Maximally supersymmetric Yang–Mills on the lattice

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Origin of Mass and Strong Coupling Gauge Theories
Kobayashi–Maskawa Institute, Nagoya University, 5 March 2015

arXiv:1405.0644, arXiv:1410.6971, arXiv:1411.0166 & more to come
with Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Context: Why lattice supersymmetry

At strong coupling...

—Supersymmetric gauge theories are particularly interesting:
Dualities, holography, confinement, conformality, ...

—Nonperturbative lattice discretization is particularly useful
Numerical analysis provides complementary approach to SCGT

Proven success for QCD; many potential susy applications:

- Compute Wilson loops, spectrum, scaling dimensions, etc.,
  complementing perturbation theory, holography, bootstrap, ...
- Further direct checks of conjectured dualities
- Predict low-energy constants from dynamical susy breaking
- Validate or refine AdS/CFT-based modelling
  (e.g., QCD phase diagram, condensed matter systems)

Context: Why not lattice supersymmetry

There is a problem with supersymmetry in discrete space-time

Recall: supersymmetry extends Poincaré symmetry
by spinorial generators $Q_\alpha^{\rm I}$ and $\overline{Q}_{\dot\alpha}^{\rm I}$ with ${\rm I} = 1, \cdots, \mathcal{N}$

The resulting algebra includes $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$

P_μ generates infinitesimal translations, which don't exist on the lattice
$\Longrightarrow$ supersymmetry explicitly broken at classical level

Consequence for lattice calculations

Quantum effects generate (typically many) susy-violating operators

Fine-tuning their couplings to restore susy is generally not practical

Exact susy on the lattice: $\mathcal{N} = 4$ SYM

In order to forbid generation of susy-violating operators
(some subset of) the susy algebra must be preserved

In four dimensions $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)
is the only known system with a supersymmetric lattice formulation

$\mathcal{N} = 4$ SYM is a particularly interesting theory

- SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ}, all massless and in adjoint rep.
- Action consists of kinetic, Yukawa and four-scalar terms
- Supersymmetric: 16 supercharges Q_α^{I} and $\overline{Q}_{\dot\alpha}^{\mathrm{I}}$ with $\mathrm{I} = 1, \cdots, 4$
  Fields and Q's transform under global SU(4) $\simeq$ SO(6) R symmetry
- Conformal: β function is zero for all 't Hooft couplings λ

Exact susy on the lattice: topological twisting

What is special about $\mathcal{N} = 4$ SYM

The 16 fermionic supercharges $Q^{\rm I}_\alpha$ and $\overline{Q}^{\rm I}_{\dot\alpha}$ fill a Kähler–Dirac multiplet:

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \gamma_\mu \mathcal{Q}_\mu + \gamma_\mu \gamma_\nu \mathcal{Q}_{\mu\nu} + \gamma_\mu \gamma_5 \mathcal{Q}_{\mu\nu\rho} + \gamma_5 \mathcal{Q}_{\mu\nu\rho\sigma}$$
$$\longrightarrow \mathcal{Q} + \gamma_a \mathcal{Q}_a + \gamma_a \gamma_b \mathcal{Q}_{ab}$$
$$\text{with } a, b = 1, \cdots, 5$$

This is a decomposition in representations of a “twisted rotation group”

$$\text{SO}(4)_{tw} \equiv \text{diag}\left[\text{SO}(4)_{\rm euc} \otimes \text{SO}(4)_R\right] \qquad \qquad \text{SO}(4)_R \subset \text{SO}(6)_R$$

In this notation there is a susy subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$

This can be exactly preserved on the lattice

Twisted $\mathcal{N} = 4$ SYM

$$\text{SO}(4)_{tw} \equiv \text{diag}\left[\text{SO}(4)_{\text{euc}} \otimes \text{SO}(4)_R\right]$$

- $\mathcal{Q}$, $\mathcal{Q}_\mu$, $\mathcal{Q}_{\mu\nu}$, ... transform with **integer spin** – no longer spinors!
- Fermion fields decompose in the same way, $\Psi^I \longrightarrow \{\eta, \psi_a, \chi_{ab}\}$
- Scalar fields transform as $\text{SO}(4)_{tw}$ vector B_μ plus two scalars ϕ, $\overline{\phi}$
  Combine with A_μ in complexified five-component gauge field

$$\mathcal{A}_a = A_a + iB_a = (A_\mu, \phi) + i(B_\mu, \overline{\phi}) \qquad \text{and similarly for } \overline{\mathcal{A}}_a$$

Complexified gauge field $\Longrightarrow$ U(N) = SU(N) $\otimes$ U(1) gauge invariance

Irrelevant in the continuum, but will affect lattice calculations

Twisted $\mathcal{N} = 4$ SYM

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In flat space twisting is just a change of variables, no effect on physics

Same lattice system also results
from orbifolding / dimensional deconstruction approach

Now we can move directly to the lattice

Twisting gives manifestly supersymmetric lattice action for $\mathcal{N} = 4$ SYM

$$S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde}\ \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$$

$\mathcal{Q}S = 0$ follows from $\mathcal{Q}^2 \cdot = 0$ and Bianchi identity

- We have exact U(N) gauge invariance
- We exactly preserve $\mathcal{Q}$, one of 16 supersymmetries
- Restoration of twisted $SO(4)_{tw}$ in continuum limit guarantees recovery of other 15 $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$

The theory is **almost** suitable for practical numerical calculations. . .

