Scale ⇒ conformal invariance
RG-cycles,
and all that ...

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with
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Two old questions (and the motivation for this work)

1.What are the possible phases of theories (limits as deep IR is approached)?
(coulomb, Higgs, confinement, conformal, ...)?
Is there, eg, a scale but not conformal phase?

2. RG flows: we know limiting flows can be IRFP (IR fixed points).
Wilson speculated one may also have limit cycles or limiting ergodic trajectories.
Are there any examples of these?

Quiz

Directions: Select the best answer.

1. Which of the following is false:

A. The trace anomaly is $T^\mu_\mu = \beta_i \mathcal{O}_i$ (up to equations of motion)

B. A theory is conformal if and only if $\beta_i = 0$

C. Scale invariance does not imply conformal invariance

D. All of the above

E. None of the above

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Explanations:

- In $d = 4$
- In flat space
- Classically scale invariant (no dimensional couplings)
- Up to Equations of Motion

Quiz-Solutions

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✓ D. All of the above

E. None of the above

In this talk I will explain these answers.

Trace Anomaly

I. Jack and H. Osborn, NPB **343**, 647 (1990).
H. Osborn, NPB **363**, 486 (1991)

Review derivation in dimensional regularization

Consider QFT with dimensionless couplings (classically scale invariant) in $d = 4 - \varepsilon$

Compute stress-energy tensor, take trace $T^\mu_\mu = \epsilon \mathcal{L} - (1)\phi \frac{\delta S_0}{\delta \phi}$

Lagrangian in terms of renormalized couplings satisfies RGE

$$\left[\hat{\beta}(g)\frac{\partial}{\partial g} - \hat{\gamma}\phi\frac{\partial}{\partial \phi} - \epsilon\right]\mathcal{L} = 0$$

where $\hat{\beta} = -\epsilon g + \beta(g) \quad \hat{\gamma} = -\epsilon + \gamma(g)$

Obtain

$$T^\mu_\mu = \hat{\beta}(g)\frac{\partial \mathcal{L}}{\partial g} - (1 + \hat{\gamma}(g))\phi\frac{\delta S_0}{\delta \phi}$$

Notes:

- Presented for scalars, trivially extended for gauge fields, spinors with yukawas
- Presented for single coupling, trivially extended for many couplings, eg

$$V = \frac{1}{4!} g_{abcd}\phi_a\phi_b\phi_c\phi_d = g_I \mathcal{O}_I$$

$$T^\mu_\mu = \hat{\beta}_I(g)\frac{\partial \mathcal{L}}{\partial g_I} - [(1+\hat{\gamma}(g))\phi]\cdot\frac{\delta S_0}{\delta\phi}$$

So what is the problem with this???

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(This is a rhetorical question, I will now answer :))

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The operator $\mathcal{O}_I = \dfrac{\partial \mathcal{L}}{\partial g_I}$ *is not finite* in general when inserted in Green functions!

Taking a derivative of the functional integral $Z = e^{iW} = \int [d\phi] e^{iS_0}$

$$\frac{\partial}{i\partial g_I} \int [d\phi] e^{iS_0} = \langle \int d^d x \; O_I \rangle$$

This is finite, but what about the *local* operator?
It can fail to be finite by a total divergence

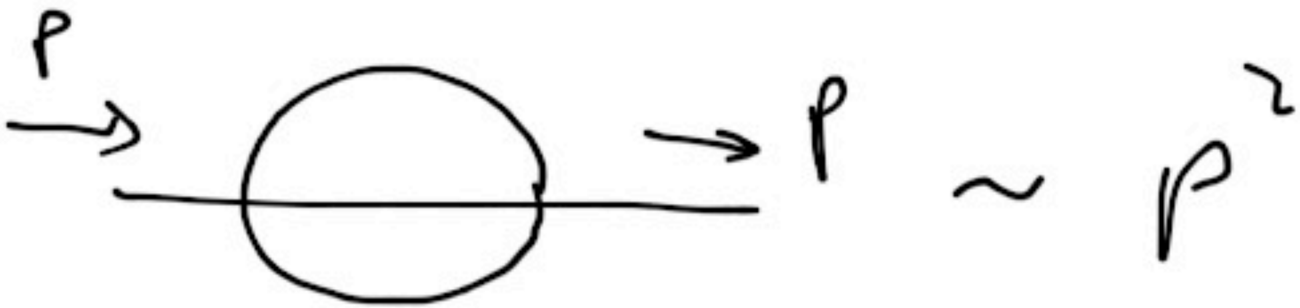
$$\langle [O_I] \rangle \equiv \langle O_I - \partial_\mu J_I^\mu \rangle = \text{finite}$$

What can this "current" be?
For example, for scalars we can have

$$J_I^\mu = -\frac{1}{2}(N_I)_{ab}(\phi_a \overleftrightarrow{\partial}^\mu \phi_b) = \partial^\mu \phi^T N_I \phi \text{ (for short)} \qquad N_I^T = -N_I, \qquad N_I = \frac{N_I^1}{\epsilon} + \frac{N_I^2}{\epsilon^2} + \cdots$$

We do not usually encounter this (need sufficient complexity)

