The Gradient Flow and the Running Coupling

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- The Gradient Flow
  - New tool for lattice calculations
  - Renormalized quantities
  - Scale setting, chiral condensate
- The Coupling Scheme
  - Several possible schemes
  - Correction terms
  - SF results

The Gradient Flow

Gradient Flow

- Flow along the gradient of the an action [1]
  e.g. the gauge action:

$$\begin{aligned}
\partial_t B_{\mu,t} &= D_{\mu,t} B_{\mu\nu,t}, \\
B_{\mu,0} &= A_\mu \\
B_{\mu\nu,t} &= \partial_\mu B_{\nu,t} - \partial_\nu B_{\mu,t} + [B_{\mu,t}, B_{\nu,t}]
\end{aligned}$$

- Diffusive flow with scale $\sqrt{8t}$
- Produces a finite, smoothed field
- Correlators of the flow field renormalized

[1] M. Luscher and P. Weisz, JHEP **1102** (2011) 051 [arXiv:1101.0963].

Gradient Flow

- On the lattice, Wilson flow

$$
\begin{aligned}
G_{\mu,t+\epsilon} &= e^{-\epsilon Z_{\mu,t}(x)} G_{\mu,t}(x), \\
Z_{\mu,t}(x) &= (1/2)\left(M - M^{\dagger}\right) - (1/6)\mathrm{Tr}\left(M - M^{\dagger}\right) \\
M &= \sum_{\nu\neq\mu}\left(\quad + \quad \right)
\end{aligned}
$$

- Stout smearing steps, continuous flow
- Cooling, flow generated by the gauge force

Gradient Flow

- For example, the field strenght: [2]

$$
\begin{aligned}
\langle E_t \rangle &= \frac{1}{4} \langle G_{\mu\nu,t} G_{\mu\nu,t} \rangle \\
&= \frac{3(N_c^2 - 1) g_{\overline{MS}}^2}{128\pi^2 t^2} \left[1 + k_1 g_{\overline{MS}}^2 + \mathcal{O}(g_{\overline{MS}}^4) \right]
\end{aligned}
$$

$$
\begin{aligned}
k_1 = 1/(4\pi) \, [\, N\,(11/3\gamma_E + 52/3 - 3\ln 3) \\
- N_f\,(2/3\gamma_E + 4/9 - 4/3\ln 2)\,]
\end{aligned}
$$

$$\mu = (8t)^{-1/2}$$

- Renormalization scale set by t

[2] M. Luscher, JHEP **1008** (2010) 071 [arXiv:1006.4518].

Scale Setting

- One of the first applications, scale setting
- Find t_0 such that

$$t_0^2 \langle E_{t_0} \rangle = 0.3$$

- A physical scale given by $l = \sqrt{8 t_0}$
- Another scale ${w_0}^3$

$$W(t) = t \frac{d}{dt} \left[t^2 \langle E(t) \rangle \right]$$
$$w_0 = \sqrt{t}|_{W(t)=0.3}$$

[3] S. Borsanyi *et al.* JHEP **1209** (2012) 010 [arXiv:1203.4469].

Extension to Fermions

- The flow equations can be extended to fermionic fields: [4]

$$\partial_t \chi_t = \Delta_t \chi_t, \qquad \partial_t \bar{\chi}_t = \bar{\chi}_t \overleftarrow{\Delta}_t,$$
$$\chi_0 = \psi \qquad \bar{\chi}_0 = \bar{\psi}$$
$$\Delta_t = D_{\mu,t} D_{\mu,t}, \qquad D_{\mu,t} = \partial_\mu + B_{\mu,t}$$

- Not automatically renormalized, but

$$\chi_{R,t} = Z_\chi{}^{1/2} \chi_t \qquad \bar{\chi}_{R,t} = \bar{\chi}_t Z_\chi{}^{1/2}$$

- Independent of t

[4] M. Luscher, JHEP **1304** (2013) 123 [arXiv:1302.5246]

Extension to Fermions

- Non-zero flow time chiral condensate:

$$\Sigma_t^{rr} = -\langle \bar{\chi}_{r,t} \chi_{r,t} \rangle , \qquad \Sigma_{R,t}^{rr} = Z_\chi \Sigma_t^{rr}$$

- To the first order

$$\Sigma_{R,t}^{rr} = \frac{2N m_{R,r}}{(4\pi)^2 t} \int_0^\infty dv \frac{e^{-vz}}{(1+v)^2}, \qquad z = 2t m_{R,r}^2$$

- Can also be related to the chiral condensate

Other Applications

- Anisotropy tuning [5]
- Energy-momentum tensor [6]
- Matching improved actions with new observables:

$$E(t) = \frac{1}{4} G_{\mu\nu}(t) G_{\mu\nu}(t),$$

$$S_t^{rs} = -\bar{\chi}_{r,t}\chi_{s,t}, \qquad P_t^{rs} = -\bar{\chi}_{r,t}\gamma_5\chi_{s,t}$$

[5] S. Borsanyi *et al.* arXiv:1205.0781.