Stabilizing numerical calculations

We need to add two deformations to the $\mathcal{Q}$-invariant action

Both deal with features required by the supersymmetric construction

Scalar potential to regulate flat directions

Gauge links $\mathcal{U}_a$ must be elements of algebra, like fermions

$\longrightarrow$ Add scalar potential $\left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - 1\right)^2$ to lift flat directions

Otherwise $\mathcal{U}_a$ can wander far from continuum form $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$

Plaquette determinant to suppress U(1) sector of U(N)

$\mathcal{U}_a$ complexified $\longrightarrow$ Add approximate SU(N) projection $\left|\det \mathcal{P}_{ab} - 1\right|^2$
where $\mathcal{P}_{ab}$ is the product of four $\mathcal{U}_a$ around the elementary plaquette

Otherwise encounter strong-coupling U(1) confinement transition

Soft susy breaking from naive stabilization

Directly adding scalar potential and plaquette determinant to action explicitly breaks supersymmetry

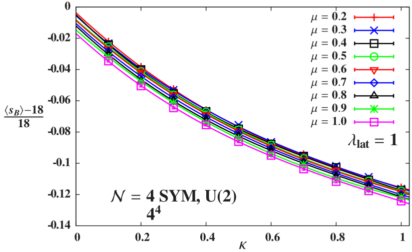
$$S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a - \frac{1}{2} \eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde}\ \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$$
$$+ \frac{N}{2\lambda_{\text{lat}}} \mu^2 \left(\frac{1}{N} \text{Tr} \left[\mathcal{U}_a \overline{\mathcal{U}}_a \right] - 1 \right)^2 + \kappa \left| \det \mathcal{P}_{ab} - 1 \right|^2$$

Breaking is **soft**

Guaranteed to vanish as $\mu, \kappa \longrightarrow 0$

Also suppressed $\propto 1/N^2$

1–10% effects in practice



New development: Supersymmetric stabilization

Possible to construct $\mathcal{Q}$-invariant scalar potential and plaquette det.

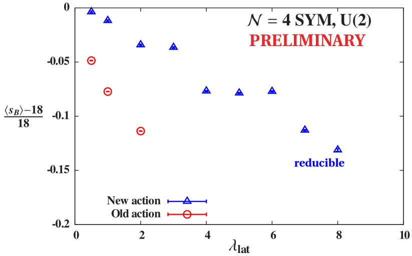
However, these result in positive vacuum energy (non-susy)

$$S = \frac{N}{2\lambda_{\text{lat}}} \mathcal{Q} \left(\chi_{ab}\mathcal{F}_{ab} + \eta \left\{ \overline{\mathcal{D}}_a \mathcal{U}_a + X \right\} - \frac{1}{2}\eta d \right) - \frac{N}{8\lambda_{\text{lat}}} \epsilon_{abcde}\ \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$$

$$X = B^2 \left(\frac{1}{N} \text{Tr} \left[\mathcal{U}_a \overline{\mathcal{U}}_a \right] - 1 \right)^2 + G \left| \det \mathcal{P}_{ab} - 1 \right|^2$$

Again effects vanish as $B, G \longrightarrow 0$

Allows access to much stronger λ with much smaller artifacts



Final thought on the lattice $\mathcal{N} = 4$ SYM formulation

more complicated action with over 100 gathers in the fermion operator:

$$
\begin{aligned}
S &= S'_{exact} + S_{closed} + S_{pot} + S_{det} \qquad (2)\\
S'_{exact} &= \frac{N}{2\lambda_{\text{lat}}} \sum_n \text{Tr}\Big[-\overline{\mathcal{F}}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)\mathcal{D}^{(+)}_{[a}\psi_{b]}(n) - \eta(n)\overline{\mathcal{D}}^{(-)}_a\psi_a(n) \\
&\quad + \frac{1}{2}\left(\overline{\mathcal{D}}^{(-)}_a\mathcal{U}_a(n) + G\sum_{a\neq b}\left|\det\left[\mathcal{P}_{ab}(n)\right] - 1\right|^2 \mathbb{I}_N + \frac{B^2}{N^2}\sum_a \left(\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - N\right)^2 \mathbb{I}_N\right)^2 \Big] \\
S_{closed} &= -\frac{N}{8\lambda_{\text{lat}}} \sum_n \text{Tr}\left[\epsilon_{abcde}\ \chi_{de}(n + \widehat{\mu}_a + \widehat{\mu}_b + \widehat{\mu}_c)\overline{\mathcal{D}}^{(-)}_c \chi_{ab}(n)\right] \\
S_{pot} &= \frac{N}{2\lambda_{\text{lat}}} B^2 \sum_n \text{Tr}\left[\eta(n)\right] \sum_a \left(\frac{1}{N}\text{Tr}\left[\mathcal{U}_a(n)\overline{\mathcal{U}}_a(n)\right] - 1\right) \text{Tr}\left[\psi_a(n)\overline{\mathcal{U}}_a(n)\right] \\
S_{det} &= \frac{N}{2\lambda_{\text{lat}}} G \sum_n \text{Tr}\left[\eta(n)\right] \sum_{a\neq b} \det\left[\mathcal{P}_{ab}(n)\right] \left\{\text{Tr}\left[\mathcal{U}_b^{-1}(n)\psi_b(n)\right] + \text{Tr}\left[\mathcal{U}_a^{-1}(n + \widehat{\mu}_b)\psi_a(n + \widehat{\mu}_b)\right]\right\}.
\end{aligned}
$$

The construction
is obviously very complicated

(For experts: $\gtrsim$100 inter-node data transfers in the fermion operator)