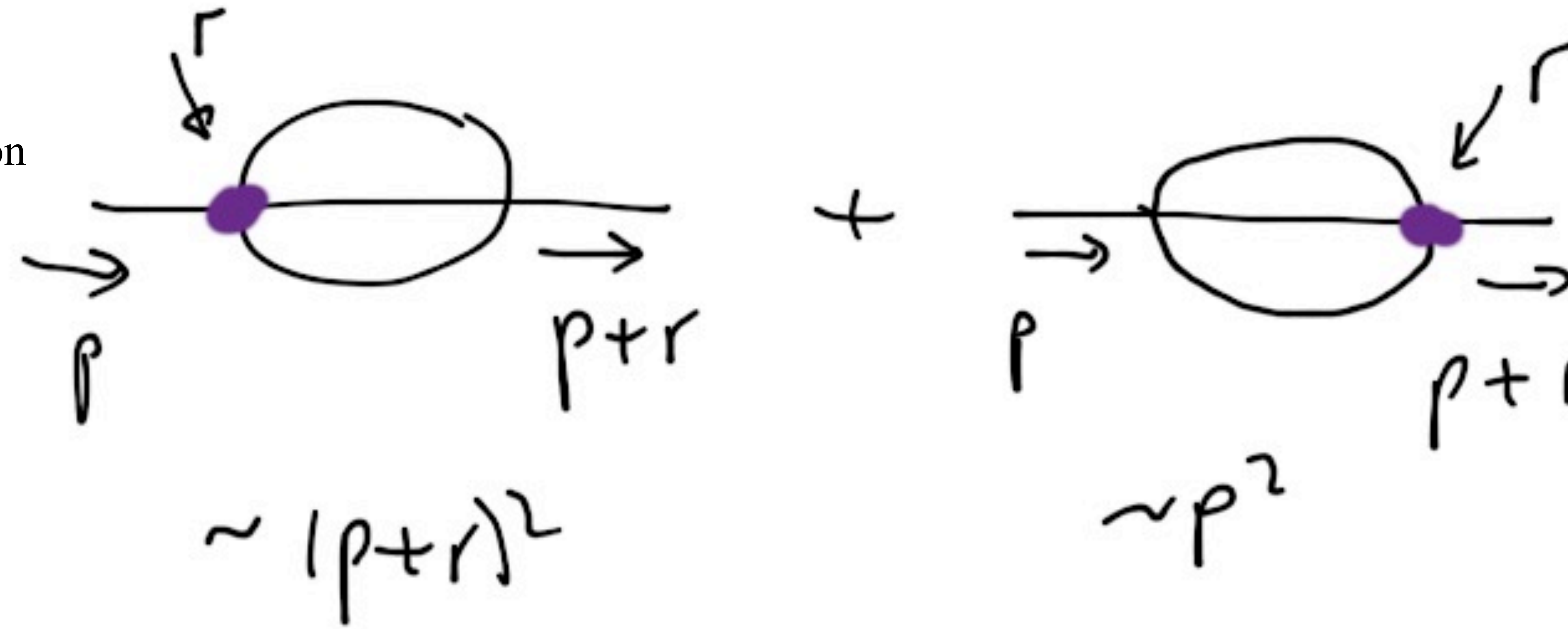
wave-
function



couterterm:

$(\partial_\mu \phi)^2$

2-pt function
with one
insertion
of O_I



Trace anomaly: $$T^{\mu}_{\mu} = \hat{\beta}(g)\frac{\partial \mathcal{L}}{\partial g} - (1+\hat{\gamma}(g))\phi\frac{\delta S_0}{\delta\phi}$$ really is:

$$T^{\mu}_{\mu} = \beta_I(g)[\mathcal{O}_I] + \partial_\mu J^\mu - (1+\gamma(g))\phi\frac{\delta S_0}{\delta\phi}$$

The “hats” have disappeared because the quantities are finite.

The current must be finite by itself.

$$J^\mu = \hat{\beta}_I J^\mu_I = \partial^\mu\phi^T\hat{\beta}_I N_I\phi = \quad \text{finite}$$

$$\hat{\beta} = -\epsilon g + \beta(g)$$

Introducing *S*, the finite part of $\hat{\beta}_I N_I$

$$N_I = \frac{N_I^1}{\epsilon} + \frac{N_I^2}{\epsilon^2} + \cdots$$

$$S = -g_I N_I^1$$

This only gives the “tree level” current.
One can show the fully renormalized current is

$$J^\mu = \partial^\mu\phi^T(S + N_I(Sg)_I)\phi = \quad \text{finite}$$

where $\quad (Sg)_{abcd} = S_{ae}g_{ebcd} + \cdots + S_{de}g_{abce}$

Compute S

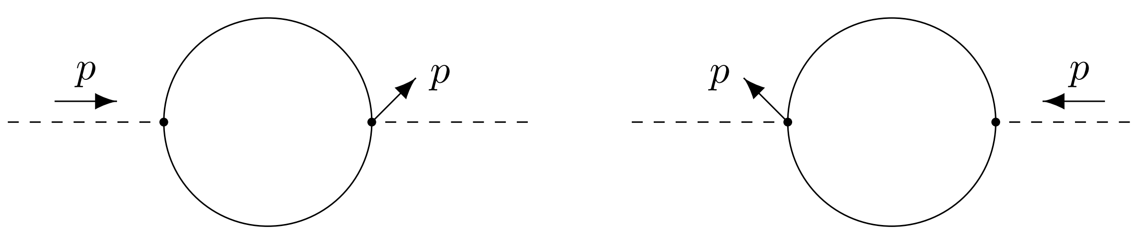
Recall $S = -g_I N_I^1$

For theory of n_s real scalars, n_f Weyl spinors (symmetry group $G_F = \mathrm{SO}(n_s) \times \mathrm{SU}(n_f)$, any subgroup possibly gauged), with

