$\tilde{S}_{\rm seed}[\tilde{g}_{\mu\nu}, \phi]$ ➡ $S_{\rm mim}[g_{\mu\nu}, \phi]$ conformal sym.

"seed" ST theory — "mimetic theory"

- mimetic = "Copying the behaviour or appearance of sb/sth else"

  Mimetic gravity can mimic dark matter (Chamseddine and Mukhanov 2013)

Mimetic theories



$$S_{\rm seed} = \int d^4x \sqrt{-g} \left[f_2 \mathcal{R} + f_3 \mathcal{G}^{\mu\nu} \nabla_\mu \nabla_\nu \phi + F\left(g_{\mu\nu}, \phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi \right) \right]$$

ADM representation of mimetic gravity

$$S_{\rm seed}[g_{\mu\nu},\phi]=\int d^4x\sqrt{-g}\left[f_2\mathcal{R}+f_3\mathcal{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi+F\left(g_{\mu\nu},\phi,\nabla_\mu\phi,\nabla_\mu\nabla_\nu\phi\right)\right]$$

Nondegenerate!

$g_{\mu\nu}\to\tilde{g}_{\mu\nu}=-Xg_{\mu\nu}$ → $S_{\rm mim}[g_{\mu\nu},\phi]$

- ADM decomposition

$$ds^2=-N^2dt^2+\gamma_{ij}\left(dx^i+N^idt\right)\left(dx^j+N^jdt\right)$$

- ADM representation of extended mimetic gravity models takes the form

Lagrange multiplier fixes

$$A_*=\frac{\dot{\phi}-N^iD_i\phi}{N}$$

$$S_{\rm mim}[g_{\mu\nu},\phi]=\int dtd^3x\left[N\sqrt{\gamma}L_{\rm M}\left(\gamma_{ij},R_{ij},\phi,A_*;V_{ij};D_i\right)+\Lambda\left(NA_*+N^iD_i\phi-\dot{\phi}\right)\right]$$

$$V_{ij}\equiv K_{ij}+\frac{\dot{A}_*-D^k\phi D_kN-N^kD_kA_*}{NA_*}\gamma_{ij}$$

$$K_{ij}=\frac{1}{2N}\left(\dot{\gamma}_{ij}-2D_{(i}N_{j)}\right)$$

D_i: 3D covariant derivative

- Highest derivatives $\dot{\gamma}_{ij}$ and $\ddot{\phi}$ appear only in the combination V_{ij} ➡ degenerate!

- In the Hamiltonian analysis, an additional primary constraint appears (next slide)

Summary of the Hamiltonian analysis

- For a technical purpose, we consider

$$S'[g_{\mu\nu},\phi] \equiv S_{\rm mim}\Big|_{V_{ij}\to B_{ij}} + \int dt d^3x\, N\lambda^{ij}(B_{ij}-V_{ij})$$

- Canonical variables ⋯ 50-dim. phase space

$$\begin{pmatrix} N & N^i & \gamma_{ij} & \phi & A_* & B_{ij} & \Lambda & \lambda^{ij} \\ \pi_N & \pi_i & \pi^{ij} & p_\phi & p_* & p^{ij} & P & P_{ij} \end{pmatrix}$$

Not all of them evolve independently (↔ constraints)

- First-class constraints ⋯ 9 in total

For short, I write "$\pi_N \approx 0$" → "π_N", etc.

$\underbrace{\pi_N, \quad \pi_i,}_{\text{4D diffeo.}} \quad \underbrace{\mathcal{H}, \quad \mathcal{H}_i,}_{\text{EM conservation}} \quad \underbrace{\bar{\mathcal{C}}}_{\text{conformal}}$ ← peculiar to mimetic gravity!