To reduce this barrier to others entering the field,
we make our efficient parallel code publicly available

github.com/daschaich/susy

Evolved from MILC lattice QCD code,
presented in arXiv:1410.6971 — CPC appeared yesterday

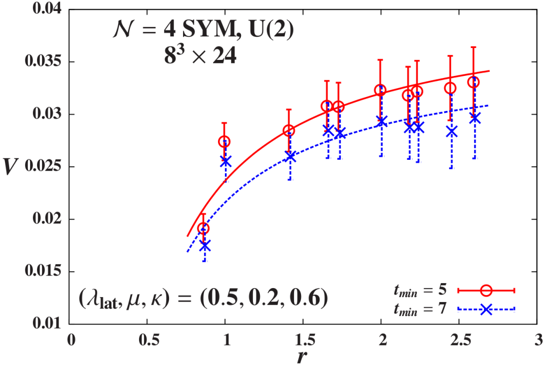
Physics result: Static potential is Coulombic at all λ

Static potential $V(r)$ from $r \times T$ Wilson loops: $W(r, T) \propto e^{-V(r)\,T}$

Fit $V(r)$ to Coulombic or confining form

$$V(r) = A - C/r$$

$$V(r) = A - C/r + \sigma r$$



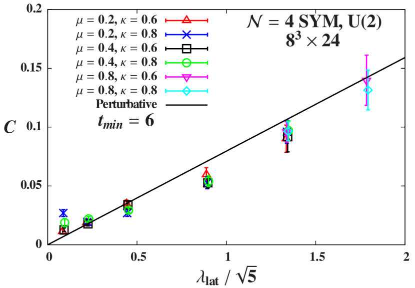
Fits to confining form always produce vanishing string tension $\sigma = 0$

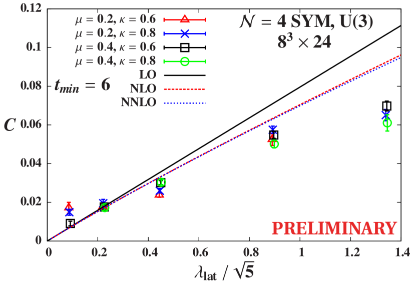
Working on standard methods to reduce noise

Coupling dependence of $V(r) = A - C/r$

—Perturbation theory predicts $C(\lambda) = \lambda/(4\pi) + \mathcal{O}(\lambda^2)$

—AdS/CFT predicts $C(\lambda) \propto \sqrt{\lambda}$ for $N \to \infty$, $\lambda \to \infty$, $\lambda \ll N$





We see agreement with perturbation theory for $N = 2$, $\lambda \lesssim 2$,
and a tantalizing discrepancy for $N = 3$, $\lambda \gtrsim 1$

No dependence on μ or $\kappa \longrightarrow$ apparently insensitive to soft $\mathcal{Q}$ breaking

Recapitulation

- Strongly coupled supersymmetric field theories very interesting to study through lattice calculations
- Practical numerical calculations possible for lattice $\mathcal{N} = 4$ SYM based on exact preservation of twisted susy subalgebra $\mathcal{Q}^2 = 0$
- The construction is complicated $\longrightarrow$ publicly-available code to reduce barriers to entry
- The static potential is always Coulombic
  For $N = 2$ $C(\lambda)$ is consistent with perturbation theory
  For $N = 3$ an intriguing discrepancy at stronger couplings
- There are many more directions to pursue in the future
  - Measuring anomalous dimension of Konishi operator
  - Understanding the (absence of a) sign problem

Thank you!

Thank you!

Collaborators

Simon Catterall, Poul Damgaard, Tom DeGrand and Joel Giedt

Funding and computing resources

Supplement: Konishi operator scaling dimension

Recall $\mathcal{N} = 4$ SYM is conformal

$\Longrightarrow$ All correlation functions decay algebraically $\propto r^{-\Delta}$

The Konishi operator is the simplest conformal primary operator

$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right] \qquad C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) = Ar^{-2\Delta_K}$$

There are many predictions for the scaling dim. $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From perturbation theory for small λ,
  related to $\lambda \to \infty$ by S duality under $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$
- From holography for $N \to \infty$ and $\lambda \to \infty$ but $\lambda \ll N$
- Bounds on max $\{\Delta_K\}$ from the conformal bootstrap program

We will add lattice gauge theory to this list

Konishi operator on the lattice

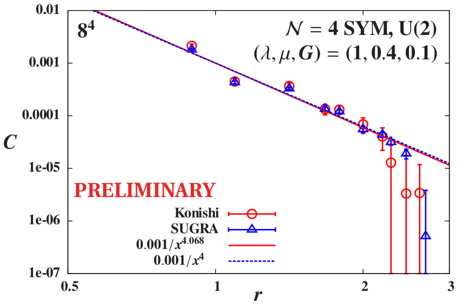
$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right] \longrightarrow \widehat{\mathcal{O}}_K = \sum_{a,\ b} \mathrm{Tr}\left[\widehat{\varphi}^{a}\widehat{\varphi}^{b}\right]$$

$$\text{with } \widehat{\varphi}^{a} = \mathcal{U}_a\overline{\mathcal{U}}_a - \frac{1}{N}\mathrm{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right]\mathbb{I}$$

$$C(r) = \widehat{\mathcal{O}}_K(x+r)\widehat{\mathcal{O}}_K(x) \propto r^{-2\Delta_K}$$

