Renormalization-Group Flows in Gauge Theories

Robert Shrock

YITP, Stony Brook University

SCGT14Mini Workshop, KMI, Nagoya University, 3/6/2014

Outline

- Beta functions, renormalization-group flows in gauge theories
- Higher-loop calculations of UV to IR evolution in asymptotically free gauge theories
- Structural properties of β and results in the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed
- Study of scheme dependence in higher-loop calculations
- UV to IR evolution in asymptotically free chiral gauge theories
- Study of RG flows in U(1) and non-Abelian gauge theories with many fermions (non-asymptotically free)
- Conclusions

This talk contains material from the following papers since SCGT12 conference:

- R. Shrock, "Higher-Loop Structural Properties of the β Function in Asymptotically Free Vectorial Gauge Theories", Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209].

- R. Shrock, "Higher-Loop Calculations of the Ultraviolet to Infrared Evolution of a Vectorial Gauge Theory in the Limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c Fixed", Phys. Rev. D 87, 116007 (2013) [arXiv:1302.5434].

- R. Shrock, "Study of Scheme Transformations to Remove Higher-Loop Terms in the β Function of a Gauge Theory", Phys. Rev. D 88, 036003 (2013) [arXiv:1305.6524].

- T. Appelquist and R. Shrock, "On the Ultraviolet to Infrared Evolution of Chiral Gauge Theories" Phys. Rev. D 88, 105012 (2013) [arXiv:1310.6076].

- R. Shrock, "Study of Possible Ultraviolet Zero of the Beta Function in Gauge Theories with Many Fermions", Phys. Rev. D 89, 045019 (2014) [arXiv:1311.5268].

together with some earlier work with T. Ryttov.

Beta functions, RG flows in gauge theories

Consider a gauge theory with gauge group G and a set of massless fermions in some representation(s) R. If $\beta < 0$ so this theory is asymptotically free (AF), then it is weakly coupled, properties are perturbatively calculable for large Euclidean momenta μ in the deep ultraviolet (UV). If $\beta > 0$, theory is infrared (IR)-free.

The renormalization-group (RG) flows in these theories as functions of μ are of fundamental field-theoretic interest. For an AF theory, study how it flows from large μ in the UV to small μ in the IR. For some fermion contents, theory may have an exact or approximate IR fixed point (zero of β).

Denote running gauge coupling at scale μ as $g = g(\mu)$, and let $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) = g(\mu)^2/(16\pi^2) = \alpha(\mu)/(4\pi)$.

The dependence of $\alpha(\mu)$ on μ is described by the renormalization group β function. For the AF case,

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ ,$$

where $dt = d \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coefficients b_1 and b_2 in β are independent of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$.

Consider vectorial AF case. As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases, eventually beyond the perturbatively calculable region. This is the case for QCD.
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha \to \alpha_{IR}$. In this class of theories, there are two further generic possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the RG; as $\mu \to 0$ and $\alpha \to \alpha_{IR}$, $\beta \to \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions.

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$.

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β fn. becomes that of a pure gauge theory, with no IR zero. Hence, if β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, since $\beta = d\alpha/dt \to 0$, $\alpha(\mu)$ varies very slowly as a function of the scale μ, i.e., there is approximately scale-invariant "walking" behavior (Yamawaki et al; Holdom, Appelquist, Wijewardhana...).

SχSB at Λ breaks the approximate dilatation symmetry and can lead to a resultant approx. Nambu-Goldstone boson (NGB), the dilaton.

Consider AF chiral gauge theory (χGT). If this has IRFP at sufficiently weak coupling, then it evolves from UV to a non-Abelian Coulomb phase in the IR. If the IRFP occurs at stronger coupling, then there are several possibilities for the UV to IR evolution.

If theory satisfies 't Hooft anomaly-matching conditions, then possible confinement without SχSB, producing massless fermions; alternatively, fermion condensates may form that break the chiral gauge symmetry and global flavor symmetries.

For a U(1) theory, or a non-Abelian theory with many fermions, $N_f >> 1$, $\beta > 0$; calculate flow from weak coupling in the IR toward the UV. Study possibility of UVFP and relate higher-loop perturbative calculations to exact results in $N_f \to \infty$ limit.

First, discuss flows for AF gauge theories. AF requires $N_f < N_{f,b1z} = 11C_A/(4T_f)$, where $C_A \equiv C_2(G)$ is quadratic Casimir invariant, $T_f \equiv T(R)$ is trace invariant. Focus here on $G = \mathrm{SU}(N_c)$; then, e.g., for $R =$ fundamental rep. (fund. rep.), $N_f < (11/2)N_c$.

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$.

At the $n = 2$ loop level, if $b_2 < 0$, then two-loop beta function, $\beta_{2\ell}$, has IR zero at $\alpha_{IR,2\ell} = -4\pi b_1/b_2$. Interval I where this occurs is

$$I: \quad N_{f,b2z} < N_f < N_{f,b1z}$$

where $N_{f,b2z} = 34C_A^2/[4T_f(5C_A + 3C_f)]$. For SU(2), I: $\quad 5.55 < N_f < 11$; for SU(3), I: $\quad 8.05 < N_f < 16.5$;

as $N_c \to \infty$, I: $\quad (34/13)N_c < N_f < (11/2)N_c$, i.e., $2.62N_c < N_f < 5.5N_c$.

(expressions evaluated for $N_f \in \mathbb{R}$, but it is understood that physical values of N_f are nonnegative integers.)

As N_f decreases from the upper to lower end of interval I, α_{IR} increases. Denote $N_{f,cr}$ as value of N_f where IR behavior changes from non-Abelian Coulomb phase with no Sχ SB to phase with SχSB.

Higher-Loop Corrections to UV $\rightarrow$ IR Evolution of Asymptotically Free Gauge Theories

Just as higher-loop calculations have been important in calculating properties of QCD, near to the UVFP, so also they are important for understanding properties of an IRFP in asymptotically free gauge theories, especially for the case where α_{IR} occurs at moderately strong coupling (Gardi, Grunberg, Karliner..).

