Some Recent Results on Strongly Coupled Gauge Theories

Robert Shrock

C. N. Yang Institute for Theoretical Physics, Stony Brook University

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Outline

- Sakata and compositeness
- Higher-loop calculations of UV to IR evolution in asymptotically free gauge theories, including IR zero of β and anomalous dimension γ_m of fermion bilinear
- Updated comparison with γ_m measurements from lattice
- Study of scheme dependence
- Some results on dynamical electroweak symmetry breaking and strongly coupled chiral gauge theories
- RG evolution in IR-free theories: U(1) and $\lambda|\vec{\phi}|^4$; question of possible UV zero in beta functions; also Yukawa theories
- Conclusions

Sakata and Compositeness

Shoichi Sakata and his Nagoya group emphasized role of compositeness and "layers" of structure in the early 1960s; this general idea was confirmed in composite structure of hadrons as color-singlet bound states of quarks and gluons.

Ziro Maki, Masami Nakagawa, and Shoichi Sakata were also the first to propose (with the two generations of leptons then known) that the neutrino weak eigenstates are linear combinations of neutrino mass eigenstates:

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

in Maki, Nakagawa, Sakata, "Remarks on the Unified Model of Elementary Particles", Prog. Theor. Phys. 28, 243-246 (1962); and Nakagawa, Okonogi, Sakata, Toyoda, "Possible Existence of a Neutrino with Mass and Partial Conservation of Muon Charge", Prog. Theor. Phys. 30, 727-729 (1963).

All of these papers were written here at Nagoya University.

Later experiments (Davis ^{37}Cl solar neutrino deficiency, SAGE, GALLEX, IMB, Kamiokande, SuperKamiokande, SNO, KamLAND, K2K...) have shown neutrino oscillations and hence neutrino masses and mixing; SuperK 1998 data especially decisive.

Sakata (1911-1970);   Maki (1929-2005);   Nakagawa (1932-2001)

Also fitting that the Kobayashi Maskawa Institute here celebrates the successful Kobayashi-Maskawa picture of CP violation with three Standard-Model (SM) quark generations (1974), made before the discovery of any 3rd-generation quarks.

2012: great discovery by ATLAS and CMS exps. at the CERN LHC of a Higgs-like scalar boson with mass 125.7 ± 0.4 GeV. From current data, this is is consistent with being the pointlike Higgs boson of the SM, and the next LHC run at 13-14 TeV will test this consistency further.

The hierarchy problem of the Higgs sector in the SM motivated extensions of the SM that could solve or avoid this problem: supersymmetry and technicolor, as well as others.

So far, LHC has not seen evidence for either supersymmetry or technicolor or other beyond-SM physics, but naturalness arguments still motivate consideration of extensions of the SM that remove the hierarchy problem. Perhaps a new discovery might be made in the next LHC run about to begin.

One possibility is the subject of these SCGT conferences, namely strongly coupled gauge (SCG) interaction(s). These are of interest in their own right and might be relevant for the observed Higgs-like boson, which would thus be composite rather than pointlike.

Koichi Yamawaki's group at Nagoya has made pioneering contributions to this area for many years.

Higher-Loop Corrections to UV $\to$ IR Evolution of Gauge Theories

Consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in representation R of G.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

The question of how this theory flows from large μ in the UV to small μ in the infrared (IR) is of fundamental field-theoretic interest (and possibly some relevance to electroweak symmetry breaking).

For some fermion contents, the (perturbatively calculated) beta function of the theory may have an exact or approximate IR fixed point (zero of β).

Notation: $g = g(\mu)$; $\alpha(\mu) = g(\mu)^2/(4\pi)$;
$a(\mu) = g(\mu)^2/(16\pi^2) = \alpha(\mu)/(4\pi)$.

Dependence of $\alpha(\mu)$ on μ described by renormalization group (RG) β function

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell$$

where $dt = d \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coeffs. b_1 and b_2 in β are indep. of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$. With $b_1 = (11C_A - 4N_f T_f)/3$, this requires $N_f < N_{f,b1z} = 11C_A/(4T_f)$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases beyond the perturbatively calculable region (as in QCD).
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha \to \alpha_{IR}$; then two possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the renorm. group (RG) as $\mu \to 0$ and $\alpha \to \alpha_{IR}$, $\beta \to \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions (Caswell, Banks-Zaks).

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$.

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β fn. becomes that of a pure gauge theory, with no IR zero. Hence, if β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, $\beta = d\alpha/dt \to 0$, so $\alpha(\mu)$ runs very slowly as a function of the scale μ, i.e., there is approximately scale-invariant (= dilatation-invariant, walking) behavior.

SχSB at Λ also breaks the approx. dilatation symmetry, leads to a resultant approx. NGB, the dilaton (Yamawaki et al., 1986; Bardeen et al..). This is not massless, since β is nonzero at $\alpha = \alpha_{cr}$ where SχSB occurs.

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$. At the $n = 2$ loop level,

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}$$

which is physical for $b_2 < 0$; this condition is met in the interval

$$I : \quad N_{f,b2z} < N_f < N_{f,b1z}$$

where

$$N_{f,b2z} = \frac{34C_A^2}{4T_f(5C_A + 3C_f)}$$

Take $G = \mathrm{SU}(N_c)$; e.g., with fermions in fund. rep.

- for SU(2), I: $5.55 < N_f < 11$;
- for SU(3), I: $8.05 < N_f < 16.5$;
- As $N_c \to \infty$ with $r = N_f/N_c$ fixed, I: $2.62 < r < 5.5$.

Denote $N_f = N_{f,cr}$ where $\alpha_{IR} = \alpha_{cr}$; $N_{f,cr}$ separates chirally symmetric IR phase at larger N_f and chirally broken IR phase at smaller N_f.

As N_f decreases and α_{IR} increases toward $\alpha_{cr} \sim O(1)$, theory becomes moderately strongly coupled, motivating higher-loop calculations of α_{IR}, and γ_m evaluated at α_{IR}, where γ_m is anomalous dimension for $\bar{\psi}\psi$ (early work by Gardi, Grunberg, Karliner).

Calculations up to 4-loop level for general fermion rep. R in Ryttov and RS, PRD83, 056011 (2011) [arXiv:1011.4542] and Pica and Sannino, PRD83, 035013 (2011) [arXiv:1011.5917]. These use calculations of b_3 and b_4 by Vermaseren, Larin, and van Ritbergen in $\overline{\mathrm{MS}}$ scheme.

Further studies in RS, PRD 87, 105005 (2013) [arXiv:1301.3209]; RS, PRD 87, 116007 (2013) [arXiv:1302.5434] and on effects of scheme transformations (discussed below). Analytic results in papers; examples of numerical results:

Numerical values of $\alpha_{IR,n\ell}$ at the $n = 2,\ 3,\ 4$ loop level for SU(2), SU(3) and fermions in fundamental representation:

N_c	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	6	11.42	1.645	2.395
2	7	2.83	1.05	1.21
2	8	1.26	0.688	0.760
2	9	0.595	0.418	0.444
2	10	0.231	0.196	0.200
3	10	2.21	0.764	0.815
3	11	1.23	0.578	0.626
3	12	0.754	0.435	0.470
3	13	0.468	0.317	0.337
3	14	0.278	0.215	0.224
3	15	0.143	0.123	0.126
3	16	0.0416	0.0397	0.0398

(Perturbative calculation not applicable if $\alpha_{IR,n\ell}$ too large.)

Some general features of these results:

- Value of IR zero of β, $\alpha_{IR,n\ell}$, decreases substantially going from $n = 2$ loop order to $n = 3$ loop order (generalizes beyond $\overline{\rm MS}$ scheme).
- Value of $\alpha_{IR,n\ell}$ increases slightly going from 3-loop to 4-loop order, but the fractional change is smaller, so
- 4-loop value, $\alpha_{IR,4\ell}$, is smaller than 2-loop value, $\alpha_{IR,2\ell}$.
- Hence, with $N_{f,cr}$ determined by $\alpha_{IR} = \alpha_{cr}$ and $\alpha_{IR,n\ell}$ increasing with decreasing N_f, these higher-loop results suggest that $N_{f,cr}$ may be smaller than the early estimate $N_{f,cr} \simeq 4N_c$ in agreement with many lattice results.
- The smaller fractional change in value of IR zero of β at higher-loop order agrees with expectation that calculation to higher-loop order should give more stable result if perturbation theory is reliable.