Consistent with
power laws using perturbative Δ

Need $\mathcal{Q}$-invariant plaquette det.
for reasonable $C(r)$ on 8^4 lattice



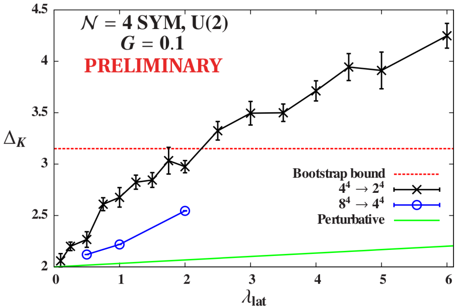
Obviously not a stable way to determine Δ_K — we have other tools

Preliminary Konishi Δ_K from Monte Carlo RG

Scaling dimension is eigenvalue of MCRG "stability matrix"

Simple trial (only statistical errors) correctly finds $\Delta_K \to 2$ as $\lambda \to 0$

Significant volume dependence $\longrightarrow$ approach perturbation theory as L increases



Need to check systematics:
different numbers of blocking steps, different operators, different G

Need to produce consistent results from independent approach(es) such as finite-size scaling

Supplement: The (absence of a) sign problem

In lattice gauge theory we compute operator expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O}\, e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\, \text{pf}\, \mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

$\text{pf}\, \mathcal{D} = |\text{pf}\, \mathcal{D}| e^{i\alpha}$ is generically complex for lattice $\mathcal{N} = 4$ SYM
$\longrightarrow$ Complicates interpretation of $\left[e^{-S_B}\, \text{pf}\, \mathcal{D}\right]$ as Boltzmann weight

Have to **reweight** "phase-quenched" (pq) calculations

$$\langle \mathcal{O} \rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O}\, e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\, |\text{pf}\, \mathcal{D}| \qquad \langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq}}{\left\langle e^{i\alpha} \right\rangle_{pq}}$$

Sign problem: This breaks down if $\left\langle e^{i\alpha} \right\rangle_{pq}$ is consistent with zero

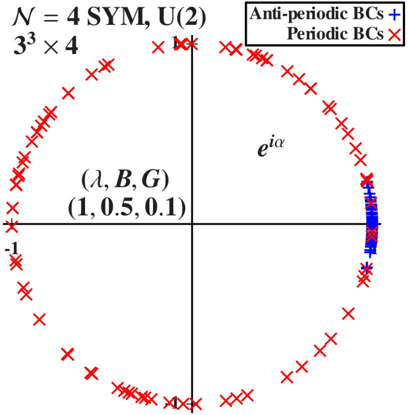
Illustration of sign problem and its absence

- With periodic temporal fermion boundary conditions
  we have an obvious sign problem, $\left\langle e^{i\alpha} \right\rangle_{pq}$ consistent with zero
- With anti-periodic BCs and all else the same $\left\langle e^{i\alpha} \right\rangle_{pq} \approx 1$
  $\longrightarrow$ phase reweighting not even necessary

Even stranger

Other $\langle \mathcal{O} \rangle_{pq}$ nearly identical despite sign problem...

Can this be understood?



Pfaffian phase dependence on volume and N

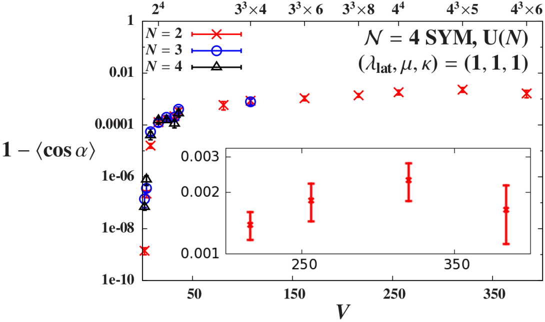
No indication of a sign problem with anti-periodic BCs

- Pfaffian $P = |P|e^{i\alpha}$ is nearly real and positive, $1 - \langle\cos(\alpha)\rangle \ll 1$
- Fluctuations in pfaffian phase don't grow with the lattice volume
- Insensitive to number of colors $N = 2, 3, 4$

Hard calculations

Each $4^3 \times 6$ measurement requires $\sim$8 days, $\sim$10GB memory

Parallel $\mathcal{O}(n^3)$ algorithm



Backup: Failure of Leibnitz rule in discrete space-time

Given that $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ is problematic,

why not try $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu$ for a discrete translation?

Here $\nabla_\mu\phi(x) = \frac{1}{a}\left[\phi(x + a\widehat{\mu}) - \phi(x)\right] = \partial_\mu\phi(x) + \frac{a}{2}\partial_\mu^2\phi(x) + \mathcal{O}(a^2)$

Essential difference between ∂_μ and ∇_μ on the lattice ($a > 0$)

$$\begin{aligned}\nabla_\mu\left[\phi(x)\chi(x)\right] &= a^{-1}\left[\phi(x + a\widehat{\mu})\chi(x + a\widehat{\mu}) - \phi(x)\chi(x)\right] \\ &= \left[\nabla_\mu\phi(x)\right]\chi(x) + \phi(x)\nabla_\mu\chi(x) + a\left[\nabla_\mu\phi(x)\right]\nabla_\mu\chi(x)\end{aligned}$$

We only recover the Leibnitz rule $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \to 0$

$\Longrightarrow$ "Discrete supersymmetry" breaks down on the lattice

(Dondi & Nicolai, "Lattice Supersymmetry", 1977)

Backup: Twisting $\longleftrightarrow$ Kähler–Dirac fermions

The Kähler–Dirac representation is related to the usual $Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{I}}_{\dot\alpha}$ by