- Second-class constraints ⋯ 26 in total

$$\bar{\pi}^{ij}, \quad \bar{p}_\phi, \quad p^{ij}, \quad P, \quad P_{ij}, \quad \varphi^{ij}$$

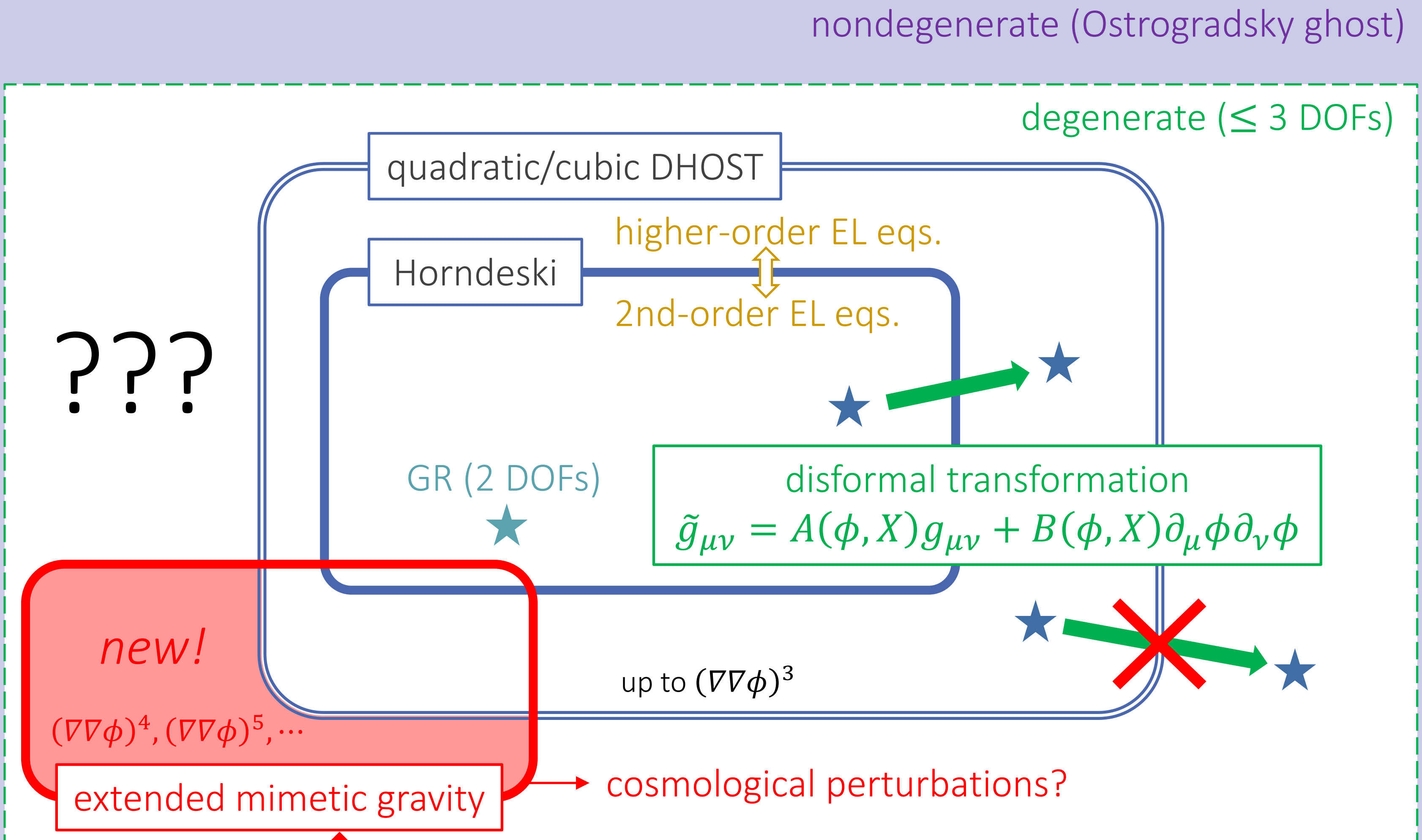
- The number of physical DOFs

$$\frac{1}{2}(50 - 9\times 2 - 26) = 3$$

50: phase-space dim.; 9: # of first-class; 26: # of second-class

No extra DOF!