With T. Ryttov, we carried out these calculations of the IR zero in β and the resultant value of the anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear, evaluated at this point, for arbitrary G and fermions in general representations R, with numerical results for $G = \mathrm{SU}(N)$ and fermion reps. including fundamental, adjoint, and rank-2 symmetric and antisymmetric tensor, in T. A. Ryttov and R. Shrock, Phys. Rev. D 83, 056011 (2011) [arXiv:1011.4542]. Related work in Pica and Sannino, Phys. Rev. D 83, 035013 (2011) [arXiv:1011.5917]; results agree.

Our calculations were done up to the highest loop order (4 loops) for which the β and γ function coefficients are known (from $\overline{MS}$ calculations by Vermaseren, Larin, and van Ritbergen).

Using the fact that $b_3 < 0$ for $N_f \in I$, we showed that the location of the zero in β (IRFP) decreases when one goes from 2-loop to 3-loop order.

We have extended this to an arbitrary scheme in (RS, Phys. Rev. D 87, 105005 (2013) [arxiv:1301.3209]).

At the 3-loop level, the expression for $\alpha_{IR,3\ell}$ is a solution of the quadratic equation $b_1 + b_2 a + b_3 a^2 = 0$ which thus involves a square root, $\sqrt{b_2^2 - 4b_1 b_3}$. If a scheme had $b_3 > 0$ in I, then, since $b_2 \longrightarrow 0$ at lower end of I, $b_2^2 - 4b_1 b_3$ would go negative, so this scheme would yield a complex, unphysical $\alpha_{IR,3\ell}$ in this region.

Since the existence of the IR zero in β at 2-loop level is scheme-independent, one may require that a scheme should maintain this property to higher-loop order, and hence that $b_3 < 0$ for $N_f \in I$. Given that $b_3 < 0$ for $N_f \in I$, we then proved that $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ holds in all such schemes, not just in $\overline{\rm MS}$.

At 4-loop level, $\alpha_{IR,4\ell}$ is determined as the physical root of the cubic equation $b_1 + b_2 a + b_3 a^2 + b_4 a^3 = 0$. We found that going from 3-loop to 4-loop level, there is only a small change, so $\alpha_{IR,4\ell} < \alpha_{IR,2\ell}$.

Our result of smaller fractional change in value of IR zero of β at higher-loop order agrees with expectation that calc. to higher loop order should give more stable result.

General result on the shift of an IR zero of β when calculated at next higher order (RS, PR D 87, 105005 (2013) [arxiv:1301.3209]). Assume fermion content is such that $b_2 < 0$, so theory has a 2-loop IR zero .

Consider a scheme in which the b_ℓ with $\ell = 3, ..., n+1$ have values that preserve the existence of the scheme-independent 2-loop IR zero of β at higher-loop level.

Use fact that theory is asymptotically free, so $\beta < 0$ for $0 < \alpha < \alpha_{IR}$, and hence $d\beta_{n\ell}/d\alpha > 0$ for $\alpha \simeq \alpha_{IR,n\ell}$.

Expand $\beta_{n\ell}$ in Taylor series around $\alpha = \alpha_{IR,n\ell}$:

$$\beta_{n\ell} = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big)$$

Now calculate β to the next-higher-loop order, i.e., $\beta_{(n+1)\ell}$, and solve for $\alpha_{IR,(n+1)\ell}$. To determine whether $\alpha_{IR,(n+1)\ell}$ is larger or smaller than $\alpha_{IR,n\ell}$, consider

$$\beta_{(n+1)\ell} - \beta_{n\ell} = -2\bar{b}_{n+1}\alpha^{n+2}$$

In a scheme where $b_{n+1} > 0$, this difference, evaluated at $\alpha = \alpha_{IR,n\ell}$, is negative, so, given that $d\beta_{n\ell}/d\alpha|_{\alpha_{IR,n\ell}} > 0$, to compensate for this, the zero shifts to the right, whereas if $b_{n+1} < 0$, the difference is positive, so the zero shifts to the left.

$$\text{If } b_{n+1} > 0 \ , \quad \text{then } \alpha_{IR,(n+1)\ell} > \alpha_{IR,n\ell}$$

$$\text{If } b_{n+1} < 0 \ , \quad \text{then } \alpha_{IR,(n+1)\ell} < \alpha_{IR,n\ell}$$

This general result is evident in our $\overline{\rm MS}$ calculations.

$$b_3 < 0, \quad \Longrightarrow \quad \alpha_{IR,3\ell} < \alpha_{IR,2\ell}$$

$$b_4 > 0, \quad \Longrightarrow \quad \alpha_{IR,4\ell} > \alpha_{IR,3\ell}$$

We have calculated the anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear at the IRFP: $\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell})$. Recall results for illustrative cases $N_c = 2, \ 3$, fermions in fund. rep.

N_c	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

As with $\alpha_{IR,n\ell}$, the fact that for a given N_c, N_f, the 3-loop and 4-loop results are closer to each other than the 2-loop and 3-loop results shows the value of including higher-loop terms to increase the accuracy of the calculation.

Thus, our higher-loop calcs. of α_{IR} and γ allow an analysis of the theory to smaller values of N_f and thus stronger couplings. Obviously, perturbative calculations are not applicable when α is too large. It is still possible that $\gamma_m \sim 1$ for $N_f \simeq N_{f,cr}$.

We have also performed these higher-loop calculations for larger fermion reps. R. In general, we find that, for a given N_c, R, and N_f, the values of $\gamma_{IR,n\ell}$ calculated to 3-loop and 4-loop order are smaller than the 2-loop value.

Extensive lattice gauge theory simulations have been performed to study IR properties of AF gauge theories with various fermion contents. It has been interesting to compare our perturbative calculations of $\gamma_{IR,n\ell}$ with lattice measurements.

For example, for SU(3), $N_f = 12$, we calculate $\gamma_{IR,2\ell} = 0.77$, $\gamma_{IR,3\ell} = 0.31$, $\gamma_{IR,4\ell} = 0.25$. Lattice results have been reported by LSD Collab. (Appelquist et al.); LHC Collab. (Kuti et al.); LatKMI Collab. (Y. Aoki et al.); Hasenfratz et al; Degrand; Itou...; values range from ~ 0.2 to ~ 0.5. Our 2-loop result is larger than lattice measurements, and our higher-loop calculations yield results closer to these measurements. This shows the value of these higher-loop calculations, since one then can achieve both analytic understanding, from continuum QFT, and a numerical understanding from the lattice, of γ_{IR}. More progress expected in future.

