Scalar-tensor theories in metric-affine formalism

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Based on arXiv: 1806.02589.

Introduction

GR is 1) a theory of massless spin-2 field
2) a theory of curved geometry

Extensions of GR (focusing on 1?)

Massless → massive: massive gravity, bigravity…

Spin-2 → other spin: scalar-tensor, vector-tensor…

We have to take care 2 when extending GR.

Physics should require how to measure the distance and the derivative.

→ **two independent objects, metric** $g_{\mu\nu}$ **and connection** $\Gamma^{\mu}_{\alpha\beta}$.

Riemannian geometry: metric is only independent object

$$\Gamma^{\mu}_{\alpha\beta} = \left\{ {\mu \atop \alpha\beta} \right\} := \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right)$$

Just a special case!

Introduction

General geometry = two independent objects, metric and connection.

$g_{\mu\nu}$ $\Gamma^{\mu}_{\alpha\beta}$

a) Metric compatibility condition (40 conditions): $\overset{\Gamma}{\nabla}_{\alpha} g_{\mu\nu} = 0$

Length and angle do not change under the parallel transport.

⇒ Riemann-Cartan geometry

b) Torsionless condition (24 conditions): $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$

No twist under the parallel transport.

⇒ Riemannian geometry: metric is only independent object

$$\Gamma^{\mu}_{\alpha\beta} = \left\{ {\mu \atop \alpha\beta} \right\} := \frac{1}{2} g^{\mu\nu} (\partial_{\alpha} g_{\beta\nu} + \partial_{\beta} g_{\alpha\nu} - \partial_{\nu} g_{\alpha\beta})$$

Riemannian geometry is a special class of the general geometry.

Metric and Metric-affine formalisms

- **Metric formalism**: Gravity is a theory of metric (= spin-2 field)
  → Gravitational theory determines only metric (Riemannian geometry)

- **Metric-affine (Palatini) formalism**: Gravity is a theory of geometry
  → Gravitational theory determines not only metric but also connection.

  No assumption on the connection.

GR in metric formalism is obtained from GR in metric-affine formalism.

(Giachetta and Mangiarotti 1997, Dadhich and Pons, 2012 for example)

$$S_{\text{gravity}}(g,\Gamma) = \int d^4x\sqrt{-g}\underbrace{\frac{M_{\text{pl}}^2}{2}\overset{\Gamma}{R}(g,\Gamma)}_{\text{EH term}} + \text{higher curvatures}$$

$$\overset{\Gamma}{R}{}^{\mu}{}_{\nu\alpha\beta}(\Gamma) := \partial_\alpha \Gamma^{\mu}_{\beta\nu} - \partial_\beta \Gamma^{\mu}_{\alpha\nu} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\mu}_{\beta\sigma}\Gamma^{\sigma}_{\alpha\nu}, \quad \overset{\Gamma}{R} := g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu}, \quad \overset{\Gamma}{R}_{\mu\nu} := \overset{\Gamma}{R}{}^{\alpha}{}_{\mu\alpha\beta}$$

Metric-affine formalism

For convenience, we introduce the distortion tensor κ

$$\kappa^{\mu}{}_{\alpha\beta} := \Gamma^{\mu}_{\alpha\beta} - \left\{ {\mu \atop \alpha\beta} \right\}$$

The metric-affine formalism = The metric formalism with κ.

$$\overset{\Gamma}{R}{}^{\mu}{}_{\alpha\beta\gamma}(\Gamma) = R^{\mu}{}_{\alpha\beta\gamma}(g) + 2\nabla_{[\alpha}\kappa^{\mu}_{\beta]\nu} + \kappa^{\mu}_{[\alpha|\sigma}\kappa^{\sigma}_{\beta]\nu}$$

$\mathcal{L}_{\mathrm{EH}} \sim M_{\mathrm{pl}}^2 R(g) + M_{\mathrm{pl}}^2 \kappa^2$: mass term of distortion

$\text{higher curvature} \supset (\nabla\kappa)^2$: kinetic term of distortion

When higher curvatures can be ignored, κ can be integrated out.

$\Rightarrow \kappa^{\mu}{}_{\alpha\beta} = 0$: Riemannian geometry $\Gamma^{\mu}_{\alpha\beta} = \left\{ {\mu \atop \alpha\beta} \right\} := \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta})$

We don't need to assume anything on the connection to get GR.

Metric or Metric-affine?

EH action in metric-affine = EH action in metric.