Relation to the known classes



$$g_{\mu\nu} \to -X g_{\mu\nu} \qquad S_{\rm seed} = \int d^4x \sqrt{-g} \left[f_2 \mathcal{R} + f_3 \mathcal{G}^{\mu\nu} \nabla_\mu \nabla_\nu \phi + F\left(g_{\mu\nu}, \phi, \nabla_\mu \phi, \nabla_\mu \nabla_\nu \phi\right) \right]$$

➤ Cosmological perturbations

- Gauge fixing: $\phi = t$ and $X = -1$ $(\to N = 1)$

$$S_{\rm mim} = \int dt d^3x \sqrt{\gamma} \left[\left(f_2 - \frac{1}{2}\dot{f}_3 \right) R + \mathcal{F}(t, K, \mathcal{K}_2, \mathcal{K}_3, \cdots, \mathcal{K}_\ell) \right],$$

$$\mathcal{K}_1 \equiv K$$

$$\mathcal{K}_n \equiv K^{i_1}_{i_2} K^{i_2}_{i_3} \cdots K^{i_n}_{i_1}$$

- Metric ansatz (flat FLRW background + perturbations)

$$N = 1, \qquad N_i = \partial_i \chi, \qquad \gamma_{ij} = a^2(t) e^{2\zeta} \left(\delta_{ij} + h_{ij} + \frac{1}{2} h_{ik} h_{kj} \right)$$

scalar pert. (χ, ζ) TT tensor pert. (h_{ij})

- Tensor quadratic action

$$S_{\rm T}^{(2)} = \int dt d^3x \frac{a^3}{4} \left[\mathcal{B} \dot{h}_{ij}^2 - \mathcal{E} \frac{(\partial_k h_{ij})^2}{a^2} \right]$$

Scalar quadratic action

$$S_{\rm S}^{(2)} = 2 \int dt d^3x a^3 \left[\frac{3\mathcal{A} + 2\mathcal{B}}{\mathcal{A} + 2\mathcal{B}} \mathcal{B} \dot{\zeta}^2 + \mathcal{E} \frac{(\partial_k \zeta)^2}{a^2} \right]$$

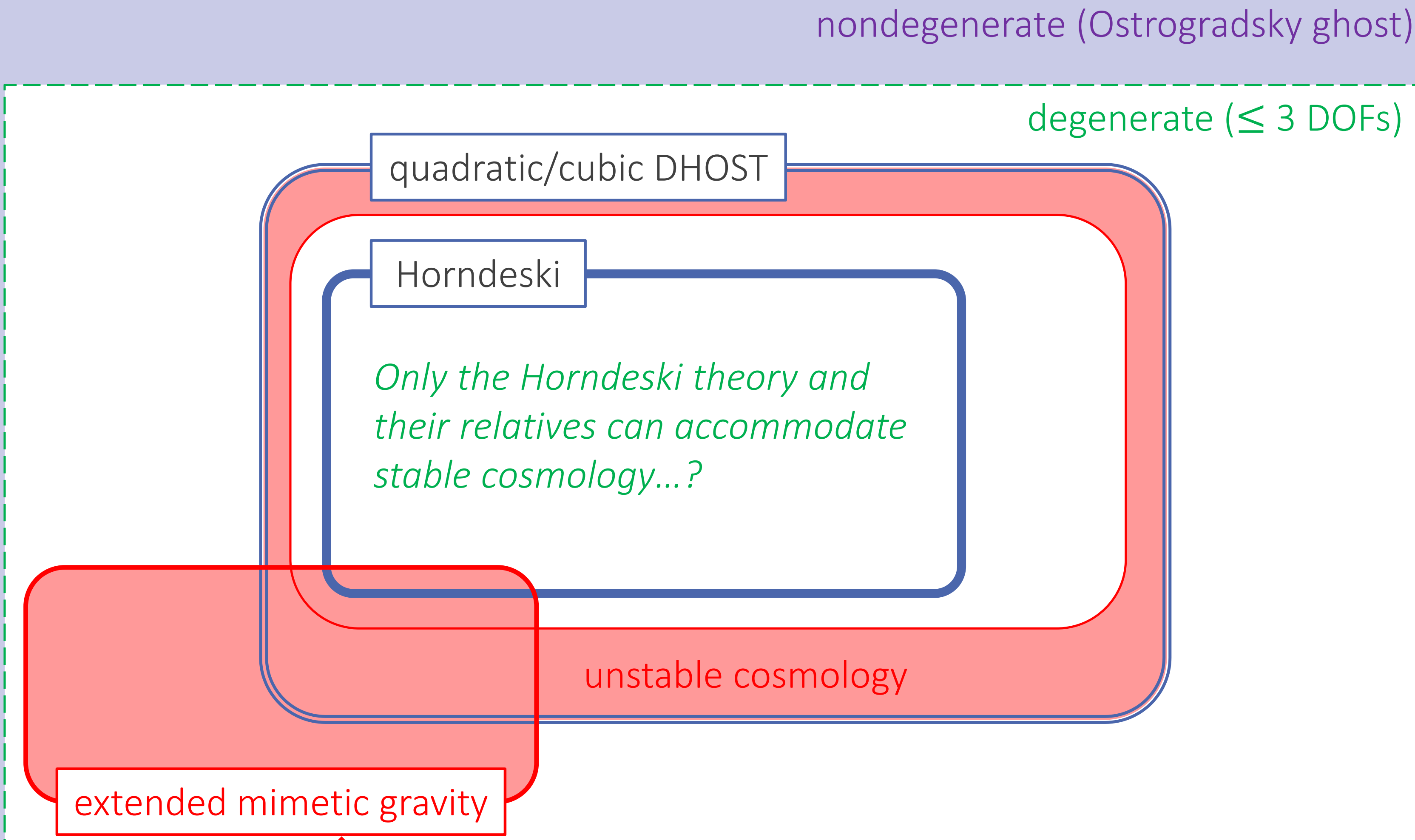
opposite sign! → gradient instabilities!