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \gamma_\mu \mathcal{Q}_\mu + \gamma_\mu\gamma_\nu \mathcal{Q}_{\mu\nu} + \gamma_\mu\gamma_5 \mathcal{Q}_{\mu\nu\rho} + \gamma_5 \mathcal{Q}_{\mu\nu\rho\sigma}$$

$$\longrightarrow \mathcal{Q} + \gamma_a \mathcal{Q}_a + \gamma_a\gamma_b \mathcal{Q}_{ab}$$

with $a, b = 1, \cdots, 5$

The 4×4 matrix involves R symmetry transformations along each row and (euclidean) Lorentz transformations along each column

$\Longrightarrow$ Kähler–Dirac components transform under "twisted rotation group"

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \mathrm{SO}(4)_R\right]$$

$\uparrow$ only $\mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$

Backup: Details of $\mathcal{Q}^2 = 0$ on the lattice

Goal: Preserve $\mathcal{Q}$ supersymmetry on the lattice

1. $\mathcal{Q}^2 \cdot = 0$
2. $\mathcal{Q}$ directly interchanges bosonic $\longleftrightarrow$ fermionic d.o.f.

Both conditions are easy to verify in five-component notation:

$$\mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$
$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$
$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

- Gauge field $\mathcal{U}_a$ and ψ_a live on links between lattice sites
  $\mathcal{U}_a$ must be elements of algebra $\mathfrak{gl}(N, \mathbb{C}) \ni \psi_a$
  $\Longrightarrow$ Non-trivial to ensure $\mathcal{U}_a \longrightarrow \mathbb{I} + \mathcal{A}_a$ in the continuum limit
- Field strength $\overline{\mathcal{F}}_{ab}$ and χ_{ab} live on diagonals of oriented faces
- Bosonic auxiliary field d and η live on sites
  Usual equation of motion: $d = \overline{\mathcal{D}}_a\,\mathcal{U}_a$

Backup: A_4^* lattice with five links in four dimensions

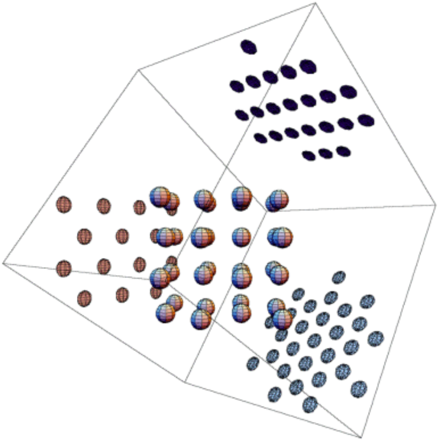
$A_a = (A_\mu, \phi)$ may remind you of dimensional reduction

On the lattice we need to treat all five $\mathcal{U}_a$ symmetrically

—Start with hypercubic lattice
in 5d momentum space

—**Symmetric** constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

—Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space

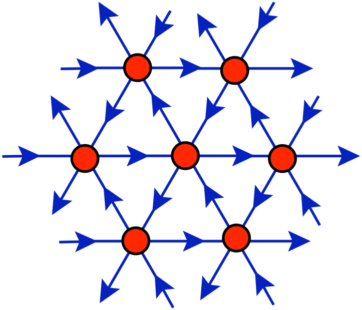


Backup: Twisted SO(4) symmetry on the A_4^* lattice

—Can picture A_4^* lattice
as 4d analog of 2d triangular lattice

—Five basis vectors are non-orthogonal
and linearly dependent

—Preserves S_5 point group symmetry



S_5 irreps precisely match onto irreps of twisted $\text{SO}(4)_{tw}$

$$\begin{aligned} \mathbf{5} = \mathbf{4} \oplus \mathbf{1}: \quad & \mathcal{U}_a \longrightarrow \mathcal{A}_\mu, \quad \phi \\ & \psi_a \longrightarrow \psi_\mu, \quad \eta_{\mu\nu\rho\sigma} \\ \mathbf{10} = \mathbf{6} \oplus \mathbf{4}: \quad & \chi_{ab} \longrightarrow \chi_{\mu\nu}, \quad \psi_{\mu\nu\rho} \end{aligned}$$

Backup: Analytic results for exact lattice susy

$$S = \frac{N}{2\lambda_{\text{lat}}}\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \eta\overline{\mathcal{D}}_a\mathcal{U}_a - \frac{1}{2}\eta d\right) - \frac{N}{8\lambda_{\text{lat}}}\epsilon_{abcde}\ \chi_{ab}\overline{\mathcal{D}}_c\chi_{de}$$

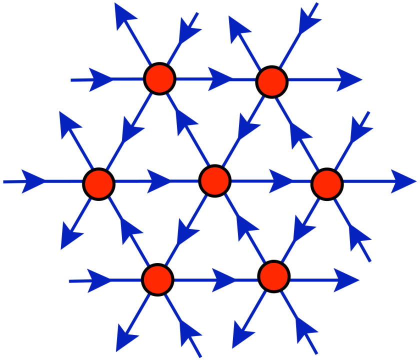
Gauge invariant, $\mathcal{Q}$ supersymmetric, S_5 symmetric

The high degree of symmetry has important consequences

- Moduli space preserved to all orders of lattice perturbation theory $\longrightarrow$ no scalar potential induced by radiative corrections
- β function vanishes at one loop (at least)
- Real-space RG blocking transformations preserve $\mathcal{Q}$ & S_5
- Only one marginal tuning to recover $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum

Backup: Hypercubic basis for A_4^* lattice

It is very convenient to represent the A_4^* lattice
as a hypercube with a backwards diagonal