However, the equivalence does not hold
if we consider either **higher curvature corrections** or **matter coupling**.

e.g., $f(R), f(R_{\mu\nu})$

e.g., Dirac field
(cf. Einstein-Cartan theory)

How is a higher derivative field?

$$\overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi \supset \kappa\partial\phi$$

We find Galileon in metric-affine $\neq$ Galileon in metric

Before discussing Galileon,
we should pay attention to a symmetry of EH and matter Lagrangians.

Projective invariance

$$\mathcal{L}_{\rm EH}(g,\Gamma) = \frac{M_{\rm pl}^2}{2}\overset{\Gamma}{R} = \frac{M_{\rm pl}^2}{2}\left(R(g) + \kappa^{\alpha}{}_{\alpha\beta}\kappa^{\beta\gamma}{}_{\gamma} - \kappa^{\alpha\beta}{}_{\gamma}\kappa^{\gamma}{}_{\alpha\beta}\right) + \text{total divergence}$$

The EH action has an additional symmetry, **"projective invariance"**,

$$\Gamma^{\mu}_{\alpha\beta} \to \Gamma^{\mu}_{\alpha\beta} + \delta^{\mu}_{\beta}U_{\alpha}(x) \quad \text{or} \quad \kappa^{\mu}{}_{\alpha\beta} \to \kappa^{\mu}{}_{\alpha\beta} + \delta^{\mu}_{\beta}U_{\alpha}(x)$$

which preserves the geodesic equation

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

and the angle for the parallel transport (a kind of conformal symmetry)

The solution of the distortion tensor is

$$\kappa^{\mu}{}_{\alpha\beta} = 0 + \text{pojective mode}$$

Projective invariance and matter

Standard matter Lagrangian are also projective invariant.

- Scalar field $\mathcal{L}_S = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$

$(F_{\mu\nu} \neq 2\overset{\Gamma}{\nabla}_{[\mu} A_{\nu]})$

- Vector field $\mathcal{L}_V = -\frac{1}{4} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta}\,,\ F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$

No coupling to distortion *κ* (trivially projective invariant)

- Dirac field $\mathcal{L}_D = i\bar{\psi}\gamma^\mu \overset{\Gamma}{\nabla}_\mu \psi - m\bar{\psi}\psi \supset -\frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta\gamma} j^5_\delta + \frac{i}{2}\kappa^{[\alpha\beta]}{}_\beta j_\alpha$

Coupling to distortion *κ* (projective invariant)

$j_\alpha = \bar{\psi}\gamma_\alpha\psi,$
$j^5_\alpha = \bar{\psi}\gamma_\alpha\gamma^5\psi,$

where $\overset{\Gamma}{\nabla}_\mu \psi = \left(\partial_\mu + \frac{1}{8}\omega^{ab}{}_\mu [\gamma_a, \gamma_b]\right)\psi$ and $\partial_\mu e^a_\nu - \Gamma^\alpha_{\nu\mu} e^a_\alpha + \omega^a{}_{b\mu} e^b_\nu = 0$

Let's assume Galileon is also projective invariant.

Galileon scalar field

- Flat spacetime (Euclidean geometry): we have only one Galileon

$$\mathcal{L}_n^{\rm gal} = \epsilon\epsilon(\partial\phi)^2(\partial\partial\phi)^{n-2} = (\partial\phi)^2\epsilon\epsilon(\partial\partial\phi)^{n-2} + \text{total divergence}$$

e.g., $$\mathcal{L}_4^{\rm gal} = \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'\gamma'}{}_{\delta}\partial_\alpha\phi\partial_{\alpha'}\phi\partial_\beta\partial_{\beta'}\phi\partial_\gamma\partial_{\gamma'}\phi$$
$$= \partial_\mu\phi\partial^\mu\phi\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha'\beta'}{}_{\gamma\delta}\partial_{\alpha'}\partial_\alpha\phi\partial_{\beta'}\partial_\beta\phi + \text{total divergence}$$

Two are same via integration by parts.

- Curved spacetime (Riemannian geometry): we have two Galileons

$$\epsilon\epsilon(\nabla\phi)^2(\nabla\nabla\phi)^{n-2} \neq (\nabla\phi)^2\epsilon\epsilon(\nabla\nabla\phi)^{n-2} + \text{total divergence}$$

GLPV (covariantized) Horndeski (covariant)

Two are not same.