In addition to $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$, it is also of interest to investigate various structural properties of the n-loop beta function $\beta_{n\ell}$, including

- the derivative $\beta'_{IR,n\ell} \equiv \frac{d\beta_{n\ell}}{d\alpha}$ evaluated at $\alpha_{IR,n\ell}$.
- the magnitude and location of the minimum in $\beta_{n\ell}$

We have calculated these structural properties analytically and numerically in RS, Phys. Rev. D87, 105005 (2013) [arXiv:1301.3209].

In quasi-scale-invariant case where $\alpha_{IR} \gtrsim \alpha_{cr}$, dilaton mass relevant in dynamical EWSB models depends on how small β is for α near to α_{IR} and hence, at n-loop order, on $\beta'_{IR,n\ell}$, via the series expansion of $\beta_{n\ell}$ around $\alpha_{IR,n\ell}$,

$$\beta_{n\ell}(\alpha) = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big)$$

Derivative of 2-loop β function at $\alpha_{IR,2\ell}$:

$$\beta'_{IR,2\ell} = -\frac{2b_1^2}{b_2} = \frac{2b_1^2}{|b_2|} = \frac{2(11C_A - 4T_f N_f)^2}{3[4(5C_A + 3C_f)T_f N_f - 34C_A^2]}$$

At 3-loop level:

$$\beta'_{IR,3\ell} = \frac{1}{|b_3|^2}\Big[-4|b_2|(b_2^2 + b_1|b_3|) + (b_2^2 + 2b_1|b_3|)\sqrt{b_2^2 + 4b_1|b_3|} \ \Big]$$

We prove a general inequality: for a given gauge group G, fermion rep. R, and $N_f \in I$ (in a scheme with $b_3 < 0$, which thus preserves the existence of the 2-loop IR zero in β at 3-loop level),

$$\beta'_{IR,3\ell} < \beta'_{IR,2\ell}$$

We carry out a similar analysis of the derivative of the 4-loop β function evaluated at $\alpha_{IR,4\ell}$, denoted $\beta'_{IR,4\ell}$, and find a similar decrease from 3-loop to 4-loop order. Some numerical values:

N_c	N_f	$\beta'_{IR,2\ell}$	$\beta'_{IR,3\ell}$	$\beta'_{IR,4\ell}$
2	7	1.20	0.728	0.677
2	8	0.400	0.318	0.300
2	9	0.126	0.115	0.110
2	10	0.0245	0.0239	0.0235
3	10	1.52	0.872	0.853
3	11	0.720	0.517	0.498
3	12	0.360	0.2955	0.282
3	13	0.174	0.156	0.149
3	14	0.0737	0.0699	0.0678
3	15	0.0227	0.0223	0.0220
3	16	0.00221	0.00220	0.00220

Illustrative figures for SU(2) with $N_f = 8$ fermions and SU(3) with $N_f = 12$ fermions:

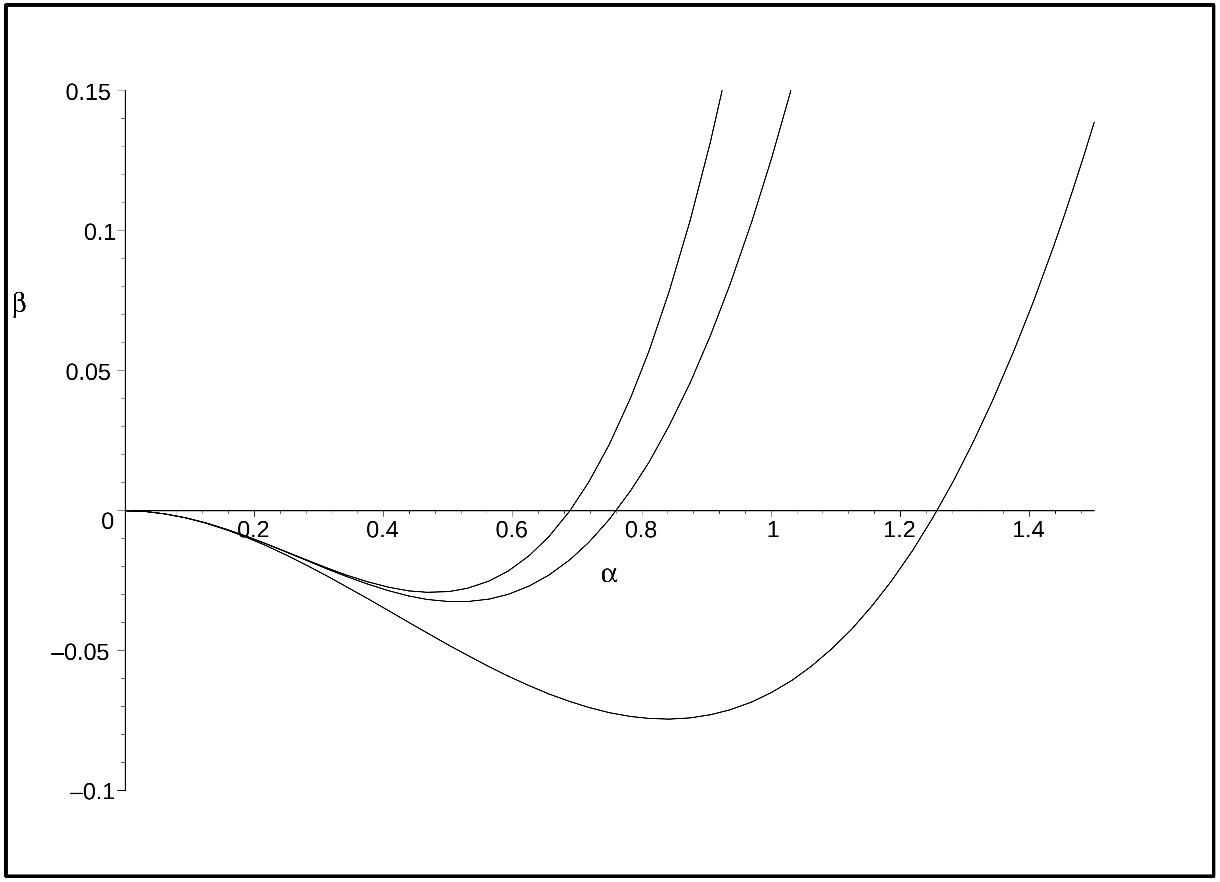

Figure 1: $\boldsymbol{\beta_{n\ell}}$ for SU(2), $\boldsymbol{N_f = 8}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