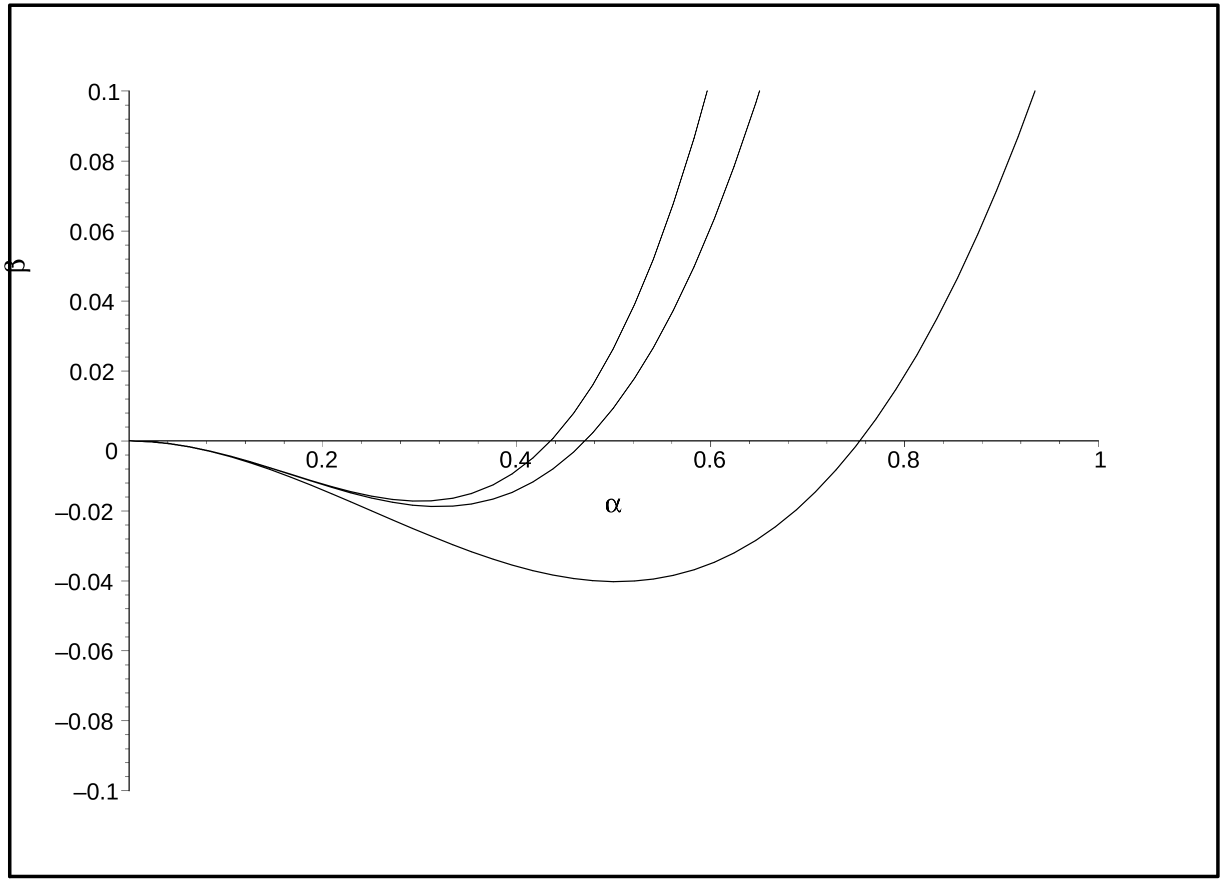

Figure 1: $\boldsymbol{\beta_{n\ell}}$ for SU(3), $\boldsymbol{N_f = 12}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

An important quantity is the anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear $\bar{\psi}\psi$. As with the IR zero of $\beta_{n\ell}$, it is useful to calculate this to higher-loop order.

Series expansion for γ_m:

$$\gamma = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop coefficient.

The 1-loop coeff. c_1 is scheme-independent; the c_ℓ with $\ell \geq 2$ are scheme-dependent and have been calculated up to 4-loop level in $\overline{\rm MS}$ scheme (Vermaseren, Larin, and van Ritbergen): $c_1 = 6C_f$, etc. for higher-loop coeffs.

Denote γ calculated to n-loop ($n\ell$) level as $\gamma_{n\ell}$ and, evaluated at the n-loop value of the IR zero of β, as

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell})$$

In the IR chirally symmetric phase, an all-order calculation of γ evaluated at an all-order calculation of α_{IR} would be an exact property of the theory.

In the chirally broken phase, just as the IR zero of β is only an approx. IRFP, so also, the γ is only approx., describing the running of $\bar{\psi}\psi$ and the dynamically generated running fermion mass near the zero of β having large-momentum (large k) behavior

$$\Sigma(k) \sim \Lambda \left(\frac{\Lambda}{k} \right)^{2-\gamma}$$

(γ bounded above as $\gamma < 2$ in general). Analytic results given in our papers; numerical results:

Illustrative numerical values of $\gamma_{IR,n\ell}$ for SU(2) and SU(3) at the $n = 2,\ 3, 4$ loop level and fermions in the fundamental representation with $N_f \in I$:

N_c	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

Plot of γ as function of N_f for SU(3):

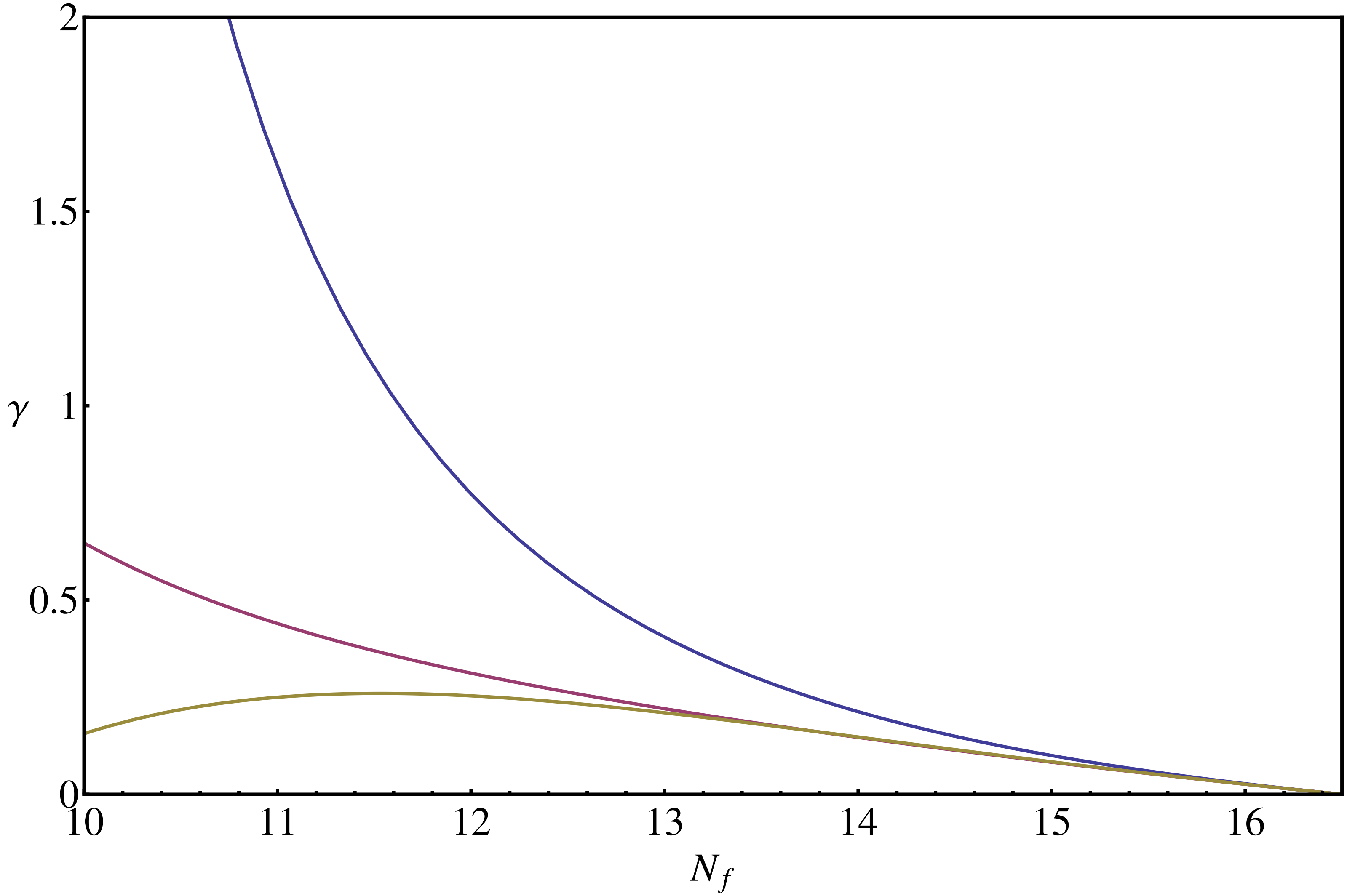

Figure 2: $\boldsymbol{n}$-loop anomalous dimension $\boldsymbol{\gamma_{IR,n\ell}}$ at $\boldsymbol{\alpha_{IR,n\ell}}$ for SU(3) with $\boldsymbol{N_f}$ fermions in fund. rep: (i) blue: $\boldsymbol{\gamma_{IR,2\ell}}$; (ii) red: $\boldsymbol{\gamma_{IR,3\ell}}$; (iii) brown: $\boldsymbol{\gamma_{IR,4\ell}}$.

We find that the 3-loop and 4-loop results are closer to each other for a larger range of N_f than the 2-loop and 3-loop results.

So our higher-loop calcs. of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ allow us to probe the theory reliably down to smaller values of N_f and thus stronger couplings.

Comparison with Lattice Measurements:

One of the most heavily studied cases on the lattice is for the gauge group SU(3) with $N_f = 12$ fermions in the fundamental representation.

For this theory, Appelquist et al. (LSD); Deuzeman, Lombardo, and Pallante; Hasenfratz et al.; DeGrand et al.; Aoki et al. (LatKMI) find that the IR behavior is chirally symmetric (Jin and Mawhinney, and Kuti et al. found it is chirally broken).