Galileon with projective invariance

- Curved spacetime (metric-affine geometry):
  Due to the projective invariance Galileon must be

$$\mathcal{L}_n^{\mathrm{gal}\Gamma}(\overset{\Gamma}{\nabla}_\mu\phi, \overset{\Gamma}{\nabla}_\mu\overset{\Gamma}{\nabla}_\nu\phi; g_{\mu\nu}) = \epsilon\epsilon(\overset{\Gamma}{\nabla}\phi)^2(\overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi)^{n-2}$$ (GLPV type)

$$\Rightarrow \mathcal{L}(g,\Gamma,\phi) = \frac{M_{\mathrm{pl}}^2}{2} g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu} + \sum_{n\geq 2}^{5}\frac{c_n}{\Lambda^{3(n-2)}}\mathcal{L}_n^{\mathrm{gal}\Gamma}$$

with $\mathcal{L}_n^{\mathrm{gal}\Gamma} = \epsilon\epsilon(\overset{\Gamma}{\nabla}\phi)^2(\overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi)^{n-2} \supset \kappa^{n-2}(\partial\phi)^n$

Up to quartic order ($n \leq 4$) → κ can be explicitly solved.

(The equation of κ becomes nonlinear when including quintic order)

After integrating out κ, we obtain…

Galileon with projective invariance

Effective action in Riemannian geometry

$$X = (\partial\phi)^2, \quad \mathcal{L}_n^{\rm galg} = \epsilon\epsilon(\nabla\phi)^2(\nabla\nabla\phi)^{n-2}, \quad \Lambda_2^4 = \Lambda_3^3 M_{\rm pl}$$

$$\mathcal{L}(g,\Gamma(g,\phi),\phi) = \frac{M_{\rm pl}^2}{2}R(g) + \frac{3(c_3^2 - 4c_2c_4)X^3/\Lambda_2^8}{1+2c_4X^2/\Lambda_2^8} + \frac{1}{1+2c_4X^2/\Lambda_2^8}\left(c_2\mathcal{L}_2^{\rm galg} + \frac{c_3}{\Lambda_3^3}\mathcal{L}_3^{\rm galg} + \frac{c_4}{\Lambda_3^6}\mathcal{L}_4^{\rm galg}\right)$$

✓ does not coincide with either covariantized or covariant Galileon.

✓ can yield the non-minimal coupling to the fermion current

$$\mathcal{L}_{\rm int} = \frac{i}{M_{\rm pl}^2(1+2c_4X^2/\Lambda_2^8)}\left[\frac{3c_3}{2\Lambda_3^3}X + \frac{c_4}{\Lambda_3^6}\left(X\phi^\beta_\beta - \phi^{\beta\gamma}\phi_\beta\phi_\gamma\right)\right]j_\alpha\phi^\alpha + \cdots$$

$$\mathcal{L}_D \supset -\frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta\gamma}j^5_\delta + \frac{i}{2}\kappa^{[\alpha\beta]}{}_\beta j_\alpha, \qquad j_\alpha = \bar\psi\gamma_\alpha\psi, \quad j^5_\alpha = \bar\psi\gamma_\alpha\gamma^5\psi, \qquad \phi_\mu = \nabla_\mu\phi, \quad \phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$$

Generalized Galileon is DHOST

In metric formalism, generalizations of Galileon = Horndeski, GLPV

A straightforward generalization (up to quadratic in connection)

$$\mathcal{L}(g,\Gamma,\phi) = f_1(\phi,X)\overset{\Gamma}{R} + f_2(\phi,X)\overset{\Gamma}{G}{}^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi$$
$$+ F_2(\phi,X) + F_3(\phi,X)\mathcal{L}_3^{\mathrm{gal}\Gamma} + F_4(\phi,X)\mathcal{L}_4^{\mathrm{gal}\Gamma}$$

= **Non-minimal couplings + generalized Galileon**

$$X := g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

where Ricci scalar $\overset{\Gamma}{R} := g^{\mu\nu}\overset{\Gamma}{R}_{\mu\nu}, \quad \overset{\Gamma}{R}_{\mu\nu} := \overset{\Gamma}{R}{}^\alpha{}_{\mu\alpha\beta}$

Einstein tensor $\overset{\Gamma}{G}{}^{\alpha\beta} := \dfrac{1}{4}\epsilon^{\gamma\alpha\mu\nu}\epsilon_\gamma{}^{\beta\mu'\nu'}\overset{\Gamma}{R}_{\mu\nu\mu'\nu'}$

Need fine-tuning of functions to be ghost-free?