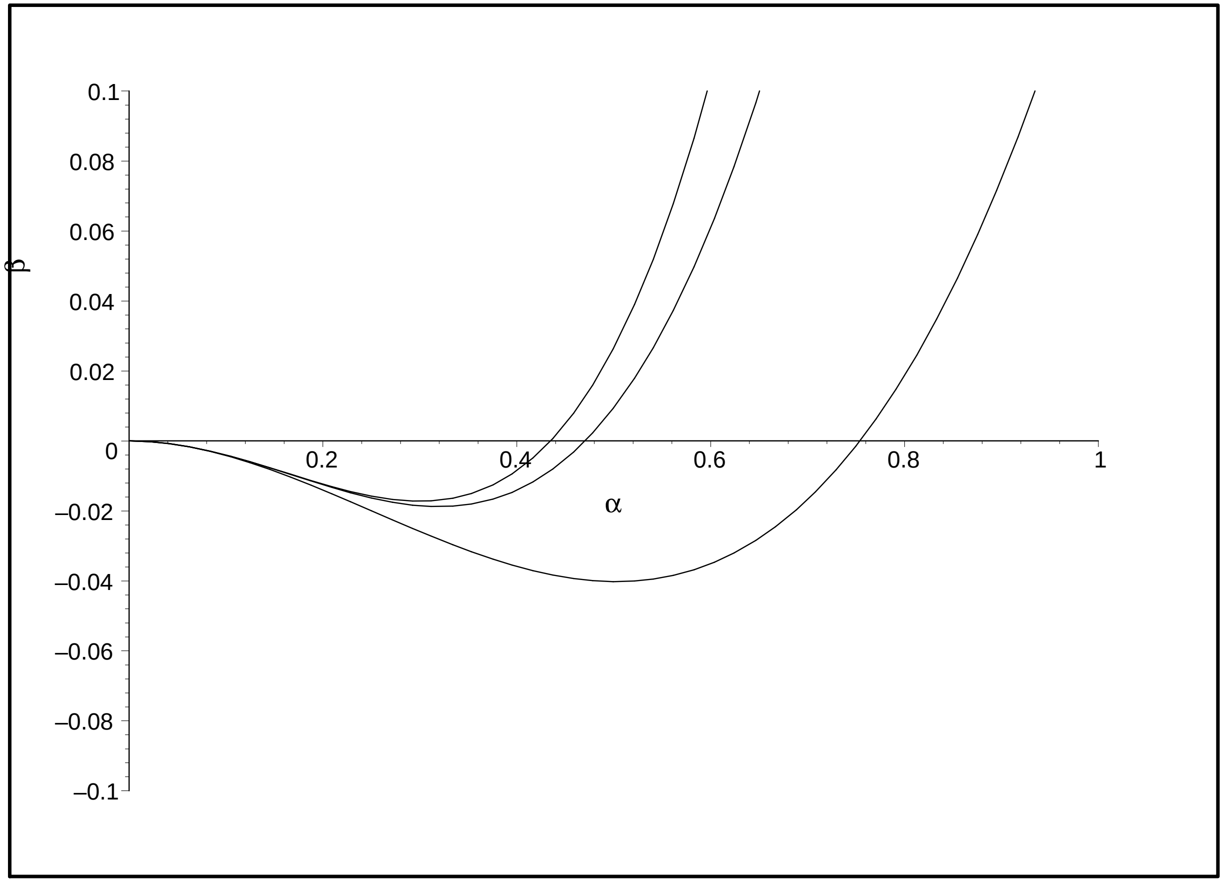

Figure 2: $\boldsymbol{\beta_{n\ell}}$ for SU(3), $\boldsymbol{N_f = 12}$, at $\boldsymbol{n = 2,\ 3,\ 4}$ loops. From bottom to top, curves are $\boldsymbol{\beta_{2\ell}}$, $\boldsymbol{\beta_{4\ell}}$, $\boldsymbol{\beta_{3\ell}}$.

Interesting property: for $R =$ fund. rep., $\alpha_{IR,n\ell}N_c$, $\gamma_{IR,n\ell}$, and other structural properties of $\beta_{n\ell}$ are similar in theories with different values of N_c and N_f if they have equal or similar values of $r = N_f/N_c$.

This motivates a study of the UV to IR evolution of an SU(N_c) gauge theory with N_f fermions in the fundamental rep. in the 't Hooft-Veneziano limit $N_c \to \infty$, $N_f \to \infty$ with $r \equiv N_f/N_c$ fixed and $\alpha(\mu)N_c \equiv \xi(\mu)$ independent of N_c. Denote this as the LNN (large N_c, large N_f) limit.

We have carried out this study in RS, Phys. Rev. D87, 116007 (2013) [arXiv:1302.5434]. Our results provide a unified quantitative understanding of the similarities in UV to IR evolution of SU(N_c) theories with different N_c and N_f but similar r.

With $\xi = \alpha N_c$ and $x = aN_c = \xi/(4\pi)$, define a rescaled beta function that is finite in the LNN limit:

$$\beta_\xi \equiv \frac{d\xi}{dt} = \lim_{LNN} \beta_\alpha N_c$$

with the expansion

$$\beta_\xi \equiv \frac{d\xi}{dt} = -8\pi x \sum_{\ell=1}^{\infty} \hat{b}_\ell x^\ell = -2\xi \sum_{\ell=1}^{\infty} \tilde{b}_\ell \xi^\ell \ ,$$

where

$$\hat{b}_\ell = \lim_{LNN} \frac{b_\ell}{N_c^\ell} \ , \quad \tilde{b}_\ell = \lim_{LNN} \frac{\bar{b}_\ell}{N_c^\ell} \quad \text{so} \quad \tilde{b}_\ell = \frac{\hat{b}_\ell}{(4\pi)^\ell}$$

1-loop and 2-loop coefficients in β_ξ:

$$\hat{b}_1 = \frac{1}{3}(11 - 2r)$$

and

$$\hat{b}_2 = \frac{1}{3}(34 - 13r)$$

Asymptotic freedom requires $r < 11/2$. The interval where $\beta_{\xi,2\ell}$ has an IR zero is $I_r: \ 34/13 < r < 11/2$, i.e., $2.615 < r < 5.500$.