[6] H. Suzuki, PTEP **2013** (2013) 8, 083B03 [arXiv:1304.0533].
L. Del Debbio *et al.* JHEP **1311** (2013) 212 [arXiv:1306.1173].

Coupling from the Gradient flow

Coupling from the Gradient flow

- $E(t)$ is related to the coupling:[7]

$$g_{GF}^2 = \frac{t^2 \langle E(t) \rangle}{N}$$

- To study scale dependence, need a single scale
- Now we have L and t, so fix

$$t = \frac{(cL)^2}{8}$$

[7] Z. Fodor *et al.* JHEP **1211** (2012) 007 [arXiv:1208.1051].
Z. Fodor *et al.* PoS LATTICE **2012** (2012) 050 [arXiv:1211.3247].

Boundary Conditions

- Periodic
- Schrödinger Functional boundary conditions[8]
  - Unique global minimum: simpler perturbation theory
  - Breaks the translation symmetry
- Twisted boundary conditions [9]
  - Milder than the SF boundary
  - Restricts the number of fermions in the fundamental representations to $N_f = 2nN_c$

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[8] P. Fritzsch and A. Ramos, JHEP **1310** (2013) 008 [arXiv:1301.4388].
P. Fritzsch and A. Ramos, arXiv:1308.4559.
J. Rantaharju, arXiv:1311.3719.

[9] A. Ramos, arXiv:1308.4558.

Boundary Conditions

- Schrödinger Functional boundary conditions:
- Periodic spatial boundary conditions, trivial time boundaries

$$U_k(x) = 1, \text{ when } x_0 = 0, L$$
$$U_\mu(x + L\hat{k}) = U_\mu(x)$$

- Fermion fields disappear at time boundaries, periodic spatial boundary conditions

$$\psi(x) = 0, \text{ when } x_0 = 0, L$$
$$\psi(x + L\hat{k}) = \psi(x).$$

The Model

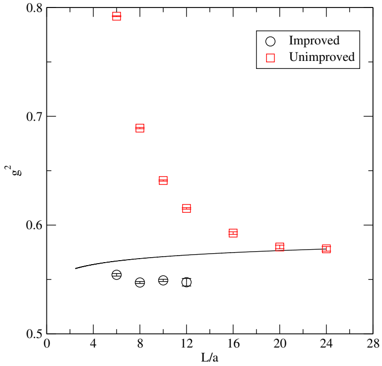
- The Model:

$$S = (1 - c_g) S_G(U) + c_g S_G(V) + S_F(V) + c_{SW} \delta S_{SW}(V)$$

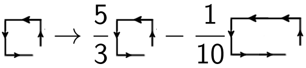
- Smeared Wilson fermion action $S_F(V)$
- and gauge action $S_G(V)$
- HEX Smearing
- Bulk correction term $\delta S_{SW}(V)$
- Here, $c_g = 0.5$ and $c_{SW} = 1$
- $N(a/L)$ in $g^2 = t^2 \langle E \rangle / N(a/L)$ measured at $\beta = 80$

Lattice Results

- Large discretization errors in previous results, corrections:



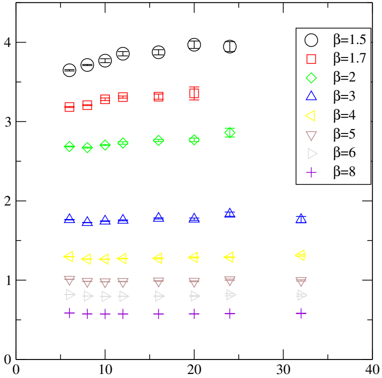
- Gradient flow and E from the Symanzik action