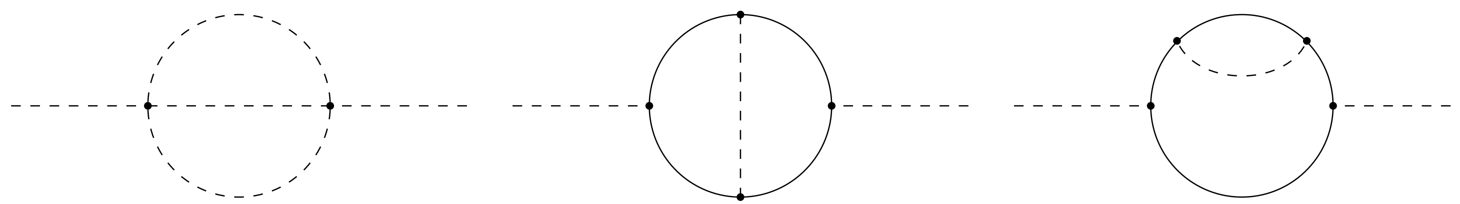
$$V = \tfrac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d + \left(\tfrac{1}{2}y_{a|ij}\phi_a\psi_i\psi_j + \text{h.c.}\right).$$

From definition, $S^T = -S$ (in the Lie algebra of $\mathrm{SO}(n_s)$)

$S = 0$ at 1- and 2-loops because individual topologies are symmetric under $a \leftrightarrow b$

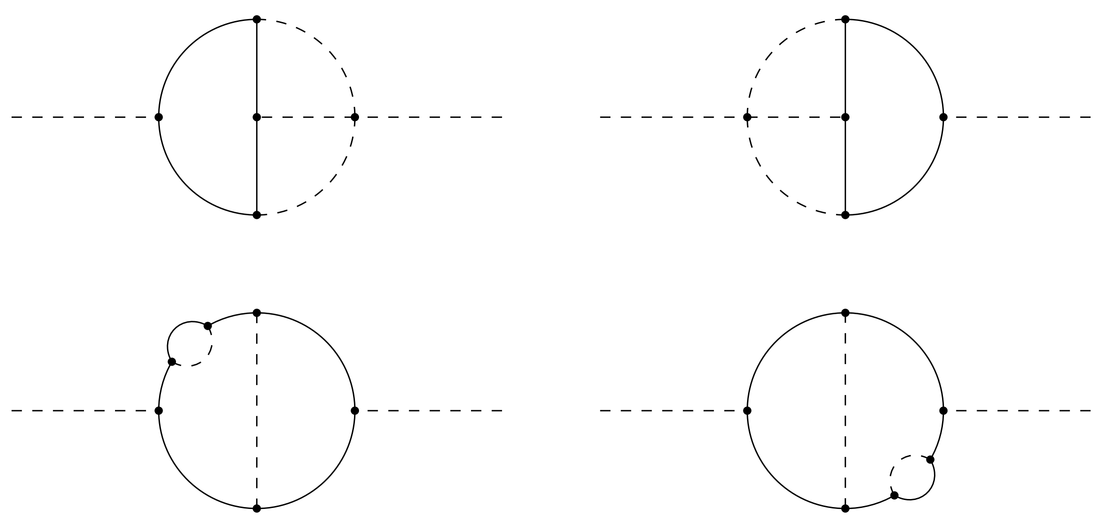


Contributions to N_I still symmetric at 2-loops



JO: $S = 0$ to 2-loops in fermion-scalar model, and to 3-loops in pure scalar model

Contributions to N_I not symmetric at 3-loops;



Be mesmerized:

$$
\begin{aligned}
(16\pi^2)^3 (N_I^1)_{ab} \partial^\mu g_I \supset & -\tfrac{1}{2}\operatorname{tr}(y_a \partial^\mu y_c^* y_d y_e^*)\lambda_{bcde} - \tfrac{1}{3}\operatorname{tr}(y_a y_c^* \partial^\mu y_d y_e^*)\lambda_{bcde} - \tfrac{1}{2}\operatorname{tr}(y_a y_c^* y_d \partial^\mu y_e^*)\lambda_{bcde} \\
& - \tfrac{5}{24}\operatorname{tr}(y_a y_c^* y_d y_e^*)\partial^\mu \lambda_{bcde} - \tfrac{1}{24}\operatorname{tr}(y_b \partial^\mu y_c^* y_d y_e^*)\lambda_{acde} - \tfrac{5}{24}\operatorname{tr}(y_b y_c^* \partial^\mu y_d y_e^*)\lambda_{acde} \\
& - \tfrac{1}{24}\operatorname{tr}(y_b y_c^* y_d \partial^\mu y_e^*)\lambda_{acde} - \tfrac{5}{24}\operatorname{tr}(\partial^\mu y_b y_c^* y_d y_e^*)\lambda_{acde} - \tfrac{7}{32}\operatorname{tr}(y_a \partial^\mu y_c^* y_d y_d^* y_b y_c^*) \\
& - \tfrac{7}{96}\operatorname{tr}(y_a y_c^* \partial^\mu y_d y_d^* y_b y_c^*) - \tfrac{23}{96}\operatorname{tr}(y_a y_c^* y_d \partial^\mu y_d^* y_b y_c^*) - \tfrac{7}{96}\operatorname{tr}(y_a y_c^* y_d y_d^* \partial^\mu y_b y_c^*) \\
& - \tfrac{7}{32}\operatorname{tr}(y_a y_c^* y_d y_d^* y_b \partial^\mu y_c^*) + \tfrac{1}{16}\operatorname{tr}(y_a \partial^\mu y_c^* y_c y_d^* y_b y_d^*) - \tfrac{5}{48}\operatorname{tr}(y_a y_c^* \partial^\mu y_c y_d^* y_b y_d^*) \\
& - \tfrac{1}{48}\operatorname{tr}(y_a y_c^* y_c \partial^\mu y_d^* y_b y_d^*) - \tfrac{7}{96}\operatorname{tr}(y_a y_c^* y_c y_d^* \partial^\mu y_b y_d^*) + \tfrac{1}{16}\operatorname{tr}(y_a y_c^* y_c y_d^* y_b \partial^\mu y_d^*) \\
& + \text{h.c.} - \{a \leftrightarrow b\},
\end{aligned}
$$

First ever computation of non-vanishing S :

$$
(16\pi^2)^3 S_{ab} = \tfrac{5}{8}\operatorname{tr}(y_a y_c^* y_d y_e^*)\lambda_{bcde} + \tfrac{3}{8}\operatorname{tr}(y_a y_c^* y_d y_d^* y_b y_c^*) + \text{h.c.} - \{a \leftrightarrow b\}
$$

SUSY result, all orders in perturbation theory:

SUSY result, all orders in perturbation theory:

$$S = 0$$

Scale without Conformal?

J. Polchinski, NPB 303, 226 (1988).
D. Dorigoni and V. S. Rychkov, 0910.1087 [hep-th].

- Condition for Scale Invariance?

$$\partial_\mu D^\mu = 0$$

where the dilatation (scale) current is given in terms of the improved stress-energy tensor

$$D^\mu = x_\nu T^{\mu\nu}$$

so that

$$\partial_\mu D^\mu = T^\mu_\mu$$

- Condition for Conformal Invariance?

$$\partial_\mu K^{\mu\nu} = -x^\nu T^\mu_\mu = 0$$

- It appears that in both cases the condition is

$$T^\mu_\mu = 0$$

- Improvements? If

$$T^{\mu}_{\mu} = \partial_\mu \partial_\nu L^{\mu\nu}$$

one can improve $T^{\mu\nu}$ so that scale and conformal still conserved.

- But! What if the unbroken symmetry is a combination of two broken symmetries? This happens in other familiar contexts:
  - For spontaneously broken symmetries, as in the SM: SU(2)×U(1) → U(1)$_{\text{EM}}$
  - For anomalous currents, as in B and L in SM, but not $B-L$

- *Look for a conserved current of the form*

$$\boxed{D^\mu = x_\nu T^{\mu\nu} - V^\mu}$$

where V^μ (the "virial current") is a non-conserved current that does not depend explicitly on coordinates.

(and which is not of the form $V^\mu = \partial_\nu L^{\mu\nu}$)

THEN: We can have

$$\partial_\mu D^\mu = T^\mu_\mu - \partial_\mu V^\mu = 0 \qquad \text{scale invariance}$$

while

$$T^\mu_\mu = \partial_\mu V^\mu \neq 0 \qquad \text{no conformal symmetry}$$

<u>A scale transformation together with a U(1) rotation is still a symmetry.</u>

Now use trace anomaly and explicit form of virial:

$$V^\mu = R_{ab}\partial^\mu \phi_a \phi_b + iP_{ij}\bar{\psi}_i \bar{\sigma}^\mu \psi_j$$

Condition for scale invariance but not conformal is then an algebraic condition:

$$\beta_I - (S\lambda)_I = (R\lambda)_I \neq 0$$

recall notation:
$(S\lambda)_{abcd} = S_{ae}\lambda_{ebcd} + \cdots + S_{de}\lambda_{abce}$

or $\quad \beta_I = (Q\lambda)_I \quad$ with $\quad R = Q - S \neq 0$

$$\beta_{abcd} = -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'} \,,$$
$$\beta_{a|ij} = -Q_{a'a}y_{a'|ij} - P_{i'i}y_{a|i'j} - P_{j'j}y_{a|ij'} \,,$$

These are not functional equations

Solution: specific values of coupling constants (and of Q and P) that satisfy these equations

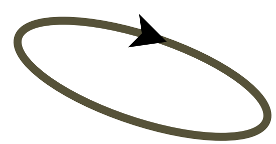
Precisely as in searching for conformal fixed points (with $Q = P = 0$)

Immediate: recurrent RG-trajectories $\quad -\frac{d\lambda_I}{dt} = \beta_I = (Q\lambda)_I \qquad \lambda_I(t) = (e^{-itQ}\lambda)_I$