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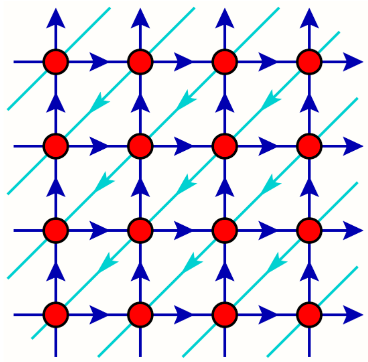
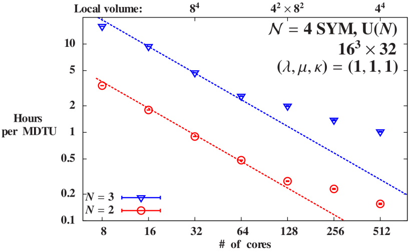
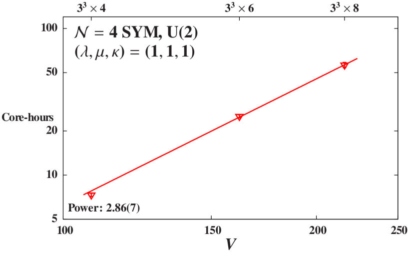


Backup: Code performance—weak and strong scaling

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(N_\Psi^3)$ pfaffian calculation (fixed local volume)
$N_\Psi \equiv 16 N^2 L^3 N_T$ is number of fermion degrees of freedom



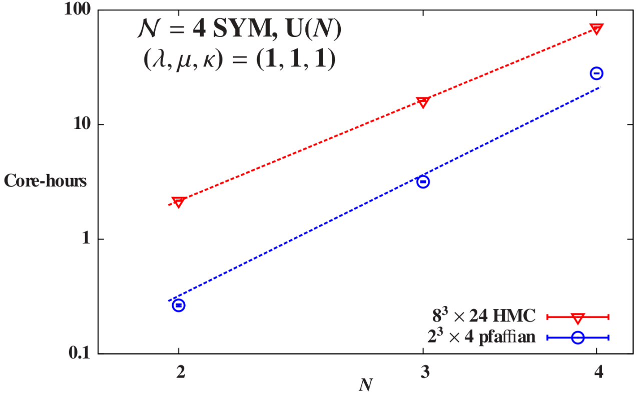


Both plots on log–log axes with power-law fits

Backup: Code performance for 2, 3 and 4 colors

Red: Find RHMC costs scaling $\sim N^5$ (recall adjoint fermion d.o.f. $\propto N^2$)

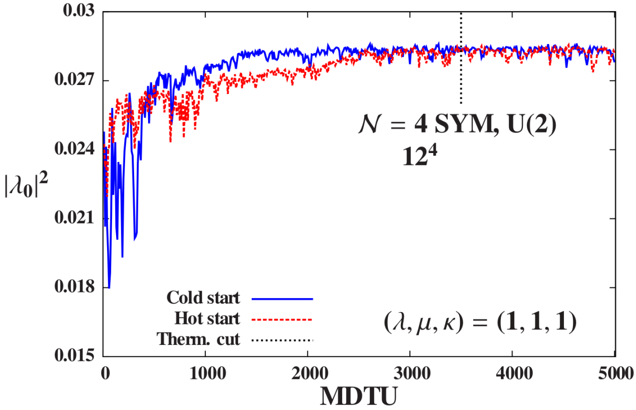
Blue: Pfaffian costs consistent with expected N^6 scaling



Backup: Thermalization

Thermalization becomes increasingly painful as N or $L^3 \times N_T$ increase

Example: Evolution of smallest $\mathcal{D}^{\dagger}\mathcal{D}$ eigenvalue $|\lambda_0|^2$



Should be possible to address this with better initial configuration

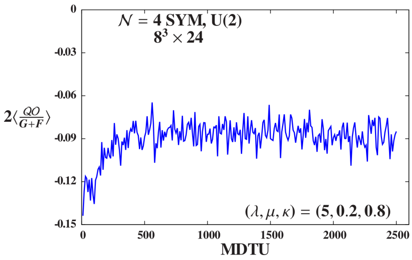
Backup: The problem with flat directions

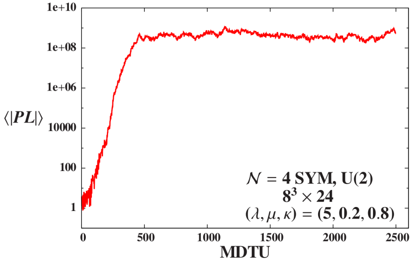
Gauge fields $\mathcal{U}_a$ can move far away from continuum form $\mathbb{I} + \mathcal{A}_a$
if $N\mu^2/(2\lambda_{\text{lat}})$ becomes too small

Example for two-color $(\lambda_{\text{lat}}, \mu, \kappa) = (5, 0.2, 0.8)$ on $8^3 \times 24$ volume

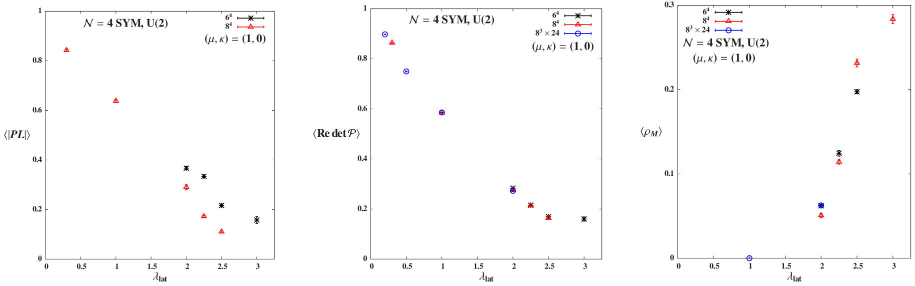
Left: Ward identity violations are stable at $\sim$9%