2-loop IR zero of $\beta_{\xi,2\ell}$ is at

$$\xi_{IR,2\ell} = \frac{4\pi(11 - 2r)}{13r - 34}$$

3-loop and 4-loop coefficients in β_ξ (in $\overline{\mathrm{MS}}$ scheme):

$$\hat{b}_3 = \frac{1}{54}(2857 - 1709r + 112r^2) = 52.9074 - 31.6481r + 2.07407r^2$$

$$\hat{b}_4 = \frac{150473}{486} - \left(\frac{485513}{1944}\right)r + \left(\frac{8654}{243}\right)r^2 + \left(\frac{130}{243}\right)r^3 + \frac{4}{9}(11 - 5r + 21r^2)\zeta(3)$$
$$= 315.492 - 252.421\,r + 46.832\,r^2 + 0.534979\,r^3$$

(where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is Riemann zeta fn.). The 3-loop β function $\beta_{\xi,3\ell}$ has an IR zero at

$$\xi_{IR,3\ell} = \frac{12\pi[-3(13r - 34) + \sqrt{C_{3\ell}}\,]}{D_{3\ell}}\ ,$$

where

$$C_{3\ell} = -52450 + 41070r - 7779r^2 + 448r^3$$

$$D_{3\ell} = -2857 + 1709r - 112r^2$$

By same type of proof as given before, we show

$$\xi_{IR,3\ell} \le \xi_{IR,2\ell}$$

Further, since $\hat{b}_4$ reverses sign from neg. to pos. as r increases through $r = 3.119$,

$$\xi_{IR,4\ell} < \xi_{IR,3\ell} \quad \text{if} \quad 2.615 < r < 3.119, \ (\text{where } \hat{b}_4 < 0),$$

$$\xi_{IR,4\ell} > \xi_{IR,3\ell} \quad \text{if} \quad 3.119 < r < 5.500, \ (\text{where } \hat{b}_4 > 0)$$

Numerical values given in next table. The magnitude of the fractional difference

$$\frac{|\xi_{IR,4\ell} - \xi_{IR,3\ell}|}{\xi_{IR,4\ell}}$$

is reasonably small.

r	$\xi_{IR,2\ell}$	$\xi_{IR,3\ell}$	$\xi_{IR,4\ell}$
2.8	28.274	3.573	3.323
3.0	12.566	2.938	2.868
3.2	7.606	2.458	2.494
3.4	5.174	2.076	2.168
3.6	3.731	1.759	1.873
3.8	2.774	1.489	1.601
4.0	2.095	1.252	1.349
4.2	1.586	1.041	1.115
4.4	1.192	0.8490	0.9003
4.6	0.8767	0.6725	0.7038
4.8	0.6195	0.5083	0.5244
5.0	0.4054	0.3538	0.3603
5.2	0.2244	0.2074	0.2089
5.4	0.06943	0.06769	0.06775

Anomalous dimension $\gamma_m \equiv \gamma$:

$$\gamma = \sum_{\ell=1}^{\infty} \hat{c}_\ell \, x^\ell = \sum_{\ell=1}^{\infty} \tilde{c}_\ell \, \xi^\ell$$

where $\hat{c}_\ell = \lim_{LNN}(c_\ell/N_c^\ell)$ and $\tilde{c}_\ell = \hat{c}_\ell/(4\pi)^\ell$. The coefficients $\hat{c}_\ell$ are

$$\hat{c}_1 = 3 \,, \qquad \hat{c}_2 = \frac{203}{12} - \frac{5}{3}r = 16.917 - 1.667r$$

$$\hat{c}_3 = \frac{11413}{108} - \left(\frac{1177}{54} + 12\zeta(3)\right)r - \frac{35}{27}r^2 = 105.676 - 36.221r - 1.296r^2$$

$$\hat{c}_4 = \frac{460151}{576} - \frac{23816}{81}r + \frac{899}{162}r^2 - \frac{83}{81}r^3 + \left(\frac{1157}{9} - \frac{889}{3}r + 20r^2 + \frac{16}{9}r^3\right)\zeta(3)$$
$$+r\Big(66 - 12r\Big)\zeta(4) + \Big(-220 + 160r\Big)\zeta(5)$$

$$= 725.280 - 412.892r + 16.603r^2 + 1.1123r^3$$

Value of n-loop γ evaluated at n-loop $\xi_{IR,n\ell}$: $\gamma_{IR,n\ell} \equiv \gamma_{n\ell}\Big|_{\xi=\xi_{IR,n\ell}}$;

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$

and so forth for higher-loop order. Numerical values:

r	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
3.6	1.853	0.5201	0.3083
3.8	1.178	0.4197	0.3061
4.0	0.7847	0.3414	0.2877
4.2	0.5366	0.2771	0.2664
4.4	0.3707	0.2221	0.2173
4.6	0.2543	0.1735	0.1745
4.8	0.1696	0.1294	0.1313
5.0	0.1057	0.08886	0.08999
5.2	0.05620	0.05123	0.05156
5.4	0.01682	0.01637	0.01638

General inequality as before: $\gamma_{IR,3\ell} < \gamma_{IR,2\ell}$.

We have studied the approach to the LNN limit and find that this is quite rapid, with leading correction terms suppressed by $1/N_c^2$. For example,

$$\alpha_{IR,2\ell} N_c = \frac{4\pi(11-2r)}{13r-34} + \frac{12\pi r(11-2r)}{(34-13r)^2 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2} + \frac{(11-2r)(18836-5331r+648r^2-140r^3)}{(13r-34)^3 N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

(We have also done an analogous study for a supersymmetric gauge theory.)

These results provide an understanding of the approximate universality that is exhibited in calculations of these quantities for different (finite) values of N_c and N_f with similar or identical values of r.