For this SU(3) theory with $N_f = 12$, we get

$$\gamma_{IR,2\ell} = 0.77, \qquad \gamma_{IR,3\ell} = 0.31, \qquad \gamma_{IR,4\ell} = 0.25$$

some lattice results (error estimates do not include all systematic uncertainties):

$\gamma = 0.414 \pm 0.016$ (Appelquist et al. (LSD Collab.), PRD 84, 054501 (2011).

$\gamma \sim 0.35$ (DeGrand, PRD 84, 116901 (2011).

$0.2 \lesssim \gamma \lesssim 0.4$ (Kuti et al. (method-dep.) arXiv:1205.1878, arXiv:1211.3548, 1211.6164, PTP, finding SχSB).

$\gamma \simeq 0.4$ (Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K.-I. Nagai, H. Ohki, A. Shibata, K. Yamawaki, and T.Yamazaki (LatKMI Collab.) PRD 86, 054506 (2012) [arXiv:1207.3060]);

$\gamma = 0.27(3)$ (Hasenfratz et al., arXiv:1207.7162; $\gamma \simeq 0.25$; Hasenfratz et al., arXiv:1310.1124).

$\gamma = 0.235(46)$ (Lombardo, Miura, Nunes, Pallante (LMNP), arXiv:1410.0298).

So 2-loop value is larger than, and the 3-loop and 4-loop values closer to, lattice data.

Thus, our higher-loop calculations of γ yield better agreement with these lattice measurements than two-loop calculations.

The LatKMI value is consistent with the LMNP value; different types of data analysis accounts for different values (explained by LatKMI group).

Schwinger-Dyson estimates suggest γ could be $\simeq 1$ in walking regime with SχSB (Yamawaki et al., Appelquist et al..,Holdom; Cohen-Georgi..). In technicolor theories, $\gamma \sim 1$ enhances SM fermion mass generation.

Lattice studies of SU(3) with $N_f = 8$ report $\gamma \sim 1$ and hence are consistent with this: Y. Aoki et al. (LatKMI), PRD 87, 094511 (2013) [arXiv:1302.6859]; and Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K. Miura, K.-I. Nagai, H. Ohki, Rinaldi, A. Shibata, K. Yamawaki, and T.Yamazaki (LatKMI), PRD 89, 111502 (2014) [arXiv:1403.5000]; Appelquist et al. (LSD) PRD 90, 114502 (2014) [arXiv:1405.4752].

The IR behavior for SU(3) with $N_f = 8$ involves too strong a coupling for our perturbative calculations to be applied.

As with our results for $\alpha_{IR,n\ell}$ the decrease that we find in $\gamma_{IR,n\ell}$ at higher loop order n, combined with the expectation that $\gamma_{IR} \sim 1$ for $N_f = N_{f,cr}$ suggests that $N_{f,cr}$ may be smaller than the early estimate $N_{f,cr} \simeq 4N_c$, again in agreement with many lattice results.

We find same trend for supersymmetric vectorial SU(N_c) gauge theory with chiral superfields in fund. rep. (SQCD), where $N_{f,cr} = (3/2)N_c$ is known, i.e., reductions in $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ at higher-loop order (Ryttov and RS, PRD85, 076009 (2012) [arXiv:1202.1297]).

Also useful to study theories with fermions in higher-dimensional reps. of gauge group (Sannino...).

e.g. SU(3) with $N_f = 2$ fermions in symmetric rank-2 tensor rep. (sextet rep.); here we calculate $\gamma_{IR,3\ell} = 1.28$ and $\gamma_{IR,4\ell} = 1.12$.

These values are consistent with $\gamma_{IR} \sim 1.5$ obtained from lattice study by Kuti group, arXiv:1205.1878; update with scalar mass: arXiv:1502.00028 finding SχSB for this theory.

N.B.: $\gamma_{IR} \lesssim 0.5$ obtained for this theory by Degrand, Shamir, Svetitsky, PRD88, 054505 (2013) [arXiv:1307.2425], finding χ sym.

Interesting property: for $R =$ fund. rep., $\alpha_{IR,n\ell} N_c$ and $\gamma_{IR,n\ell}$ are similar in theories with different values of N_c and N_f if they have equal or similar values of $r = N_f/N_c$.

This motivates a study of the UV to IR evolution of an SU(N_c) gauge theory with N_f fermions in the fundamental rep. in the 't Hooft-Veneziano (HV) limit $N_c \to \infty$, $N_f \to \infty$ with

$$r \equiv \frac{N_f}{N_c} \quad \text{and} \quad \alpha(\mu) N_c \equiv \xi(\mu) \quad \text{finite}$$

(RS, Phys. Rev. D87, 116007 (2013) [arXiv:1302.5434]).

Define a rescaled beta function that is finite in the this limit:

$$\beta_\xi \equiv \frac{d\xi}{dt} = \lim_{HV} \beta_\alpha N_c$$

Interval of r where $\beta_{\xi,2\ell}$ has an IR zero is

$$I_r : \quad \frac{34}{13} < r < \frac{11}{2}\ , \ i.e., \quad 2.615 < r < 5.500$$

2-loop IR zero of $\beta_{\xi,2\ell}$ is at

$$\xi_{IR,2\ell} = \frac{4\pi(11-2r)}{13r-34}$$

Value of n-loop γ evaluated at n-loop $\xi_{IR,n\ell}$: $\gamma_{IR,n\ell} \equiv \gamma_{n\ell}\big|_{\xi=\xi_{IR,n\ell}}$;

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$

We find that corrections to the HV limiting forms go like $1/N_c^2$ and hence this limit is approached rather rapidly as N_c and N_f increase. For example,

$$\alpha_{IR,2\ell}N_c = \frac{4\pi(11-2r)}{13r-34} + \frac{12\pi r(11-2r)}{(34-13r)^2 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$
$$+\frac{(11-2r)(18836-5331r+648r^2-140r^3)}{(13r-34)^3 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

Results for $\gamma_{IR,n\ell}$ up to 4-loop level in this limit:

r	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
3.6	1.853	0.5201	0.3083
3.8	1.178	0.4197	0.3061
4.0	0.7847	0.3414	0.2877
4.2	0.5366	0.2771	0.2664
4.4	0.3707	0.2221	0.2173
4.6	0.2543	0.1735	0.1745
4.8	0.1696	0.1294	0.1313
5.0	0.1057	0.08886	0.08999
5.2	0.05620	0.05123	0.05156
5.4	0.01682	0.01637	0.01638

These results provide an understanding of similarities in $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ in theories having different values of N_c and N_f with similar or identical values of r.

Study of Scheme Dependence in Calculation of IR Fixed Point

Since coeffs. b_n in $\beta_{n\ell}$, and hence also $\alpha_{IR,n\ell}$, are scheme-dependent for $n \geq 3$, it is important to assess the effects of this scheme dependence (RS, PRD 88, 036003 (2013) [arXiv:1305.6524]; RS, PRD 90, 045011 (2014) [arXiv:1405.6244]; Choi and RS, PRD 90, 125029 (2014) [arXiv:1411.6645]; Ryttov and RS, PRD 86, 065032 (2012) [arXiv:1206.2366] and PRD 86, 085005 (2012) [arXiv:1206.6895]).

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$ of the form

$$a = a' f(a')$$

with $f(0) = 1$ since ST has no effect in limit of zero coupling.

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s$$

where $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite.

The Jacobian $J = da/da' = d\alpha/d\alpha' = 1 + \sum_{s=1}^{s_{max}} (s+1) k_s (a')^s$, satisfying $J = 1$ at $a = a' = 0$.

After the scheme transformation is applied, the beta function in the new scheme is given by

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha}\frac{d\alpha}{dt} = J^{-1}\,\beta_{\alpha}$$

with the expansion

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_{\ell}(a')^{\ell} = -2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_{\ell}(\alpha')^{\ell}$$

where $\bar{b}'_{\ell} = b'_{\ell}/(4\pi)^{\ell}$.

We calculate the b'_{ℓ} as functions of the b_{ℓ} and k_s. At 1-loop and 2-loop, this yields the well-known results

$$b'_1 = b_1\ , \quad b'_2 = b_2$$

We find

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2)b_1\ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3)b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2) b_3 + (-k_1^3 + 3k_1 k_2 - k_3) b_2$$
$$+ (4k_1^4 - 11k_1^2 k_2 + 6k_1 k_3 + 4k_2^2 - 3k_4) b_1$$

etc. at higher-loop order.

A physically acceptable ST must satisfy several conditions:

- C_1: the ST must map a (real positive) α to a real positive α', since a map taking $\alpha > 0$ to $\alpha' = 0$ would be singular, and a map taking $\alpha > 0$ to a negative or complex α' would violate the unitarity of the theory.
- C_2: the ST should not map a moderate value of α, where perturbation theory is applicable, to a value of α' so large that pert. theory is inapplicable.
- C_3: J should not vanish (or diverge) or else there would be a pole in $\beta_{\alpha'}$
- C_4: Existence of an IR zero of β is a scheme-independent property, so the ST should satisfy the condition that β_α has an IR zero if and only if $\beta_{\alpha'}$ has an IR zero.

These conditions can always be satisfied by an ST near the UVFP at $\alpha = \alpha' = 0$, but they are not automatic, and can be quite restrictive at an IRFP.