→ Don't need! This action yields class N-1/Ia of DHOST = ghost-free

Generalized Galileon is DHOST

Integrating out κ,

$$\mathcal{L}(g,\Gamma(g,\phi),\phi) = fR(g) + P + Q_1 g^{\mu\nu}\phi_{\mu\nu} + Q_2\phi^\mu\phi_{\mu\nu}\phi^\nu + C^{\mu\nu,\rho\sigma}\phi_{\mu\nu}\phi_{\rho\sigma}$$

$$C^{\mu\nu,\rho\sigma} = \alpha_1 g^{\rho(\mu}g^{\nu)\sigma} + \alpha_2 g^{\mu\nu}g^{\rho\sigma} + \frac{1}{2}\alpha_3(\phi^\mu\phi^\nu g^{\rho\sigma} + \phi^\rho\phi^\sigma g^{\mu\nu}) + \frac{1}{2}\alpha_4(\phi^\rho\phi^{(\mu}g^{\nu)\sigma} + \phi^\sigma\phi^{(\mu}g^{\nu)\rho}) + \alpha_5\phi^\mu\phi^\nu\phi^\rho\phi^\sigma$$

$$\phi_\mu = \nabla_\mu\phi,\ \phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$$

where $f = f_1 - \frac{1}{2}f_2 X\,,\quad P = F_2 + \frac{3X(f_{1\phi} - F_3 X)^2}{2f_1 - f_2X + 2F_4X^2}\,,$

$$Q_1 = -2f_\phi + \frac{4f_1(f_{1\phi} - F_3X)}{2f_1 - f_2X + 2F_4X^2}\,,\quad Q_2 = \frac{2f_\phi}{X} - \frac{4(f_1 - 3f_{1X})(f_{1\phi} - F_3X)}{X(2f_1 - f_2X + 2F_4X^2)}\,,$$

$$\alpha_1 = -\alpha_2 = -\frac{f_2}{2} - \frac{f_1(f_2 - 2F_4X)}{2f_1 - f_2X + 2F_4X^2}\,,\quad \alpha_3 = 2f_{2X} + \frac{4f_1F_4 + (4f_{1X} - f_2)(f_2 - 2F_4X)}{2f_1 - f_2X + 2F_4X^2}\,,$$

$$\alpha_4 = -2f_{2X} + 2f_1^{-1}f_{1X}(3f_{1X} - f_2) + f_1^{-2}f_{1X}X(f_{1X}f_2 - 4f_1f_{2X}) + \frac{f_2^2 - 4f_1F_4 - 2f_2F_4X}{2f_1 - f_2X + 2F_4X^2}\,,$$

$$\alpha_5 = -f_1^{-2}f_{1X}(f_{1X}f_2 - 4f_1f_{2X}) + \frac{2f_{1X}\{4f_1F_4 + (3f_{1X} - f_2)(f_2 - 2F_4X)\}}{f_1(2f_1 - f_2X + 2F_4X^2)}\,,$$

$$X := g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

satisfy the degeneracy conditions (belonging in class N-1/Ia)

Same number of arbitrary functions as class N-1/Ia of DHOST.

Generalized Galileon is DHOST

Class N-1/Ia DHOST (projective invariant)

$$\mathcal{L}(g,\Gamma,\phi) = f_1(\phi,X)\overset{\Gamma}{R} + f_2(\phi,X)\overset{\Gamma}{G}{}^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi$$
$$+ F_2(\phi,X) + F_3(\phi,X)\mathcal{L}_3^{\mathrm{gal}\Gamma} + F_4(\phi,X)\mathcal{L}_4^{\mathrm{gal}\Gamma}$$

which may predict the non-minimal coupling to fermions

$$\mathcal{L}_D \supset -\frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\kappa_{\alpha\beta\gamma}j^5_\delta + \frac{i}{2}\kappa^{[\alpha\beta]}{}_\beta j_\alpha \to \mathcal{L}_{\mathrm{int}} = \frac{i}{2}\kappa^{[\alpha\beta]}{}_\beta(g,\phi)j_\alpha$$

where $\kappa_{[\alpha\beta\gamma]}(g,\phi) = 0\,,\quad \kappa^{[\alpha\beta]}{}_\beta(g,\phi) \neq 0$

$$j_\alpha = \bar{\psi}\gamma_\alpha\psi,$$
$$j^5_\alpha = \bar{\psi}\gamma_\alpha\gamma^5\psi,$$

However, the projective invariance cannot protect ghost-freeness.