$e^{-tQ} \in G_F$ group of internal global transformations ("flavor" symmetry of kinetic terms)

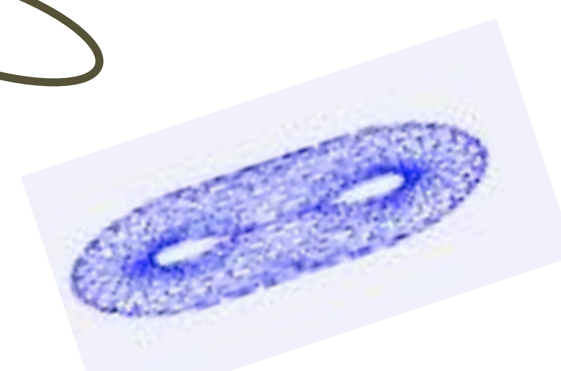
As a function of t: one parameter trajectory in compact space

◉ Trajectory closes



$\Rightarrow$

◉ Trajectory comes arbitrarily close to initial point (Poincare recurrence)



We look for solutions using perturbation theory:

At 1-loop the form of the beta-functions (ie, which monomials appear) suffices to show that $Q = P = 0$.

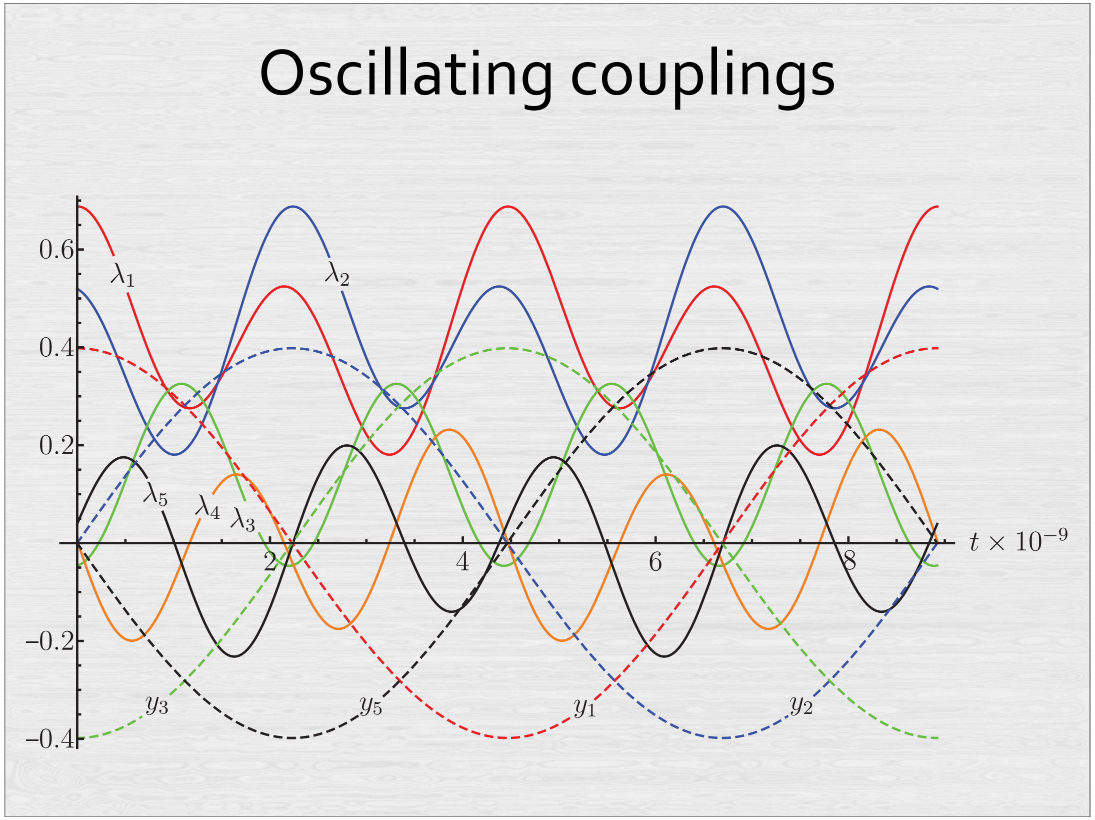
At 2-loops this happens by detailed cancellation among terms (and by non-linearity one has a reminder, Q = order 3-loops)

The 3-loop beta functions are not known.

We have computed the necessary terms (not the complete beta-function)

Summary of findings:

- In $d = 4 - \varepsilon$, no gauge fields (like Wilson-Fisher fixed points)
  - No non-trivial solution to all orders in perturbation theory for any n_f if $n_s < 2$
  - Solutions with $P = 0$ but $Q \neq 0$ at 3-loops in:
    - $n_f = 1, n_s = 2,$ with unbounded tree-level potential
    - $n_f = 2, n_s = 2,$ with bounded tree-level potential
- In $d = 4$, SU(3)-YM $n_f = 2 + 2$ (anti-)fundamentals, $n_s = 2$ neutral (Caswell-Banks-Zaks FP)
  - Solutions with $P = 0$ but $Q \neq 0$ at 3-loops.
  - Unbounded tree-level potential