For example, consider the ST (dependent on a parameter r)

$$a = \frac{\tanh(ra')}{r}$$

with inverse

$$a' = \frac{1}{2r} \ln\left(\frac{1+ra}{1-ra}\right)$$

(e.g., for $r = 4\pi$, $\alpha = \tanh \alpha'$). This is acceptable for small a, but if $a > 1/r$, i.e., $\alpha > 4\pi/r$, it maps a real α to a complex α' and hence is physically unacceptable.

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. For example, we have used a $\sinh$ transformation (depending on a parameter r):

$$a = \frac{\sinh(ra')}{r}$$

with inverse

$$a' = \frac{1}{r} \ln\left[ra + \sqrt{1+(ra)^2}\,\right]$$

Written in the form $a = a' f(a')$, this has the transformation function

$$f(a') = \frac{\sinh(ra')}{ra'}$$

This satisfies $f(0) = 1$ and also approaches the identity map as $r \to 0$. With no loss of generality, take $r \geq 0$.

The Jacobian is $J = \cosh(ra')$, which always satisfies C_3, i.e., is nonsingular.

Taylor series expansion of $f(a')$ has coefficients $k_s = 0$ for odd s and

$$k_2 = \frac{r^2}{6}, \quad k_4 = \frac{r^4}{120}, \quad k_6 = \frac{r^6}{5040}, \quad k_8 = \frac{r^8}{362880},$$

etc. for $s \geq 10$. Thus, for small $|r|a'$,

$$a = a'\left[1 + \frac{(ra')^2}{6} + O\Big((ra')^4\Big)\right]$$

so (for $a \neq 0$) $a' < a$ for $|r| > 0$.

Illustrative results with this $\sinh$ scheme transformation follow. We denote the IR zero of $\beta_{\alpha'}$ at the n-loop level as $\alpha'_{IR,n\ell} \equiv \alpha'_{IR,n\ell,r}$.

For SU(3) gauge theory with $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, and:

$$\alpha_{IR,3\ell,\overline{\rm MS}} = 0.435, \quad \alpha'_{IR,3\ell,r=3} = 0.434, \quad \alpha'_{IR,3\ell,r=6} = 0.433,$$
$$\alpha_{IR,4\ell,\overline{\rm MS}} = 0.470, \quad \alpha'_{IR,4\ell,r=3} = 0.470, \quad \alpha'_{IR,4\ell,r=6} = 0.467,$$

Thus, we find moderately small scheme dependence in the value of the IR zero at 3-loop and 4-loop level for moderate α and r.

Construction and application of two new scheme transformations in Choi and RS, PRD 90, 125029 (2014) [arXiv:1411.6645] confirms and extends these results:

$$S_{L_r}: \quad a = \frac{\ln(1+ra')}{r}$$

$$S_{Q_r}: \quad a = \frac{a'}{1-ra'}$$

where again, r is a parameter (some details on supplementary slides at end).

Since the coefficients b_ℓ at loop order $\ell \geq 3$ in the beta function are scheme-dependent, one might expect that it would be possible, at least in the vicinity of zero coupling (UVFP in an asymp. free theory; IRFP in an IR-free theory) to construct a scheme transformations that would set $b'_\ell = 0$ for some range of $\ell \geq 3$, and, indeed a ST that would do this for all $\ell \geq 3$, so that $\beta_{\alpha'}$ would consist only of the 1-loop and 2-loop terms ('t Hooft scheme).

We have constructed an explicit scheme transformation that can do this in the vicinity of zero coupling constant. However, we have also shown that it is much more difficult to try to do this at a zero of β away from the origin (IR zero for an asymp. free theory; UV zero for an IR-free theory).

Specifically, we construct a scheme transformation, denoted S_{R,m,k_1}, that removes the terms in the beta function from loop order 3 up to $m+1$, inclusive, for small coupling. In the limit $m \to \infty$, this transforms to the 't Hooft scheme.

To construct this ST, first, we take advantage of the property that in b'_ℓ, the ST coefficient $k_{\ell-1}$ appears only linearly. For example, $b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1$, etc. for higher-$\ell$ b'_ℓ. So solve eq. $b'_3 = 0$ for k_2, obtaining

$$k_2 = \frac{b_3}{b_1} + \frac{b_2}{b_1} k_1 + k_1^2$$

This determines $S_{R,2,k_1}$.

To get $S_{R,3,k_1}$, substitute this k_2 into expression for b'_4 and solve eq. $b'_4 = 0$, obtaining

$$k_3 = \frac{b_4}{2b_1} + \frac{3b_3}{b_1} k_1 + \frac{5b_2}{2b_1} k_1^2 + k_1^3$$

This determines $S_{R,3,k_1}$. We continue this procedure iteratively to calculate S_{R,m,k_1} for higher m. In general, the equation $b'_\ell = 0$ is a linear equation for $k_{\ell-1}$, so one is guaranteed a unique solution.

So the ST S_{R,m,k_1} has nonzero k_s, $s = 1, ..., m$ and in the transformed beta function, sets $b'_\ell = 0$ for $\ell = 3, ..., m+1$. The coefficients k_s for this ST depend on the b_n in the original beta function for $n = 1, ..., m+1$, and on the parameter k_1.

In addition to the successful application near the origin, $\alpha = 0$, we have shown that this ST S_{R,m,k_1} can be applied over part, but not all, of the interval I where the 2-loop beta function has an IR zero.

Some Results on Dynamical Electroweak Symmetry Breaking and Strongly Coupled Chiral Gauge Theories

Although the Higgs-like scalar discovered at the LHC is consistent with being the SM Higgs, naturalness arguments still motivate studies of extensions of the SM, including possible dynamical electroweak symmetry breaking (EWSB) models with technicolor (TC).

Recall that a TC theory features an asymptotically free vectorial TC gauge symmetry and a set of TC-nonsinglet, SM-nonsinglet fermions, $\{F\}$ (Weinberg, Susskind, 1979).

The TC theory becomes strongly coupled at the TeV scale, confining and producing technifermion condensates $\langle \bar{F} F \rangle$, with associated spontaneous chiral symmetry breaking (SχSB) and dynamical EWSB.

A crucial property of viable TC theories is quasi-scale-invariant (i.e., walking) behavior (Yamawaki et al., 1986; Holdom, 1986; Appelquist et al., 1986). The Higgs-like scalar is then the technidilaton resulting from the SSB of the approximate scale invariance of the WTC theory.

This is one major motivation for the intensive lattice studies of quasi-scale-invariant gauge theories by many groups, with results on γ_{IR} and obtaining a light composite scalar.

To give masses to SM fermions, one embeds the TC theory in a larger, extended technicolor (ETC) theory. With an SU(N_{TC}) TC gauge group, the SU(N_{ETC}) ETC theory gauges the SM fermion generation index and combines it with the TC gauge index, so

$$N_{ETC} = N_{TC} + N_{gen.}$$

The ETC theory is an asymptotically free chiral gauge theory that becomes strongly coupled and self-breaks in $N_{gen.}$ stages down to TC via formation of various condensates. Reasonably UV-complete ETC theories have been constructed that exhibit the requisite self-breaking (e.g., Appelquist and RS, PLB 548, 204 (2002); PRL 90, 201801 (2003); Appelquist, Piai, RS, PRD 69, 015002 (2004)).

TC/ETC theories face many challenges, including precision EW constraints, flavor-changing neutral current processes, t-b mass splitting, CKM mixing, ability to produce a light, Higgs-like scalar, ability to produce very small neutrino masses, etc.

After the original 1986 papers on a technidilaton, there have been many studying how its properties compare with those of the SM Higgs, e.g., Goldberger, Grinstein, Skiba, PRL 100, 111802 (2008); Appelquist and Bai, PRD82, 071701 (2010); Hashimoto and Yamawaki, PRD83, 015008 (2011)... (refs. in arXiv:1501.06454).

TC fit to Higgs as a technidilaton: Matsuzaki and Yamawaki, PRD 85, 095020 (2012); PRD 86, 115004 (2012); PLB 719, 378 (2013), favoring $N_{TC} = 4$ (discussed in Matsuzaki's talk).

Recent result on TC/ETC model-building: Kurachi, RS, Yamawaki, arXiv:1501.06454; we construct an ETC theory in which we embed one-family TC. The SM-singlet part of the ETC theory has a chiral fermion in the antisymmetric rank-2 tensor rep. of SU(N_{ETC}), $\genfrac{}{}{0pt}{}{\Box}{\Box}$ plus $(N_{ETC} - 4)$ copies ("flavors") of chiral fermions in the conjugate fundamental rep. $\overline{\Box}$ (an anomaly-free set):

$$\psi_R^{ij} = \psi_R^{[ij]} : \genfrac{}{}{0pt}{}{\Box}{\Box}$$

$$\chi_{i,s,R} : \overline{\Box}, \quad 1 \le s \le N_{ETC} - 4$$

where i, j = ETC gauge indices and s = flavor index.

The ETC gauge interaction is asymptotically free and, at a scale denoted Λ_1, leads to fermion condensation in the channel

$$\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \bar{\Box} \rightarrow \Box \ ,$$

breaking SU(N_{ETC}) to SU($N_{ETC}-1$). The associated condensate is

$$\langle \sum_{j=2}^{N_{ETC}} \psi_R^{1j\ T} C \, \chi_{j,1,R} \rangle \ ,$$

where, by notation convention, we take the ETC gauge index $i=1$ in ψ_R^{ij} and the flavor index $s=1$ in $\chi_{j,s,R}$.