Ghost-free ST theories
Horndeski, GLPV, DHOST

$\xrightarrow{\text{true}}$
$\xleftarrow[\text{not true}]{}$

Projective invariant ST theories

$(\mathcal{L}_3^{\mathrm{gal}})^2$: Ostrogradsky ghost

Specific models

- Einstein tensor coupling

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2}\overset{\Gamma}{R}(g,\Gamma) - \frac{1}{2}\left(g^{\mu\nu} - \frac{\overset{\Gamma}{G}{}^{\mu\nu}(g,\Gamma)}{M^2}\right)\partial_\mu\phi\partial_\nu\phi\,.$$

After integrating out κ

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2}R(g) - \frac{1}{2}\left(g^{\mu\nu} - \frac{G^{\mu\nu}(g)}{M^2}\right)\partial_\mu\phi\partial_\nu\phi - \frac{1}{4M^4M_{\rm pl}^2(2-X/M^2M_{\rm pl}^2)}\mathcal{L}_4^{\rm galg}$$

Additional term

- Kinetic coupling to Ricci scalar

$$\mathcal{L}(\overset{\Gamma}{R},X) = f(X)\overset{\Gamma}{R} + P(X) \;\Leftrightarrow\; \mathcal{L} = f(X)R(g) + P(X) + \frac{6f_X^2}{f}\phi^\alpha\phi^\beta\phi_{\alpha\gamma}\phi^\gamma_\beta$$

Generalized k-essence? = Simplest theory of DHOST?

$$X := g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$\phi_\mu = \nabla_\mu\phi,\;\; \phi_{\mu\nu} = \nabla_\mu\nabla_\nu\phi$$

Summary

- Metric-affine formalism: metric and connection are independent.

  No assumption on connection is needed to obtain GR.

- The covariant Galileon is unique due to the projective invariance.

$$\mathcal{L}(g,\Gamma,\phi) = \frac{M_{\rm pl}^2}{2} g^{\mu\nu} \overset{\Gamma}{R}_{\mu\nu} + \sum_{n\geq 2}^{5} \frac{c_n}{\Lambda^{3(n-2)}} \mathcal{L}_n^{\rm gal\Gamma}$$

with $\mathcal{L}_n^{\rm gal} = \epsilon\epsilon(\overset{\Gamma}{\nabla}\phi)^2(\overset{\Gamma}{\nabla}\overset{\Gamma}{\nabla}\phi)^{n-2}$

- Class N-1/Ia DHOST is

$$\begin{aligned}\mathcal{L}(g,\Gamma,\phi) = {} & f_1(\phi,X)\overset{\Gamma}{R} + f_2(\phi,X)\overset{\Gamma}{G}{}^{\mu\nu}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi \\ & + F_2(\phi,X) + F_3(\phi,X)\mathcal{L}_3^{\rm gal\Gamma} + F_4(\phi,X)\mathcal{L}_4^{\rm gal\Gamma}\end{aligned}$$

  Complicated structure is obtained by integrating out κ.

- These theories may predict fermion-scalar coupling. $\mathcal{L}_{\rm int} = \frac{i}{2}\kappa^{[\alpha\beta]}{}_\beta(g,\phi)j_\alpha$

Discussions

- Phenomenology?

  Inflation or dark energy/matter in specific models?

  Non-minimal fermion-scalar coupling? $\mathcal{L}_{\rm int} = \frac{i}{2}\kappa^{[\alpha\beta]}{}_{\beta}(g,\phi)j_\alpha$

- Further extensions?

  Nonlinear terms of connection?

  $\overset{\Gamma}{G}{}^{\mu\nu\alpha\beta} := \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\rho'\sigma'}\overset{\Gamma}{R}_{\rho\sigma\rho'\sigma'}$

  Quintic Galileon: $F_5(\phi,X)\mathcal{L}_5^{\rm gal\Gamma}$

  Fab Four: $f_3(\phi,X)\overset{\Gamma}{G}{}^{\mu\alpha\nu\beta}\overset{\Gamma}{\nabla}_\mu\phi\overset{\Gamma}{\nabla}_\nu\phi\overset{\Gamma}{\nabla}_\alpha\overset{\Gamma}{\nabla}_\beta\phi\,,\ \ f_4(\phi,X)\overset{\Gamma}{G}{}^{\mu\nu\alpha\beta}\overset{\Gamma}{R}_{\mu\nu\alpha\beta}$

  $\Rightarrow \mathcal{L}(g,\Gamma(g,\phi),\phi) \supset \sum_{i,j}^{\infty}(\nabla\phi)^i(\nabla\nabla\phi)^j$???

  Ghost or Beyond DHOST?

  Higher curvatures? → κ may be dynamical.