S vs Q

- $S = S(g)$, defined everywhere in theory space (ie, space of coupling constants)
- Q = constant (one for each cycle solution), defined by solving for existence of cycle
- Both appear first at third order in the loop expansion
- Both in Lie algebra of SO(n_s) in G_F

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Theorem: In perturbation theory

- On cycles $S = Q$ (up to a symmetry generator)
- On fixed points $S = 0$ (up to a symmetry generator)

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Outline of proof:

- On cycle a quantity A remains constant simply because it is G_F invariant
- A satisfies a strong c-theorem in perturbation theory, grows like $\left(\beta - (S\lambda)\right)^2$
- On cycle, $\beta = (Q\lambda)$, the two statements imply $(Q\lambda) - (S\lambda) = 0$

Scale IS Conformal

Legal Disclosure: In $d = 4$, unitary, local, renormalizable, perturbative, interacting QFT with well defined correlators of stress-energy tensor

Recall, the condition for scale but not conformal is $\beta_I - (S\lambda)_I = (R\lambda)_I \neq 0$

or $\beta_I = (Q\lambda)_I$ with $R = Q - S \neq 0$

We just showed $(Rg)_I = 0$ (*R* may not vanish but then it generates a symmetry)

Hence: scale + Poincare + unitarity + no-nonsense $\Rightarrow$ conformal

See also: Luty, Polchinski, Rattazzi

Quiz-Solutions Explained

Directions: Select the best answer.

1. Which of the following is false:

A. The trace anomaly is $T^\mu_\mu = \beta_i \mathcal{O}_i$ (up to equations of motion) $T^\mu_\mu = \beta_I(g)[\mathcal{O}_I] + \partial_\mu J^\mu$

B. A theory is conformal if and only if $\beta_i = 0$ $B_I = \beta_I - (Sg)_I = 0$

C. Scale invariance does not imply conformal invariance Yes it does (at least perturbatively)

D. All of the above

E. None of the above

After Thoughts, Conclusions, ...

- Is B_I the new beta-function? No.
  But it does coincide with the beta-function in *one* particular scheme. This scheme cannot be obtained by a transformation $g \to f(g)$ alone (not the usual stuff). In this scheme the anomalous dimension matrix of the scalar fields is non-symmetric.

- Perturbative positivity of $\chi^g{}_{IJ}$ is used for proof of a-theorem and that scale implies conformal.
  New work by Komargodsky and Schwimmer (also Luty, Polchinski, Rattazzi) suggests there is a non-perturbative a-theorem. Positivity there is from optical theorem. Suggests a connection that may establish $S = Q$ and "scale implies conformal" non-perturbatively.

- Explicit form of S allows to find examples of Q much more simply:
  - Solve for zeroes of <u>*1-loop*</u> beta-function $\beta_I = 0$
  - Plug value of "fixed point" into our 3-loop expression for $S(g)$
  - If $S = 0$ a FP, else $Q = S$

- What is the role of *S?* Are these "cyclic CFT" different form normal, FP CFT?
  Two point function look completely "normal" in cyclic CFT.

- Many other open questions ($d = 4 - \varepsilon$?, flows between FP and cyclic?, $d = 4$ with bounded potential? ...)

The End

Additional Slides

Explanations

A. The trace anomaly is $T^\mu_\mu = \beta_i \mathcal{O}_i$ (up to equations of motion)

We mean:

- In $d = 4$
- In flat space, else we would have additional terms involving the Riemann tensor quadratically

$$T^\mu_\mu = \beta_i \mathcal{O}_i - \frac{a}{16\pi^2} G + \frac{c}{16\pi^2} F - \frac{b}{16\pi^2} R^2$$

$$F = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - \frac{4}{d-2} R^{\mu\nu} R_{\mu\nu} + \frac{2}{(d-2)(d-1)} R^2, \qquad \text{(square Weyl)}$$

$$G = \frac{2}{(d-3)(d-2)} (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2), \qquad \text{(Euler density)}$$

- No dimensional couplings (no masses or cubic scalar couplings)
- There are in fact equations of motion terms on the RHS

$$T^\mu_\mu = \beta_i \mathcal{O}_i + \Delta\phi \frac{\delta}{\delta\phi} S_0$$

where Δ is the dimension of ϕ

B. A theory is conformal if and only if $\beta_i = 0$

We mean a theory that is conformal classically (question would be pointless is there are masses).