The fermions ψ_R^{1j} and $\chi_{j,1,R}$ with $2 \le j \le N_{ETC}$ involved in this condensate gain dynamical masses of order Λ_1 and are integrated out of the SU($N_{ETC}-1$) low-energy effective theory (LEET) applicable at scales $\mu < \Lambda_1$.

This SU($N_{ETC}-1$) theory is again asymptotically free, with a gauge coupling that continues to grow, and we infer that at a lower scale, Λ_2, there is again condensation in the $\begin{smallmatrix}\Box\\\Box\end{smallmatrix} \times \bar{\Box} \rightarrow \Box$ channel, breaking SU($N_{ETC}-1$) to SU($N_{ETC}-2$).

The associated condensate is

$$\langle \sum_{j=3}^{N_{ETC}} \psi_R^{2j\ T} C \chi_{j,2,R} \rangle,$$

where, by notation convention, we take the gauge index $i = 2$ in ψ_R^{ij} and the flavor index $s = 2$ in $\chi_{j,s,R}$. The fermions ψ_R^{2j} and $\chi_{j,2,R}$ with $3 \le j \le N_{ETC}$ involved in this condensate gain dynamical masses of order Λ_2 and are integrated out of the SU($N_{ETC} - 2$) LEET operative at $\mu < \Lambda_2$.

This sequential self-breaking of the SU(N_{ETC}) theory continues iteratively in $N_{ETC} - 4$ stages, using the $N_{ETC} - 4$ flavors of $\chi_{j,s,R}$ fermions, reducing the original SU(N_{ETC}) ETC gauge symmetry to the (vectorial) SU(N_{TC}) subgroup symmetry, with the broken indices being generation indices.

Hence,

$$N_{gen.} = N_{ETC} - 4$$

Substituting this in the eq. $N_{ETC} = N_{gen.} + N_{TC}$, we get $N_{TC} = 4$.

We have thus determined N_{TC} from the structure of the specific ETC theory in which our TC theory is embedded. (Note that $N_{gen.}$ cancels out in the algebra.) Setting $N_{gen.} = 3$, we thus get an SU(7) ETC gauge group.

Interestingly, this value $N_{TC} = 4$ agrees with the preferred value obtained by Matsuzaki and Yamawaki from their technidilaton fit to the Higgs-like scalar.

This ETC model naturally accounts for the mass hierarchy in the SM fermion generations, since the SM fermion masses in the i'th generation result from exchange of ETC vector bosons with mass Λ_i and, in the ETC boson propagators,

$$\Lambda_1^{-2} \ll \Lambda_2^{-2} \ll \Lambda_3^{-2}$$

Resultant running fermion mass $m_{f_i}(p)$ is constant up to Λ_i and has the power-law decay $m_{f_i}(p) \propto (\Lambda_i/p)^2$ for $p \gg \Lambda_i$ (Christensen, RS, PRL 94, 241801 (2005)).

The SU(4)$_{TC}$ theory has one SM family of technifermions and the (SM-singlet) □ $\equiv A$ fermion, which is self-conjugate in SU(4). At the $\sim$ TeV scale, the □ fermion forms a condensate in the channel $A \times A \rightarrow 1$, which is invariant under TC and the SM. Technifermion condensates $\langle \bar{F} F \rangle$ cause EWSB in the usual way.

In general, a technidilaton-like composite scalar in this theory contains $\bar{F}F$, AA, and techni-glue components.

In order for this model to be viable, it must exhibit walking behavior and must have considerable suppression of the EW S parameter (e.g., Kurachi, RS, Yamawaki, PRD76, 035003 (2007)). The 60 PNGBs and the techni-vector mesons should also gain sufficiently large masses $\gtrsim$ few TeV to agree with current LHC bounds (Matsuzaki and Yamawaki; op. cit., Kurachi, Matsuzaki, Yamawaki, PRD90, 055028 (2014); PRD90, 095013 (2014)) (Matsuzaki's talk; PDG LHC review: Chivukula, Narain, Womersley).

This model motivates lattice studies of SU(4) with $N_f = 2(N_c + 1) = 8$ Dirac fermions in the fundamental rep.

Further ingredients are needed to account for actual SM fermion masses and mixings, e.g., t-b mass splitting

The next run of the LHC should provide a stringent test of this model.

In addition to this phenomenological application, this model is of field-theoretic interest for the insight that it provides on how the structure of a low-energy effective field theory - here the TC theory - is determined by its embedding in an ultraviolet completion, the ETC theory.

The sequential chiral gauge symmetry breaking via condensate formation in this model is typical of strongly coupled chiral gauge theories, χGTs (early work: Georgi, Dimopoulos, Raby, Susskind).

Some recent studies of patterns of UV to IR evolution in asymptotically free χGTs: Appelquist and RS, PRD 88, 105012 (2013) [arXiv:1310.6076]; Y. Shi and RS, PRD 91, 045004 (2015) [arXiv:1411.2042].

Analyze beta function for possible IR zero at weak or stronger coupling.

For strongly coupled χGT, use most attractive channel (MAC) guide: condensates form preferentially in channel $R_1 \times R_2 \to R_{cond.}$ with largest $\Delta C_2 = C_2(R_1) + C_2(R_2) - C_2(R_{cond.})$, ($R =$ fermion rep., $C_2(R) =$ Casimir invariant); also use vacuum alignment arguments.

If resultant IR theory is weakly coupled (e.g., massless NGBs, gauge-singlet fermions), interesting to count perturbative degrees of freedom in fields, test conjecture that $f_{UV} \ge f_{IR}$, where $f = 2N_v + (7/4)N_f + N_s$ (v, f, s refer to massless spin 1, 1/2, and 0 fields) (Appelquist, Cohen, Schmaltz, RS, 1999).

Ideally, one would use lattice for fully nonperturbative method, but fermion doubling makes it difficult to put χGTs on lattice.

Studies of RG Flows in Infrared-Free Gauge Theories

If the β function of a theory is positive near zero coupling, then this theory is IR-free; as μ increases from the IR to the UV, the coupling grows. It is of interest to investigate whether an IR-free theory might have a UV fixed point (UV zero of β).

In addition to performing perturbative calculations of β to search for such a UVFP in an IR-free theory, one can use large-N methods. An explicit example is the O(N) nonlinear σ model in $d = 2 + \epsilon$ spacetime dimensions. From an exact solution of this model in the limit $N \to \infty$ in 1976, we found that (for small ϵ)

$$\beta(\lambda) = \frac{d\lambda}{dt} = \epsilon\lambda\Big(1 - \frac{\lambda}{\lambda_c}\Big) \ , \ i.e., \quad \beta(x) = \frac{dx}{dt} = \epsilon x\Big(1 - \frac{x}{x_c}\Big)$$

where λ is the effective coupling, $\lambda_c = 2\pi\epsilon/N$; $x = \lim_{N\to\infty} \lambda N$, $x_c = 2\pi\epsilon$ (Bardeen, B. W. Lee, and RS, PRD14, 985 (1976); Brézin and Zinn-Justin, PRB 14, 3110 (1976)). Thus this theory has a UVFP at x_c, so that if initial value of $x < x_c$, then $x \nearrow x_c$ as $\mu \to \infty$.

There has long been interest in RG properties of $d = 4$ QED and, more generally, U(1) gauge theory (early work: Gell-Mann and Low; Johnson, Baker, and Willey; Adler; Yamawaki, Miransky,..).

Consider a vectorial U(1) theory with N_f massless Dirac fermions of charge q. With no loss of generality, set $q = 1$. Write β function as

$$\beta_\alpha = 2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell$$

The 1-loop and 2-loop coefficients are

$$b_1 = \frac{4N_f}{3} \, , \qquad b_2 = 4N_f$$

These coefficients have the same sign, so the two-loop beta function, $\beta_{\alpha,2\ell}$, does not have a UV zero, and this is the maximal scheme-independent information about it. The coefficients have been calculated up to five loops in the $\overline{\mathrm{MS}}$ scheme.

The 3-loop coefficient (deRafael and Rosner) is negative:

$$b_3 = -2N_f\Big(1 + \frac{22N_f}{9}\Big)$$

Hence, $\beta_{\alpha,3\ell}$ has a UV zero, namely,

$$\alpha_{UV,3\ell} = 4\pi a_{UV,3\ell} = \frac{4\pi[9 + \sqrt{3(45 + 44N_f)}\ ]}{9 + 22N_f}$$

The 4-loop coefficient (Gorishny et al.) is negative: numerically,

$$b_4 = -N_f\,(46 + 82.97533 N_f + 5.06996 N_f^2)$$

Recently, b_5 has been calculated (Kataev, Larin; Baikov et al., 2012, 2013). Numerically,

$$b_5 = N_f(846.6966 + 798.8919 N_f - 148.7919 N_f^2 + 9.22127 N_f^3)$$

which is positive for all $N_f > 0$.

In RS, PRD 89, 045019 (2014) [arXiv:1311.5268], we have investigated whether the n-loop beta function for this U(1) gauge theory has a UV zero for n up to 5 loops, for a large range of N_f. Our results are given in the table (dash means no UV zero).