Right: Polyakov loop wanders off to $\sim 10^9$





Backup: Lattice phase due to U(1) sector



1. Polyakov loop collapses $\Longrightarrow$ confining phase
   (**not** present in continuum $\mathcal{N} = 4$ SYM)

2. Plaquette determinant is variable in U(1) sector
   Drops at same coupling λ as Polyakov loop

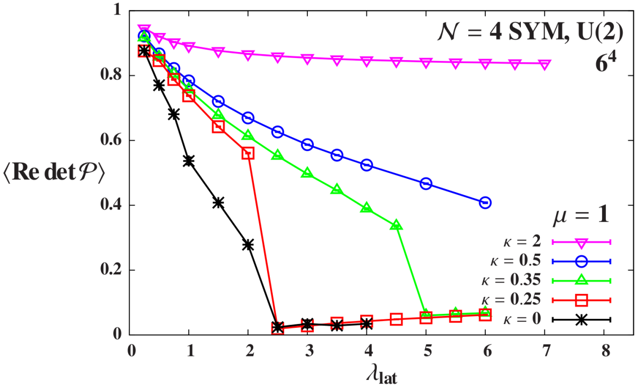
3. ρ_M is density of U(1) monopole world lines (DeGrand & Toussaint)
   Non-zero when Polyakov loop and plaquette det. collapse

Backup: Suppressing the U(1) sector

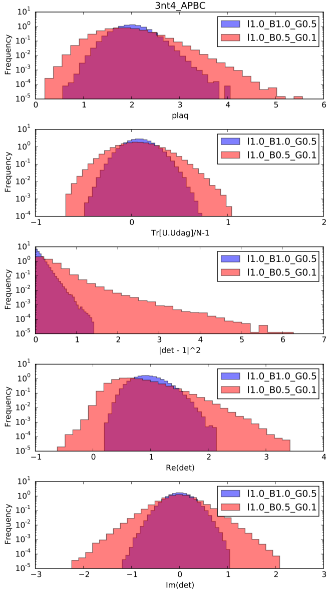
$\Delta S = \kappa \left| \det \mathcal{P} - 1 \right|^2$ suppresses the lattice strong-coupling phase

Produces $2\kappa F_{\mu\nu} F^{\mu\nu}$ term in U(1) sector

$\Longrightarrow$ QED critical $\beta_c = 0.99 \longrightarrow$ critical $\kappa_c \approx 0.5$



Backup: Plaquette and determinant distributions



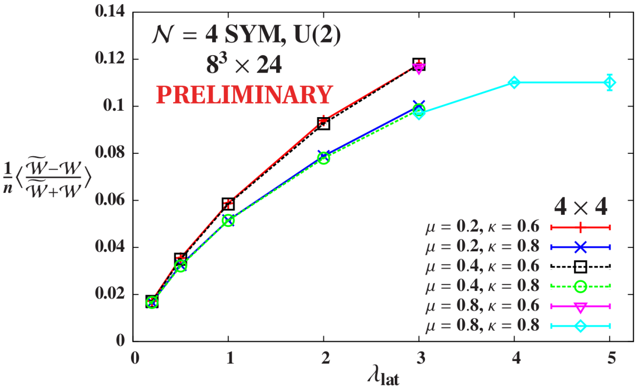
Larger couplings *B* and *G* produce the desired sharper peaks

Price: Larger Ward identity violations and larger computational costs

Backup: Restoration of $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries

Restoration of the other 15 $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ in the continuum limit follows from restoration of R symmetry (motivation for A_4^* lattice)

Modified Wilson loops test R symmetries at non-zero lattice spacing

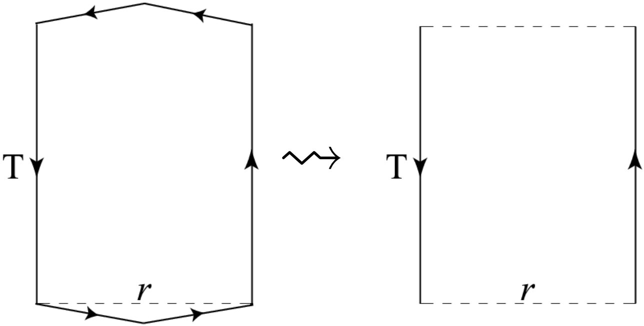


Backup: $\mathcal{N} = 4$ static potential from Wilson loops

Extract static potential $V(r)$

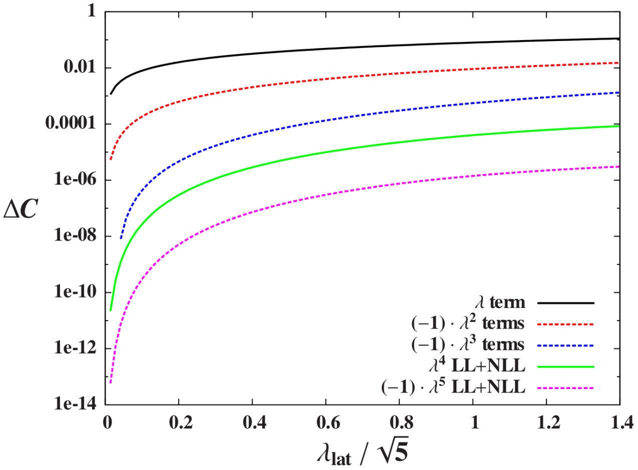
from $r \times T$ Wilson loops: $W(r, T) \propto e^{-V(r)\,T}$