- Correction to the flow time

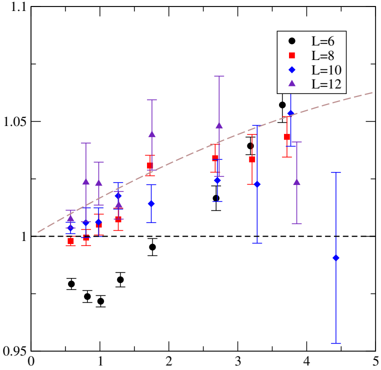
$$t = c^2 L^2/8 - d a^2,$$
$$c = 0.6, \; d = 0.5$$

SU(2) with SF boundaries



- Preliminary
- SU(2) with 8 flavors
- Relatively stable results
- Allows for large lattice sizes
- Measurements still at low coupling

SU(2) with SF boundaries



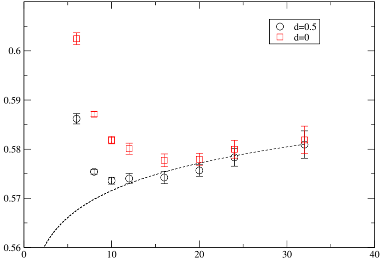
- Preliminary
- Scale dependence described by the step scaling function

$$\Sigma(u, \mathrm{a}/L) = \frac{g^2_{GF}(g_0, 2L/\mathrm{a})\big|_{g^2_{GF}(g_0, L/\mathrm{a})=u}}{u}$$

$$\sigma(u) = \lim_{\mathrm{a}\to 0} \Sigma(u, \mathrm{a}/L)$$

- The Gradient Flow
  - Renormalized quantities
  - Scale setting
  - Chiral condensate
  - Space-time symmetries
  - Improved actions
- The Coupling Scheme
  - Several possible schemes
  - Correction terms
  - Promising SF results

The effect of d



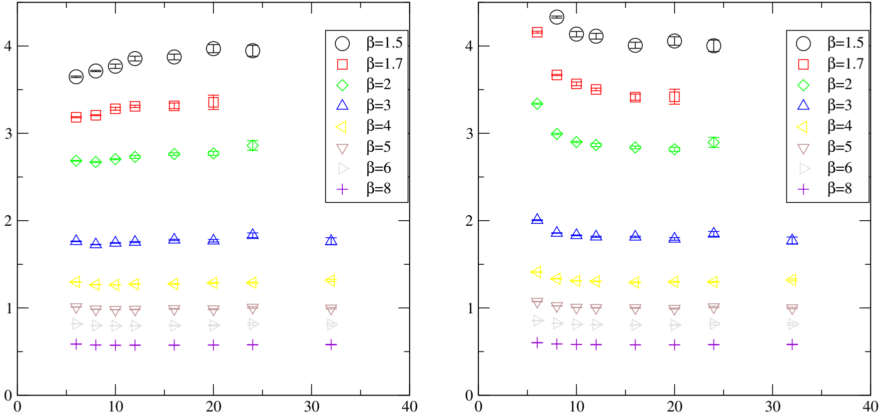
- The effect of d in the flow time
- SU(2) with 2 adjoint fermions, $\beta = 8$

$$t = c^2 L^2/8 - da^2,$$

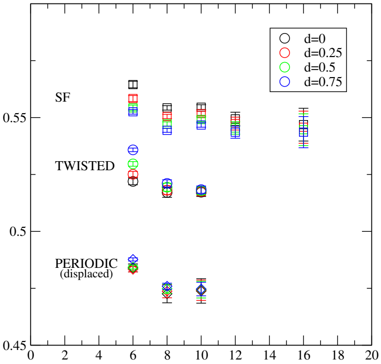
twisted: $d = 0$

SF,periodic: $d = 0.5$

Compared to $d = 0$



The effect of d



- The effect of d in the flow time
- SU(2) with 2 adjoint fermions, $\beta = 8$

$$t = c^2 L^2/8 - da^2,$$

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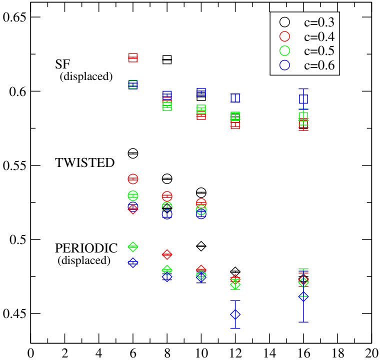
The effect of c



- The effect of c in the flow time
- Labels different schemes in the continuum

$$t = c^2L^2/8 - da^2,$$

twisted: $d = 0$

SF,periodic: $d = 0.5$