N_f	$\alpha_{UV,2\ell}$	$\alpha_{UV,3\ell}$	$\alpha_{UV,4\ell}$	$\alpha_{UV,5\ell}$
1	—	10.2720	3.0400	—
2	—	6.8700	2.4239	—
3	—	5.3689	2.0776	—
4	—	4.5017	1.8463	—
5	—	3.9279	1.67685	2.5570
10	—	2.5871	1.2135	1.3120
20	—	1.7262	0.8483	—
100	—	0.7081	0.33265	—
500	—	0.3038	0.1203	—
10^3	—	0.2127	0.07678	—
10^4	—	0.016614	0.016965	—

A necessary condition for the perturbatively calculated β function to yield evidence for a stable UV zero is that it should remain present when one increases the loop order and the fractional change in the value should decrease going from n to $n+1$ loops.

We find that the UV zeros that we have calculated at $\ell = 3, \ 4, \ 5$ loop order for a large range of N_f values do not satisfy this necessary condition. Hence, our results do not give evidence for a UVFP in U(1) gauge theory for general N_f. We find similar conclusions for an SU(N) gauge theory with N_f larger than the asympt. free range.

RG Flows in the O(N) $\lambda|\vec{\phi}|^4$ Theory

We have carried out a similar study, again up to 5-loop order, of another IR-free theory, namely O(N) $\lambda|\vec{\phi}|^4$ theory (in $d = 4$) to search for a possible UV zero of the beta function, in RS, Phys. Rev. D 90, 065023 (2014) [arXiv:1408.3141].

Interaction term: $\mathcal{L}_{int} = -\frac{\lambda}{4!}(\vec{\phi}^2)^2$

$$\beta \text{ function :} \quad \beta_a = \frac{da}{dt} = a \sum_{\ell=1}^{\infty} b_\ell \, a^\ell \quad \text{where} \quad a = \frac{\lambda}{16\pi^2}$$

Coefficients:

$$b_1 = \frac{1}{3}(N+8) \ , \quad b_2 = -\frac{1}{3}(3N+14)$$

$$b_3 = \frac{11}{72}N^2 + \left(\frac{461}{108} + \frac{20\zeta(3)}{9}\right)N + \frac{370}{27} + \frac{88\zeta(3)}{9}$$

Numerically,

$$b_3 = 0.15278N^2 + 6.93976N + 24.4571$$

and so forth for b_4 and b_5 (calculated in $\overline{\text{MS}}$ scheme)

Although the two-loop beta function has a UV zero, it occurs at too large a value of the coupling for the perturbative calculation to be reliable, as shown by the fact that when one calculates to higher-loop order, the 3-loop beta function has no UV zero, and the 4-loop and 5-loop beta functions differ considerably from the 2-loop and 3-loop beta functions where the 2-loop function has a zero.

We have studied this further with scheme transformations and Padé approximants.

We thus conclude that in the range of λ where the perturbative calculation of the n-loop beta function is reliable, the theory does not exhibit evidence of a UV zero up to the level of $n = 5$ loops.

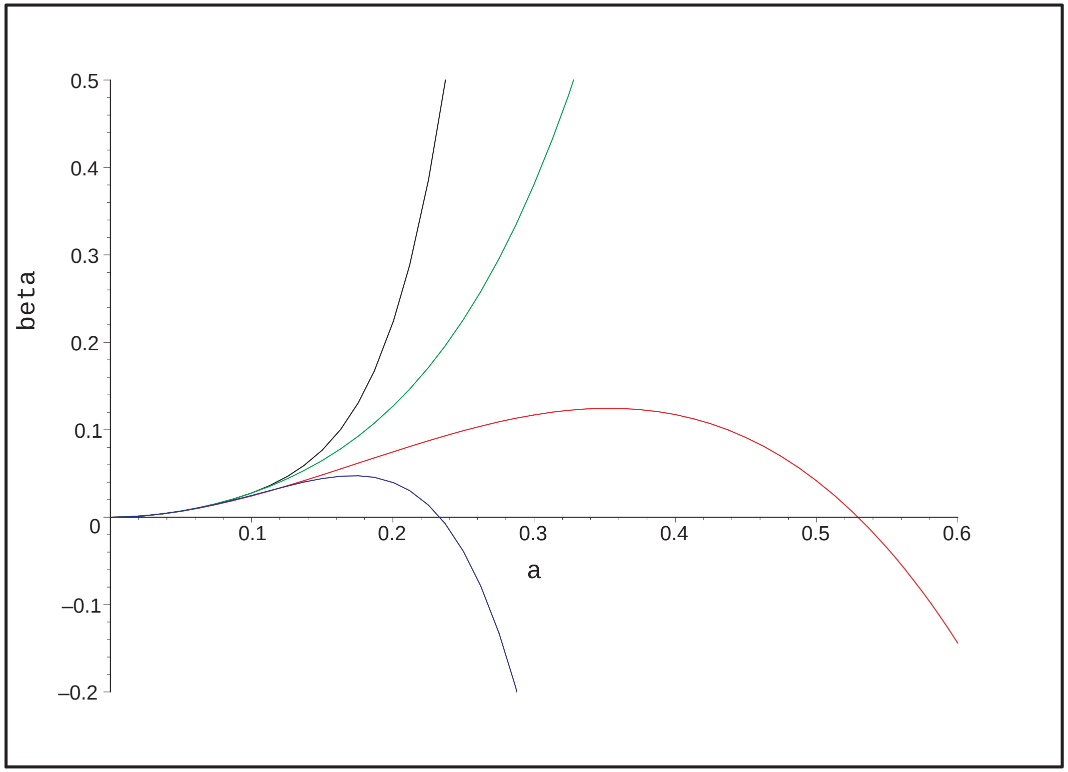

Figure 3: Plot of the n-loop β function $\beta_{a,n\ell}$ as functions of a for $N = 1$ and (i) $n = 2$ (red), (ii) $n = 3$ (green), (iii) $n = 4$ (blue), and $n = 5$ (black). At $a = 0.18$, going from bottom to top, the curves are for $n = 4$, $n = 2$, $n = 3$, and $n = 5$.

N	$a_{UV,2\ell}$	$a_{UV,3\ell}$	$a_{UV,4\ell}$	$a_{UV,5\ell}$
1	0.5294	—	0.2333	—
2	0.5000	—	0.2217	—
3	0.4783	—	0.2123	—
4	0.4615	—	0.2044	—
5	0.4483	—	0.1978	—
6	0.4375	—	0.1920	—
7	0.4286	—	0.1869	—
8	0.42105	—	0.1823	—
9	0.4146	—	0.1783	—
10	0.4091	—	0.1746	—
100	0.3439	—	0.1012	—
1000	0.3344	—	0.07241	0.02276
3000	0.3337	—	0.5475	0.008850
10^4	0.3334	—	—	0.003460

RG Flows in a Yukawa Theory

With E. Mølgaard, we have calculated RG flows for Yukawa theories in Mølgaard and RS, PR D 89, 105007 (2014) [arXiv:1403.3058].

To study flows in simple context, use the (one-gen.) leptonic sector of the SM with the gauge fields turned off . This has a global chiral symmetry group: $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$, forbidding bare fermion mass terms.

fermions: ψ_L: fund. rep. of SU(2)$_L$ with U(1)$_Y$ charge Y_ψ; χ_R: singlet of SU(2)$_L$ with U(1)$_Y$ charge Y_χ; scalar ϕ: fund. rep. of SU(2) with U(1)$_Y$ charge $Y_\phi = Y_\psi - Y_\chi$ so Yukawa term $y\bar{\psi}_L \chi_R \phi + h.c.$ allowed by symmetry.

RG flows depend on y and the quartic scalar coupling λ. Beta functions (with $dt = d\ln\mu$):

$$\beta_y = \frac{dy}{dt}\,, \qquad \beta_\lambda = \frac{d\lambda}{dt}$$

Convenient variables: $a_y = y^2/(4\pi)^2$ and $a_\lambda = \lambda/(4\pi)^2$. Corresponding beta functions: $\beta_{a_y} = da_y/dt = (2y)(4\pi)^{-2}\,\beta_y$ and $\beta_{a_\lambda} = da_\lambda/dt = (4\pi)^{-2}\,\beta_\lambda$.

As before compare calculations to different loop orders; calculate β_y and β_λ to loop orders (1,1), (1,2), (2,1), (2,2), then integrate to get the RG flows.

For small a_y and a_λ, the RG flow is to the IR-free zero of both beta functions at $a_y = a_\lambda = 0$, i.e., $y = \lambda = 0$.

For larger y and λ, the flows show further structure.

Comparison of these different loop-order RG flows yields info. on the extent of the region in a_y and a_λ where the perturbative calculations agree with each other and hence may be reliable.