Coulomb gauge trick from lattice QCD reduces A_4^* lattice complications



Backup: Perturbation theory for Coulomb coefficient

For range of λ_{lat} currently being studied, the perturbative series for the U(3) Coulomb coefficient appears well convergent

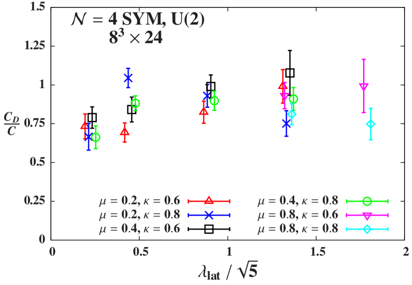


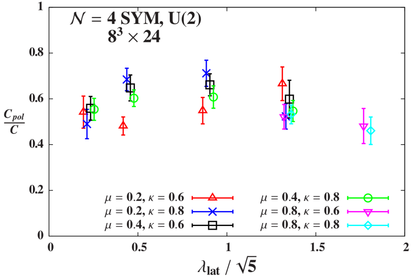
Backup: More tests of the U(2) static potential

Left: Projecting Wilson loops from U(2) $\longrightarrow$ SU(2)

$\Longrightarrow$ factor of $\frac{N^2-1}{N^2} = 3/4$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$





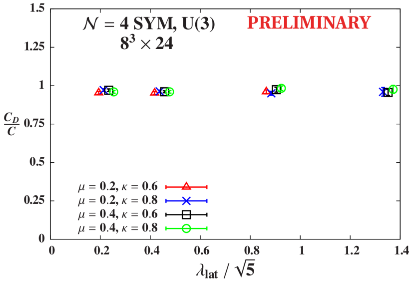
Both expected factors present, although (again) noisily

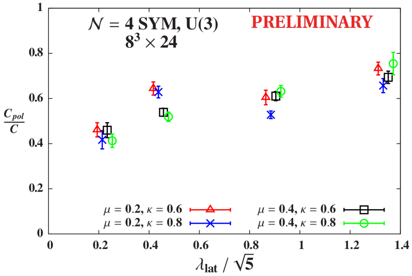
Backup: More tests of the U(3) static potential

Left: Projecting Wilson loops from U(3) $\longrightarrow$ SU(3)

$\Longrightarrow$ factor of $\frac{N^2-1}{N^2} = 8/9$

Right: Unitarizing links removes scalars $\Longrightarrow$ factor of $1/2$



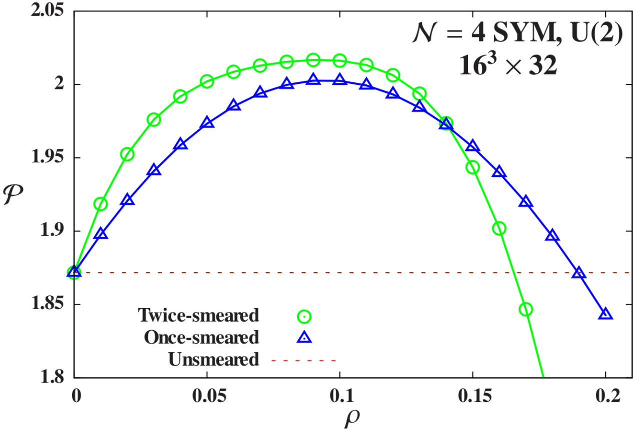


Ratios look slightly higher than expected,

less noise in SU(3)-projected results

Backup: Smearing for noise reduction

Smearing may reduce noise in static potential (etc.) measurements
—Stout smearing implemented and tested
—APE or HYP (without unitary projection) may work better for Konishi



Backup: Konishi operator on the lattice

$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right]$$

On the lattice the scalars Φ^{I} are twisted
and wrapped up in the complexified gauge field $\mathcal{U}_a$

Given $\mathcal{U}_a \approx \mathbb{I} + \mathcal{A}_a$ the most obvious way to extract the scalars is

$$\widehat{\varphi}^a = \mathcal{U}_a\overline{\mathcal{U}}_a - \frac{1}{N}\mathrm{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right]\mathbb{I}$$

This is still twisted, so all $\{a, b\}$ contribute to R-singlet Konishi

$$\widehat{\mathcal{O}}_K = \sum_{a,\ b} \mathrm{Tr}\left[\widehat{\varphi}^a\widehat{\varphi}^b\right]$$

Backup: Scaling dimensions from Monte Carlo RG

Couplings flow under RG blocking transformation R_b

n-times-blocked system is $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Consider linear expansion around fixed point $H^\star$ with couplings $c_i^\star$

$$c_i^{(n)} - c_i^\star = \sum_j \left. \frac{\partial c_i^{(n)}}{\partial c_j^{(n-1)}} \right|_{H^\star} \left(c_j^{(n-1)} - c_j^\star \right) \equiv \sum_j T_{ij}^\star \left(c_j^{(n-1)} - c_j^\star \right)$$

$T_{ij}^\star$ is the "stability matrix"

Eigenvalues of $T_{ij}^\star$ are scaling dimensions of corresponding operators

Backup: Pfaffian phase dependence on λ_{lat}, μ, κ

We observe little dependence on κ

Fluctuations in phase grow as λ_{lat} increases