Here,

$$\mathcal{A} \equiv \sum_{m=1}^{\ell} \sum_{n=1}^{\ell} mn H^{m+n-2} \mathcal{F}_{mn}, \qquad \mathcal{B} \equiv \sum_{n=2}^{\ell} \frac{n(n-1)}{2} H^{n-2} \mathcal{F}_n, \qquad \mathcal{E} \equiv f_2 - \frac{1}{2}\dot{f}_3$$

$$\mathcal{F}_{mn} \equiv \frac{\partial^2 \mathcal{F}}{\partial \mathcal{K}_m \partial \mathcal{K}_n}, \qquad \mathcal{F}_n \equiv \frac{\partial \mathcal{F}}{\partial \mathcal{K}_n}$$

➢ "No go" for beyond Horndeski theories?



$$g_{\mu\nu} \to -X g_{\mu\nu}$$

$$S_{\rm seed} = \int d^4x\sqrt{-g}\left[f_2\mathcal{R} + f_3\mathcal{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi + F\left(g_{\mu\nu},\phi,\nabla_\mu\phi,\nabla_\mu\nabla_\nu\phi\right)\right]$$

➢ Conclusions

- How can we go beyond the quadratic/cubic DHOST class via disformal transformation?

  ➡ Consider a noninvertible transformation of nondegenerate theories!

  (∵ Invertible transformations cannot change the DOFs) KT, Motohashi, Suyama, and Kobayashi 2017

- "Extended mimetic gravity" KT and Kobayashi 2017

  Perform $\tilde{g}_{\mu\nu} = -X g_{\mu\nu}$ on theories with 4 DOFs:

$$\tilde{S}_{\rm seed}[\tilde{g}_{\mu\nu}, \phi] = \int d^4x \sqrt{-\tilde{g}} \left[f_2 \tilde{\mathcal{R}} + f_3 \tilde{\mathcal{G}}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi + F\left(\tilde{g}_{\mu\nu}, \phi, \tilde{\nabla}_\mu \phi, \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi\right) \right]$$

  functions of $(\phi, \tilde{X})$ $\quad \tilde{X} \equiv \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$

  ➡ The resultant theory $S_{\rm mim}[g_{\mu\nu}, \phi]$ has only 3 DOFs due to the conformal sym.

- Cosmological perturbations suffer from gradient instabilities

- Similar extension? Phenomenology?