Not identically but at a point in theory space $\beta_i(g_*) = 0$

C. Scale invariance does not imply conformal invariance

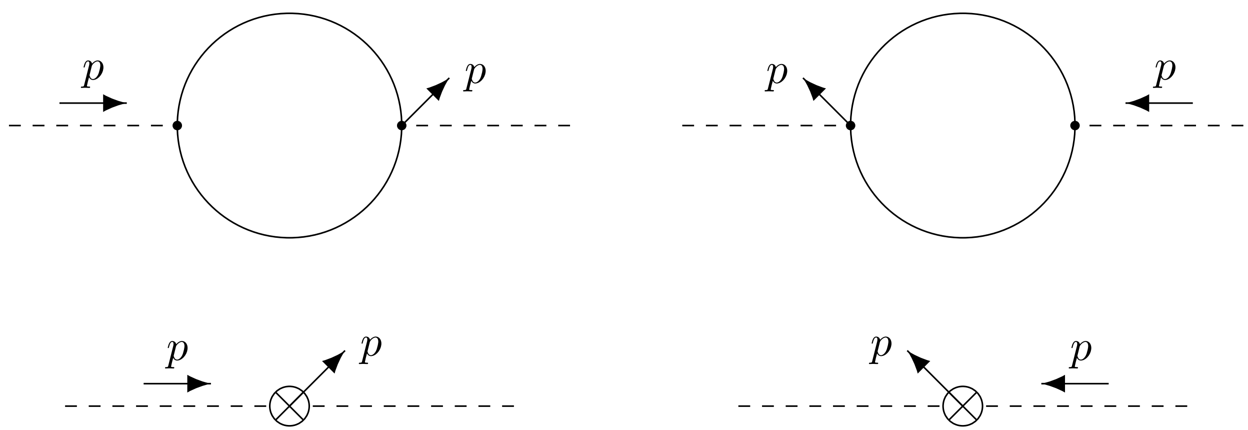
Again, for $d = 4$. For $d = 2$ it was shown (Polchinski, 1987) scale implies conformal.

Unparticle physics uses (except for scalar unparticles) scale invariance without conformal. This gives amplified effects because it allows for

(i) Non-conserved vectors of dimension 3
(ii) Smaller dimensions than allowed by CFT, amplifying the effect of unparticles (eg, vectors of dim<3)

Example: n_s real scalars, n_f Weyl spinors with

$$V = \frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d + (\tfrac{1}{2}y_{a|ij}\phi_a\psi_i\psi_j + \text{h.c.}).$$



A simple 1-loop computation gives:

$$(N_{c|ij})_{ab} = -\frac{1}{16\pi^2\epsilon}\,\frac{1}{2}(y^*_{a|ij}\delta_{bc} - y^*_{b|ij}\delta_{ac})$$

(The index "I" now runs over ($abcd$) and ($a|ij$))

Use EOM?

Trace anomaly:

$$
\begin{aligned}
T_\mu{}^\mu(x) = {} & \gamma_{aa'} D^2\phi_a\phi_{a'} - \gamma^*_{i'i}\bar{\psi}_i i\bar{\sigma}^\mu D_\mu\psi_{i'} + \gamma_{ii'} D_\mu\bar{\psi}_i i\bar{\sigma}^\mu\psi_{i'} \\
& - \tfrac{1}{4!}(\beta_{abcd} - \gamma_{a'a}\lambda_{a'bcd} - \gamma_{b'b}\lambda_{ab'cd} - \gamma_{c'c}\lambda_{abc'd} - \gamma_{d'd}\lambda_{abcd'})\phi_a\phi_b\phi_c\phi_d \\
& - \tfrac{1}{2}(\beta_{a|ij} - \gamma_{a'a}y_{a'|ij} - \gamma_{i'i}y_{a|i'j} - \gamma_{j'j}y_{a|ij'})\phi_a\psi_i\psi_j + \text{h.c.}\,.
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu D^\mu(x) = {} & (\gamma_{aa'} + Q_{aa'})D^2\phi_a\phi_{a'} - (\gamma^*_{i'i} + P^*_{i'i})\bar{\psi}_i i\bar{\sigma}^\mu D_\mu\psi_{i'} + (\gamma_{ii'} + P_{ii'})D_\mu\bar{\psi}_i i\bar{\sigma}^\mu\psi_{i'} \\
& -\tfrac{1}{4!}(\beta_{abcd} - \gamma_{a'a}\lambda_{a'bcd} - \gamma_{b'b}\lambda_{ab'cd} - \gamma_{c'c}\lambda_{abc'd} - \gamma_{d'd}\lambda_{abcd'})\phi_a\phi_b\phi_c\phi_d \\
& -\tfrac{1}{2}(\beta_{a|ij} - \gamma_{a'a}y_{a'|ij} - \gamma_{i'i}y_{a|i'j} - \gamma_{j'j}y_{a|ij'})\phi_a\psi_i\psi_j + \text{h.c.}\,,
\end{aligned}
$$

Using the EOM to eliminate anomalous dimensions is on the same footing as using EOM on the virial current

Diagrams that can contribute to Q in the $n_s = n_f = 2$ model

2-loops

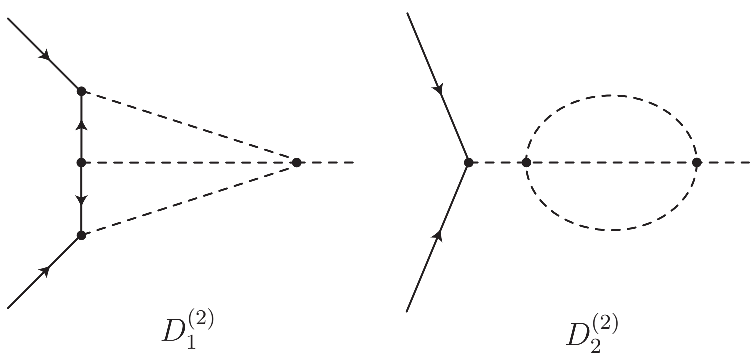


Fig. 1: Diagrams that contribute to q at two-loop order.