Study of Scheme Dependence in Higher-Loop Calculations

Since coeffs. b_n with $n \geq 3$ in $\beta_{n\ell}$ are scheme-dependent, it is important to assess the effects of this scheme dependence in higher-loop calculations. We have given new results in RS, Phys. Rev D 88, 036003 (2013) [arXiv:1305.6524], extending the earlier studies in Ryttov and RS, PRD 86, 065032 (2012) [arXiv:1206.2366] and PRD 86, 085005 (2012) [arXiv:1206.6895].

A physically acceptable ST must satisfy several conditions, which are easily met in the vicinity of a UVFP of an AF theory at $\alpha = 0$ or the IRFP of an IR-free theory at $\alpha = 0$, but are significant constraints if the fixed point occurs at $\alpha \sim O(1)$.

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. An example is $a = (1/r)\sinh(ra')$, with inverse $a' = (1/r)\ln[ra + \sqrt{1+(ra)^2}\,]$. We find reasonably small scheme-dependence for moderate α_{IR}. Our studies give a quantitative assessment of the scheme dependence of an IR zero of β in such a theory.

Since the b_ℓ with $\ell \geq 3$ are scheme-dependent, one might expect that it would be possible, at least in the vicinity of $\alpha = \alpha' = 0$, to construct scheme transformations that would set $b'_\ell = 0$ for some range of $\ell \geq 3$, and, indeed a ST that would do this for all $\ell \geq 3$, so that $\beta_{\alpha'}$ would consist only of the 1-loop and 2-loop terms ('t Hooft scheme).

We have constructed scheme transformations denoted $S_{R,m}$ with $m \geq 2$ that remove the terms in the beta function at loop order $\ell = 3$ to $\ell = m + 1$, inclusive, and, in the limit, $m \to \infty$, a ST that can transform to the 't Hooft scheme in the vicinity of $\alpha = \alpha' = 0$.

We have quantitatively determined the range of applicability of these scheme transformations in an AF SU(N_c) gauge theory with N_f fermions.

UV to IR Evolution in Asymptotically Free Chiral Gauge Theories

With T. Appelquist, we have obtained new results on the UV to IR evolution of asymptotically free chiral gauge theories (χGTs) in T. Appelquist and RS, Phys. Rev. D 88, 105012 (2013) [arXiv:1310.6076]. (These theories are free of any triangle anomalies in gauged currents.) There are no fermion masses in the Lagrangians for these theories.

Some of these χGTs have IRFPs, and if these occur at sufficiently weak coupling, then the theories flow from the UV to a non-Abelian Coulomb phase in the IR, with no confinement or SχSB.

For theories with an IR zero in β at strong coupling or no IR zero in β, the flow from the UV leads to a strong coupling among the elementary fermions and gluons in the IR. Several possible types of behavior can occur, including

- confinement without any SχSB, leading to massless gauge-singlet composite fermions, if anomalies in global flavor symmetries are matched between the fermions in the Lagrangian and the composite fermions ('t Hooft anomaly-matching conditions)

- formation of bilinear fermion condensates, which break global flavor symmetries and can break the chiral gauge symmetries in sequential stages

N.B.: sequential self-breaking of a strongly coupled chiral gauge theory was used heavily in earlier constructions of moderately UV-complete ETC models, e.g., Appelquist and RS, Phys. Lett. B 548, 204 (2002); Phys. Rev. Lett. 90, 201801 (2003); Appelquist, Piai, and RS, Phys. Rev. D 69, 015002 (2004). The sequential nature of this symmetry breaking was the key to producing a hierarchy of different ETC scales and hence a generational hierarchy in the SM fermion masses.

If several possible bilinear fermion condensation channels are possible, *a priori*, a natural criterion for deciding which occurs is the most attractive channel (MAC) criterion. For chiral fermions in reps. R_1 and R_2, consider a condensation channel (c) $R_1 \times R_2 \to R_c$. The MAC criterion is that the channel in which condensation actually occurs is the one that maximizes

$$\Delta C_2 \equiv C_2(R_1) + C_2(R_2) - C_2(R_c)$$

The MAC is motivated by its application to QCD, which explains why only $3 \times \bar{3} \to 1$ occurs (it is the MAC), and not $3 \times \bar{3} \to 8$, $3 \times 3 \to \bar{3}_a$, or $3 \times 3 \to 6_s$.

Of course since the condensation is nonperturbative, while the MAC is semiperturbative, it is only a rough guide.

Second-order phase transitions in statistical mechanics and condensed matter physics motivate the notion that the UV to IR flow involves a Wilsonian thinning of degrees of freedom (d.o.f.) in the fields. In an approach using finite temperature T as the RG scale μ, and in a weakly interacting regime, one can enumerate these field degrees of freedom via the coefficient f in the free energy density

$$F(T) = f\frac{\pi^2}{90}T^4$$

where

$$f = 2N_V + \frac{7}{4}N_F + N_S \ ,$$

where N_V is the number of massless gauge fields, N_F is the number of massless chiral components of fermion fields, and N_S is the number of massless real scalar fields. Define

$$\lim_{\mu\to\infty} f \equiv f_{UV} \ , \qquad \lim_{\mu\to 0} f \equiv f_{IR} \ , \qquad \Delta f \equiv f_{UV} - f_{IR}$$

A conjectured d.o.f. inequality (DFI) (Appelquist, Cohen, Schmaltz, Shrock, 1999) motivated by the idea of thinning of d.o.f. in a UV to IR flow is

$$\Delta f \geq 0$$

Lattice gauge simulations can be used to test the DFI conjecture for a vectorial gauge theory (VGT), but to do this for a chiral gauge theory requires putting a χGT on the lattice, which is challenging.

A further conjecture is that if various different condensation channels are possible, a $priori$, the one that occurs is the one that maximizes Δf (DFM conjecture).

Consider, e.g., a theory with $G = \mathrm{SU}(N)$ and massless chiral fermions including a symmetric, rank-2 tensor (S) $\psi_L^{ab} \equiv \psi_L^{(ab)}$, together with $N+4$ copies (flavors) of the conjugate fundamental $(\bar{F})$ representation, $\chi_{a,i,L}$, $i = 1, ..., N+4$, where $a, b, ..$ are SU(N_c) gauge indices; $i, j, ...$ are copy indices. We use $N+4$ $\bar{F}$ fermions to cancel gauge anomalies, since $Anom(S) + (N+4)Anom(\bar{F}) = 0$.