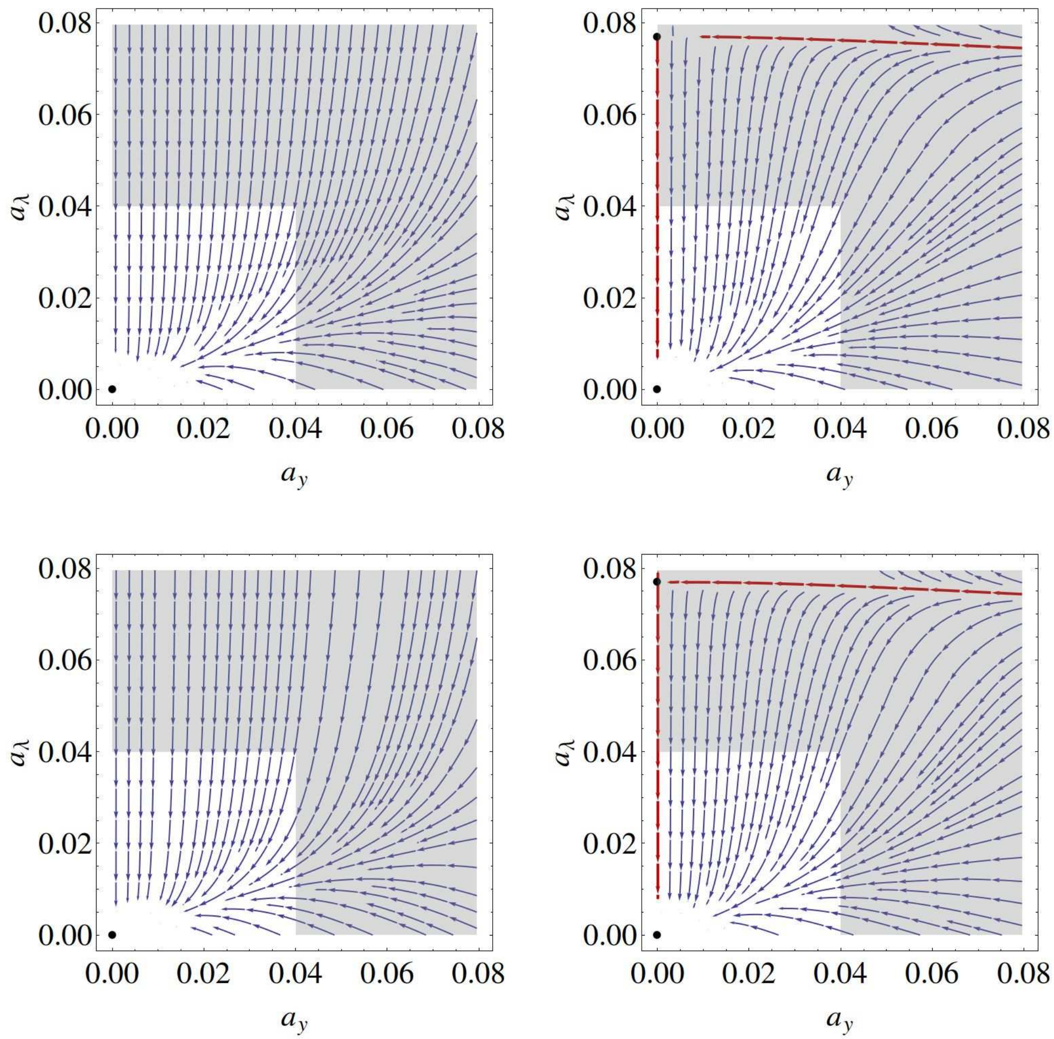

Figure 4: RG flows obtained via integration of beta functions $\beta_{a_y,\ell}$ and $\beta_{a_\lambda,\ell'}$ for small a_y and a_λ, calculated for loop orders (ℓ,ℓ'): (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). Arrows are flows from UV to IR.

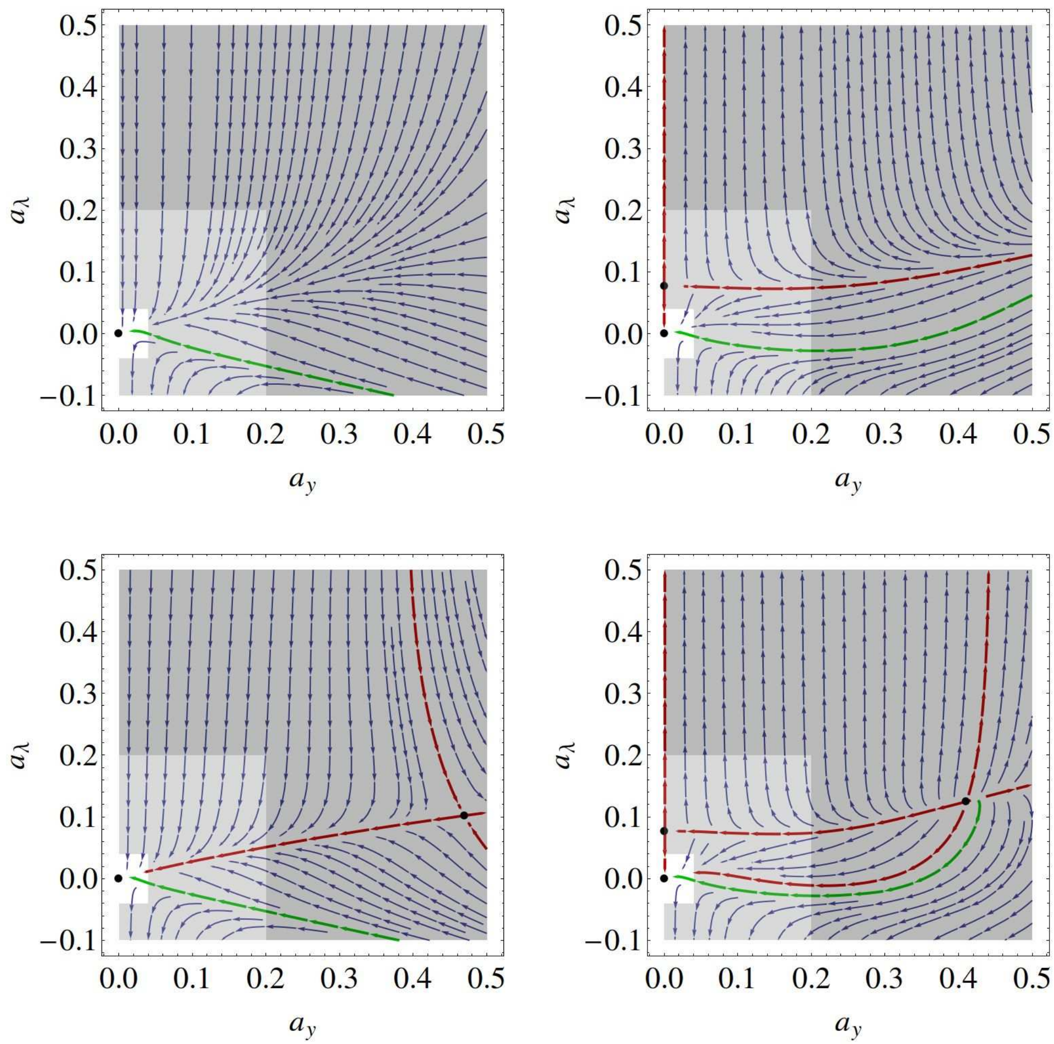

Figure 5: RG flows obtained via integration of beta functions $\boldsymbol{\beta_{a_y,\ell}}$ and $\boldsymbol{\beta_{a_\lambda,\ell'}}$ for moderate $\boldsymbol{a_y}$ and $\boldsymbol{a_\lambda}$, calculated for loop orders $\boldsymbol{(\ell,\ell')}$: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). Arrows are flows from UV to IR.

Conclusions

- Understanding the UV to IR evolution of an asymptotically free gauge theory and behavior associated with an exact or approximate IR fixed point of RG is of fundamental field-theoretic interest and may have relevance to physics beyond the Standard Model.

- Our higher-loop calcs. give info. on this UV to IR flow and on determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$; interesting comparison with γ_{IR} from lattice.

- We have investigated effects of scheme-dependence of IR zero in the beta function in higher-loop calculations.

- We have discussed an SU(4) technicolor model in which $N_{TC} = 4$ is derived from embedding of technicolor in an extended technicolor model.

- We have carried out analyses of RG flows other theories: IR-free theories including U(1) gauge theory, nonabelian gauge theory with $N_f > N_{f,b1z}$, $\lambda|\vec{\phi}|^4$, Yukawa models.

Thanks to the organizers of this KMI SCGT15 conference, thanks also to Koichi Yamawaki and KMI members for warm hospitality during a visit here in Jan. 2015, and thank you for your attention.

Supplementary slides:
Values of $\bar{b}_\ell = b_\ell/(4\pi)^\ell$ for $N_c = 3$, where interval I is $8.05 < N_f < 16.5$:

N_c	N_f	$\bar{b}_1$	$\bar{b}_2$	$\bar{b}_3$	$\bar{b}_4$
3	0	0.875	0.646	0.720	1.173
3	1	0.822	0.566	0.582	0.910
3	2	0.769	0.485	0.450	0.681
3	3	0.716	0.405	0.324	0.485
3	4	0.663	0.325	0.205	0.322
3	5	0.610	0.245	0.091	0.194
3	6	0.557	0.165	-0.016	0.099
3	7	0.504	0.084	-0.118	0.039
3	8	0.451	0.004	-0.213	0.015
3	9	0.398	-0.076	-0.303	0.025
3	10	0.345	-0.156	-0.386	0.072
3	11	0.292	-0.236	-0.463	0.154
3	12	0.239	-0.317	-0.534	0.273
3	13	0.186	-0.397	-0.599	0.429
3	14	0.133	-0.477	-0.658	0.622
3	15	0.080	-0.557	-0.711	0.852
3	16	0.0265	-0.637	-0.758	1.121

Remark on 3-loop analysis: since $\beta_{3\ell} = -[\alpha^2/(2\pi)](b_1 + b_2 a + b_3 a^2)$, $\beta_{3\ell} = 0$ away from $\alpha = 0$ formally at two values of α:

$$\alpha = \frac{2\pi}{b_3}\Big(- b_2 \pm \sqrt{b_2^2 - 4b_1 b_3}\ \Big)$$

Since $b_2 \to 0$ at lower end of interval I, and since $b_1 > 0$, it is necessary that $b_3 < 0$ for $N_f \in I$ in order to have $b_2^2 - 4b_1 b_3 > 0$ and hence a physical IR zero of $\beta_{3\ell}$.

Since the existence of the IR zero in β at 2-loop level is scheme-independent, one may require that a scheme should maintain this property to higher-loop order, and hence that $b_3 < 0$ for $N_f \in I$.

Since $b_2 < 0$ and with $b_3 < 0$, one can write

$$\alpha = \frac{2\pi}{|b_3|}\Big(- |b_2| \mp \sqrt{b_2^2 + 4b_1|b_3|}\ \Big)$$

soln. with $+$ sqrt is $\alpha_{IR,3\ell}$ (soln. with $-$ sqrt is unphysical).

Given that $b_3 < 0$ for $N_f \in I$, a simple algebraic proof yields the result $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ (RS, Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209]).
So the inequality $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ holds more generally than just in $\overline{\mathrm{MS}}$ scheme.

Construction and application of new scheme transformations, in Choi and RS, PRD 90, 125029 (2014) [arXiv:1411.6645]:

S_{L_r} scheme transformation:

$$S_{L_r}: \quad a = \frac{\ln(1+ra')}{r}$$

where r is a (real) parameter; corresponding transformation function:

$$f(a') = \frac{\ln(1+ra')}{ra'}$$

$$\text{Inverse}: \quad a' = \frac{e^{ra}-1}{r} \, , \quad \text{Jacobian}: J = \frac{1}{1+ra'} = e^{-ra}$$

Here $f(a')$ has the Taylor series expansion

$$f(a') = 1 + \sum_{s=1}^{\infty} \frac{(-ra')^s}{s+1} \, ,$$

i.e., coefficients are $k_s = (-r)^s/(s+1)$.

So for small $|r|a'$,

$$a = a'\Big[\,1 - \frac{ra'}{2} + O\Big((ra')^2\Big)\,\Big]$$

so (for $a \neq 0$), $a' > a$ if $r > 0$ and $a' < a$ if $r < 0$.

Note that for a given s, these k_s are much larger than those for the $\sinh$ ST, so for a given value of r, the S_{L_r} ST is farther from the identity than the $\sinh$ ST.

Allowed range of r: condition C_1 requires that the argument of the log must be positive, which yields the lower bound $r > -1/a'$ (also required by condition C_3 that $J > 0$). If $r > 0$, this inequality is obviously satisfied, so consider negative r.

Substitute relation for a' into $r > -1/a'$; get $r > r/(1 - e^{ra})$. Since r is assumed negative, can rewrite this as $-|r| > -|r|/(1 - e^{-|r|a})$, i.e., $1 < 1/(1 - e^{-|r|a})$, which is always satisfied.

Thus, r may be positive or negative, and the actual range of r is determined by the combination of the conditions C_i, $i = 1, ..4$.

Illustrative results with this S_{L_r} scheme transformation: We again denote the IR zero of $\beta_{\alpha'}$ at the n-loop level as $\alpha'_{IR,n\ell} \equiv \alpha'_{IR,n\ell,r}$.

For SU(3) with $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, and:

$$\alpha'_{IR,3\ell,r=-2} = 0.429, \quad \alpha'_{IR,3\ell,r=-1} = 0.432, \quad \alpha'_{IR,3\ell,r=0} = \alpha_{IR,3\ell,\overline{\rm MS}} = 0.435,$$
$$\alpha'_{IR,3\ell,r=1} = 0.438, \quad \alpha'_{IR,3\ell,r=2} = 0.441,$$

$$\alpha'_{IR,4\ell,r=-2} = 0.450, \quad \alpha'_{IR,4\ell,r=-1} = 0.460, \quad \alpha'_{IR,4\ell,r=0} = \alpha_{IR,4\ell,\overline{\rm MS}} = 0.470,$$
$$\alpha'_{IR,4\ell,r=1} = 0.482, \quad \alpha'_{IR,4\ell,r=2} = 0.496$$

Again, we find rather small scheme dependence in the value of the IR zero of beta at $n = 3$ and $n = 4$ loop level with this scheme transformation for moderate α and r.

We have also considered scheme transformation involving rational transformation functions; for example,

$$S_{Q_r}: \quad a = \frac{a'}{1 - ra'}$$

where r is a (real) parameter; corresponding transformation function:

$$f(a') = \frac{1}{1 - ra'}$$

$$\text{Inverse}: \quad a' = \frac{a}{1 + ra}, \quad \text{Jacobian}: J = \frac{1}{(1 - ra')^2} = (1 + ra)^2$$

Here $f(a')$ has the Taylor series expansion

$$f(a') = 1 + \sum_{s=1}^{\infty} (ra')^s ,$$

i.e., coefficients are $k_s = r^s$. So for small $|r|a'$,

$$a = a'\Big[\, 1 + ra' + O\Big((ra')^2\Big)\, \Big] .$$

Here, $a' < a$ if $r > 0$ and $a' > a$ if $r < 0$.

Allowed range of r: since $a' = a/(1 + ra)$, condition C_1 requires that denom. be positive, and hence that $r > -1/a$; and since $a = a'/(1 - ra')$, C_1 requires $r < 1/a'$. Substituting above relation for a' yields $r < a^{-1} + r$, which is always valid.

So, as with the S_{L_r} ST, actual range of r determined by combination of the conditions C_i, $i = 1, ..4$.

For the S_{Q_r} scheme transformation, as with the S_{L_r} ST, we find that the shift in the IR zero of the beta function at 3-loop and 4-loop level is small for moderate α and r.

These results are in agreement with our previous ones for the $\sinh$ scheme transformation.

Our studies provide a quantitative evaluation of scheme-dependent effects in calculations of the IR zero in the beta function. We have constructed scheme transformations that are physically acceptable over the required range of α_{IR} values and have found reasonably small scheme-dependence in the value of the IR zero of β for moderate α_{IR} and ST-parameter r.

RG Flows in U(1) Theory

In addition to our analysis of the beta function up to 5-loop order, we have carried out an analysis in the limit

$$N_f \to \infty \quad \text{with finite} \quad y(\mu) \equiv N_f\, a(\mu) = \frac{N_f\, \alpha(\mu)}{4\pi}$$

We denote this as the LNF (large-N_f) limit; analogous to $N \to \infty$ limit in nonlinear σ model.

We set $b_1 = b_{1,1} N_f$ with $b_{1,1} = 4/3$. Further,

$$b_\ell = \sum_{k=1}^{\ell-1} b_{\ell,k}\, N_f^k \quad \text{for } \ell \ge 2 \ ,$$

where the $b_{\ell,k}$ are independent of N_f.

Hence,

$$b_\ell \propto N_f^{\ell-1} \quad \text{for} \ \ \ell \ge 2 \quad \text{as} \ \ N_f \to \infty$$

We thus define the finite quantities

$$\check{b}_\ell \equiv \frac{b_\ell}{N_f^{\ell-1}} \quad \text{for } \ell \ge 2$$

so

$$\lim_{N_f \to \infty} \check{b}_\ell = b_{\ell,\ell-1} \quad \text{for } \ell \geq 2$$

We define a rescaled β function that is finite in the LNF limit as $\beta_y \equiv \beta_\alpha N_f$. Then

$$\beta_y = 8\pi b_{1,1}\, y^2 \left[1 + \frac{1}{b_{1,1} N_f} \sum_{\ell=2}^{\infty} b_\ell\, y^{\ell-1} \right]$$

The condition that the n-loop β_y, $\beta_{y,n\ell}$, has a zero at $y \neq 0$ is the equation

$$1 + \frac{1}{b_{1,1} N_f} \sum_{\ell=2}^{n} b_\ell\, y^{\ell-1} = 0$$

In the LNF limit, of the $n-1$ roots of this equation, the relevant one has the approximate form

$$y_{UV,n\ell} \sim \left(-\frac{b_{1,1} N_f}{b_{n,n-1}} \right)^{\frac{1}{n-1}}$$

Hence, $\beta_{y,n\ell}$ has a zero for $y \neq 0$ in the LNF limit if and only if $b_{n,n-1} < 0$, which is not, in general true. Further, even if it were true for a given loop order n, in the LNF limit, $\lim_{N_f \to \infty} y_{UV,n\ell} = \infty$.

One can reexpress β_y as a series in powers of $\nu \equiv 1/N_f$:

$$\beta_y = 8\pi b_{1,1}\, y^2 \left[1 + \sum_{s=1}^{\infty} F_s(y)\nu^s\right]$$

An exact integral representation of $F_1(y)$ is known (cf. Holdom, 2010). We have used this representation to determine the signs of $b_{n,n-1}$ up to $n = 24$ loops. We find that these signs are scattered, and show no indication of an onset of negative signs.

Thus, we do not find evidence of a UVFP in a U(1) gauge theory with N_f massless charged fermions for large N_f.