To this irreducibly chiral set of fermions we add a vectorlike set consisting of p copies of $(F + \bar{F})$, $\chi_{a,i,L}$, $i = N+4+1, ...N+4+p$ and $\omega^a_{j,L}$, $j = 1, ..., p$.

This theory satisfies the conditions that would allow the formation of a set of massless gauge-singlet composite fermions (Bars and Yankielowicz).

The AF property $\beta < 0$ requires that $p < p_{b1z} = (9/2)N - 3$, and we assume this.

For a range of $p < p_{b1z}$, this theory has a (scheme-independent) two-loop zero of β, at

$$\alpha_{IR,2\ell} = \frac{8\pi N(9N-6-2p)}{-39N^3+90N^2-3N-36+p(26N^2-6)}$$

If $\alpha_{IR,2\ell}$ is small, then the UV flow leads to (i) a non-Abelian Coulomb phase in the IR, and the DFI is satisfied. As p decreases, $\alpha_{IR,2\ell}$ increases.

For sufficiently small p and large IR coupling, there are several IR possibilities, including

(ii) a phase with confinement and a massless sector consisting of gauge-singlet composite fermions, with no gauge or chiral symmetry breaking.

(iii) sequential condensation in the respective $S \times \bar{F} \to F$ (MAC) channel, breaking the SU(N) gauge symmetry completely, leaving a set of massless elementary fermions and massless composite NGBs in the IR.

The fact that the MAC is $S \times \bar{F} \to F$ follows since ΔC_2 is greater for this channel than for $F \times \bar{F} \to 1$:

$$\Delta C_2(S \times \bar{F} \to F) - \Delta C_2(F \times \bar{F} \to 1) = \frac{N-1}{N} > 0$$

The $S \times \bar{F} \to F$ condensation breaks the SU(N) gauge symmetry to SU($N-1$). Denote the scale where this happens as Λ_N. The fermion field components involved in the condensate gain dynamical masses of order Λ_N and are integrated out in the low-energy effective field theory operative for $\mu < \Lambda_N$. There is then sequential condensation in the $S \times \bar{F} \to F$ channel in this SU$(N-1)$ theory, and so on to lower scales.

Two other possible types of UV to IR evolution are

(iv) condensation of the p vectorlike fermions in the channel $F \times \bar{F} \to 1$ followed by confinement with massless composite fermion formation, no further chiral symmetry breaking, and no gauge-symmetry breaking, so that the IR effective FT consists of the massless composite fermions together with massless NGBs

(v) condensation of the p vectorlike fermions in the channel $F \times \bar{F} \to 1$, followed by condensation in the $S \times \bar{F} \to F$ channel, again breaking the SU(N) gauge symmetry completely, so that the IR particle content consists of massless NGBs and massless elementary fermions

We find that the MAC prediction for the most likely type of UV to IR evolution differs from the prediction from the Δf maximization (DFM) conjecture. The MAC predicts that (iii) will occur, but this would actually violate the $\Delta f \geq 0$ (DFI) conjecture for a range of larger values of p. For example, in the LNN limit $N \to \infty$, $p \to \infty$ holding $r = p/N$ fixed (with $r < 11/2$ to maintain AF), (iii) would violate DFI for $15/8 < r < 11/2$, i.e., for $1.875 < r < 5.5$.

Further, (iii) does not minimize f_{IR}, i.e., maximize Δf; instead, (iv), namely, $F \times \bar{F} \to 1$ condensation followed by confinement with massless gauge-singlet fermions, maximizes Δf and hence (iv) is favored by the DFM conjecture.

More details in our paper. These results show that the methods that one currently has for analyzing the UV to IR evolution in asymptotically free chiral gauge theories do not, in general, agree in their predictions. There is a great opportunity here for further progress in understanding this evolution better.

Study of RG Flows in Gauge Theories with Many Fermions

If the β function of a theory is positive near zero coupling, then this theory is IR-free; as μ increases from the IR to the UV, the coupling grows. It is of interest to investigate whether a non-AF theory of this type might have a UV fixed point (UV zero of β).

In addition to performing perturbative calculations of β to search for such a UVFP in an IR-free theory, one can use large-N methods. An explicit example is the O(N) nonlinear σ model in $d = 2 + \epsilon$ spacetime dimensions. From an exact solution of this model in the limit $N \to \infty$, one finds that (for small ϵ)

$$\beta(\lambda) = \epsilon\lambda\Big(1 - \frac{\lambda}{\lambda_c}\Big) \ ,$$

where λ is the effective coupling and $\lambda_c = 2\pi\epsilon/N$ (W. Bardeen, B. W. Lee, and R. Shrock, Phys. Rev. D 14, 985 (1976); E. Brézin and J. Zinn-Justin, Phys. Rev. B 14, 3110 (1976)). Thus this theory has a UVFP at λ_c, so that if initial value of $\lambda < \lambda_c$, then $\lambda \nearrow \lambda_c$ as $\mu \to \infty$.

There has long been interest in RG properties of $d = 4$ QED and, more generally, U(1) gauge theory (Gell-Mann and Low; Johnson, Baker, and Willey; Adler; Miransky; Yamawaki,...).

Consider a vectorial U(1) theory with N_f massless Dirac fermions of charge q. With no loss of generality, set $q = 1$. Write β function as

$$\beta_\alpha = 2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell$$

The 1-loop and 2-loop coefficients are

$$b_1 = \frac{4N_f}{3} , \qquad b_2 = 4N_f$$

These coefficients have the same sign, so the two-loop beta function, $\beta_{\alpha,2\ell}$, does not have a UV zero, and this is the maximal scheme-independent information about it. The coefficients have been calculated up to five loops in the $\overline{MS}$ scheme.

The 3-loop coefficient (deRafael and Rosner, 1974) is negative:

$$b_3 = -2N_f\Big(1 + \frac{22N_f}{9}\Big)$$

Hence, $\beta_{\alpha,3\ell}$ has a UV zero, namely,

$$\alpha_{UV,3\ell} = 4\pi a_{UV,3\ell} = \frac{4\pi[9 + \sqrt{3(45 + 44N_f)}\,]}{9 + 22N_f}$$

The 4-loop coefficient is (Gorishny, Kataev, Larin, Surguladze, 1991)

$$b_4 = N_f\Big[-46 + \Big(\frac{760}{27} - \frac{832\zeta(3)}{9}\Big)N_f - \frac{1232}{243}N_f^2\Big] ,$$

Numerically,

$$b_4 = -N_f\,(46 + 82.97533N_f + 5.06996N_f^2]$$

This is negative for all $N_f > 0$.

Recently, b_5 has been calculated (Kataev, Larin, 2012; Baikov, Chetyrkin, Kühn, Rittinger, Sturm, 2012, 2013). Numerically,

$$b_5 = N_f(846.6966 + 798.8919N_f - 148.7919N_f^2 + 9.22127N_f^3$$

which is positive for all $N_f > 0$.

Using these results we have investigated whether $\beta_{\alpha,n\ell}$ has a UV zero for n up to 5 loops for a large range of N_f. Our results are given in the table. The notation — means that for the given loop order and value of N_f, $\beta_{n\ell}$ has no UV zero.

N_f	$\alpha_{UV,2\ell}$	$\alpha_{UV,3\ell}$	$\alpha_{UV,4\ell}$	$\alpha_{UV,5\ell}$
1	—	10.2720	3.0400	—
2	—	6.8700	2.4239	—
3	—	5.3689	2.0776	—
4	—	4.5017	1.8463	—
5	—	3.9279	1.67685	2.5570
6	—	3.5156	1.5455	1.8469
7	—	3.2027	1.4397	1.6243
8	—	2.9555	1.3519	1.4851
9	—	2.7545	1.2776	1.3863
10	—	2.5871	1.2135	1.3120
20	—	1.7262	0.8483	—
100	—	0.7081	0.33265	—
500	—	0.3038	0.1203	—
10^3	—	0.2127	0.07678	—
10^4	—	0.016614	0.016965	—

A necessary condition for the perturbatively calculated β function to yield evidence for a stable UV zero is that it should remain present when one increases the loop order and the fractional change in the value should decrease going from n to $n+1$ loops.

As is evident from the table, we do not find that the UV zeros that we have calculated at $\ell = 3,\ 4,\ 5$ loop order for a large range of N_f values satisfy this necessary condition. Hence, these calculations do not give evidence for a UVFP in this theory.

We have also carried out an analysis in the limit

$$N_f \to \infty \quad \text{with finite} \quad y(\mu) \equiv N_f\, a(\mu) = \frac{N_f\, \alpha(\mu)}{4\pi}$$

We denote this as the LNF (large-N_f) limit; analogous to $N \to \infty$ limit in nonlinear σ model.

We set $b_1 = b_{1,1} N_f$ with $b_{1,1} = 4/3$. Further,

$$b_\ell = \sum_{k=1}^{\ell-1} b_{\ell,k}\, N_f^k \quad \text{for } \ell \ge 2 \ ,$$

where the $b_{\ell,k}$ are independent of N_f.

Hence,

$$b_\ell \propto N_f^{\ell-1} \quad \text{for} \ \ \ell \ge 2 \quad \text{as} \ \ N_f \to \infty$$

We thus define the finite quantities

$$\check{b}_\ell \equiv \frac{b_\ell}{N_f^{\ell-1}} \quad \text{for } \ell \ge 2$$

so

$$\lim_{N_f \to \infty} \check{b}_\ell = b_{\ell,\ell-1} \quad \text{for } \ell \geq 2$$

We define a rescaled β function that is finite in the LNF limit as $\beta_y \equiv \beta_\alpha N_f$. Then

$$\beta_y = 8\pi b_{1,1}\, y^2 \left[1 + \frac{1}{b_{1,1} N_f} \sum_{\ell=2}^{\infty} b_\ell\, y^{\ell-1} \right] .$$

The condition that the n-loop β_y, $\beta_{y,n\ell}$, has a zero at $y \neq 0$ is the equation

$$1 + \frac{1}{b_{1,1} N_f} \sum_{\ell=2}^{n} b_\ell\, y^{\ell-1} = 0 \ .$$

In the LNF limit, of the $n-1$ roots of this equation, the relevant one has the approximate form

$$y_{UV,n\ell} \sim \left(-\frac{b_{1,1} N_f}{b_{n,n-1}} \right)^{\frac{1}{n-1}}$$

Hence, $\beta_{y,n\ell}$ has a zero for $y \neq 0$ in the LNF limit if and only if $b_{n,n-1} < 0$. However, even if this condition were to be met, it follows that, for fixed finite loop order n, in the LNF limit, $\lim_{N_f \to \infty} y_{UV,n\ell} = \infty$.

One can reexpress β_y as a series in powers of $\nu \equiv 1/N_f$:

$$\beta_y = 8\pi b_{1,1}\, y^2 \left[1 + \sum_{s=1}^{\infty} F_s(y)\nu^s\right]$$

An exact integral representation of $F_1(y)$ is known (cf. Holdom, 2010). We have used this representation to determine the signs of $b_{n,n-1}$ up to $n = 24$ loops. We find that these signs are scattered, and show no indication of an onset of negative signs.

Thus, we do not find evidence of a UVFP in a U(1) gauge theory with N_f massless charged fermions for large N_f. Further nonperturbative results, such as calculations of $F_s(y)$ for $s \geq 2$, would give more information on this question.

We have also studied an SU(N) non-Abelian gauge theory with N_f massless fermions in a given representation for N_f. This theory is IR-free, and we again we do not find evidence of a UVFP.

Conclusions

- We have reported further results on the UV to IR evolution of an asymptotically free gauge theory and the nature of the IR behavior, in particular, on effects of including higher-loop terms in calculations and structural properties of the β function.

- Results on the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed provide understanding of similarities in UV to IR flows in theories with different N_c and N_f but similar r.

- Effects of scheme-dependence of IR zero in higher-loop calculations have been quantitatively studied.

- The varieties of UV to IR evolution in asymptotically free chiral gauge theories have been studied comparatively using different methods, including the MAC, a conjectured Δf inequality, and a Δf maximization conjecture.

- We have studied RG flows in U(1) and non-Abelian gauge theories with N_f fermions for large N_f.