3-loops

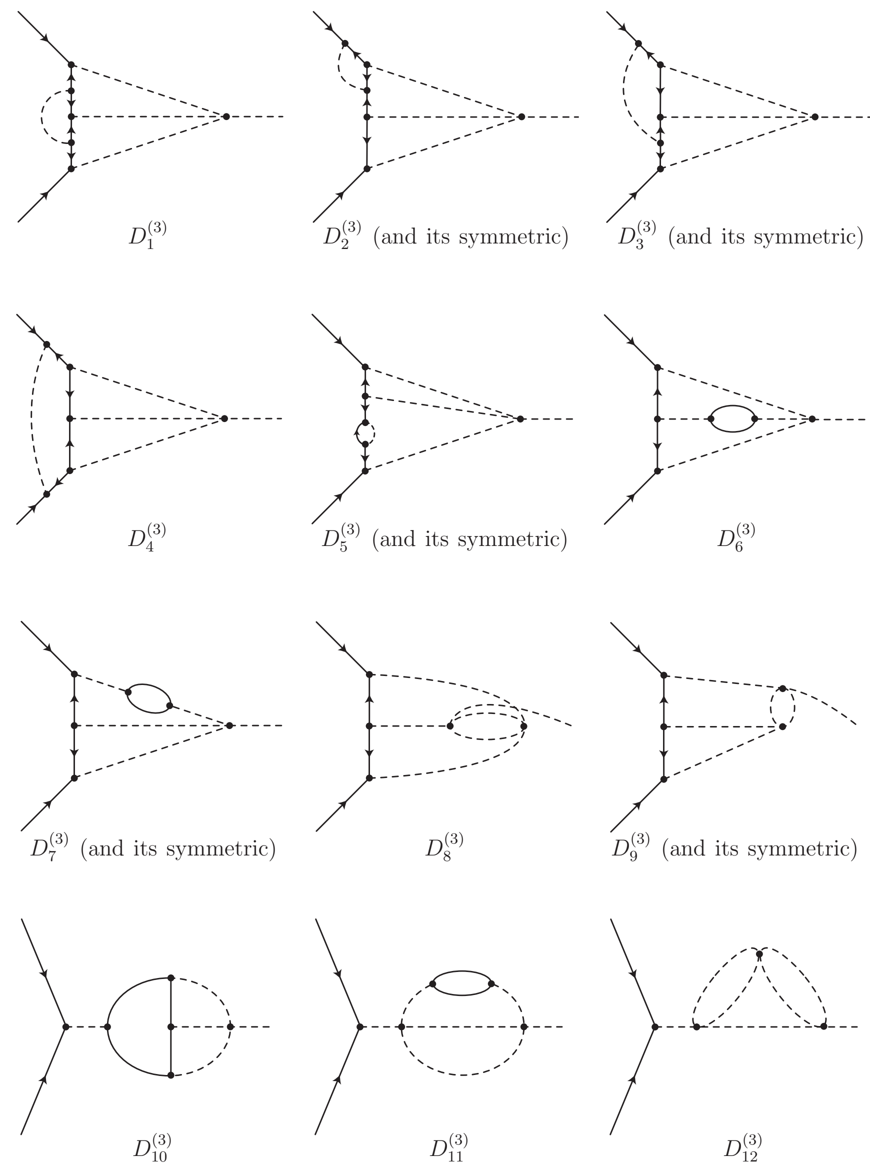


Fig. 2: Diagrams that contribute to q at three-loop order.

The proof uses one of the consistency conditions of Jack & Osborn

$$8B_I \partial_I \tilde{A} = \chi^g_{IJ} B_I B_J$$

Here

$\tilde{A} \equiv a + \frac{1}{8} w_I B_I$

$B_I = \beta_I - (Sg)_I$

a = coefficient of Euler density in Weyl Anomaly (of "a-theorem" fame)

$\chi^g{}_{IJ}$ and w_I are defined in JO

$\tilde{A}$ is a G_F scalar $\Rightarrow$ $(\omega g)_I \partial_I \tilde{A} = 0$ for any ω in the Lie algebra of G_F

On a cycle or a FP B_I is of this form: $B_I = \beta_I - (Sg)_I = (Qg)_I - (Sg)_I = ([Q-S]g)_I$

(for FP use $Q = 0$)

$\Rightarrow$ On a cycle or FP $\chi^g_{IJ} B_I B_J = 0$

To lowest order in the loop expansion $\chi^g{}_{IJ}$ is positive definite (a "metric" in theory space)

(only place perturbative assumption enters)

$\Rightarrow$ $B_I = 0$ $\Rightarrow$ $(Sg)_I = (Qg)_I$ $\Rightarrow$ $S = Q + \Delta Q$ with $(\Delta Q g)_I = 0$, ie, a symmetry

Q.E.D.

Note: perturbative c-theorem follows from same consistency condition (Jack & Osborn)

Trace Anomaly II

Follow Jack and Osborn's derivation of trace anomaly

- Consider QFT in curved background with space-time dependent couplings
  - Curved background: can then
    - Obtain stress-energy tensor by taking derivative w.r.t metric

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_0}{\delta g_{\mu\nu}}$$

    - Study Weyl variations: these encode both dilatations and conformal transformations

$$g_{\mu\nu}(x) \to e^{-2\sigma(x)} g_{\mu\nu}(x)$$

  - Space-time dependent couplings
    - Sources for *finite* composite operators $[O_I(x)] = \frac{\delta S_0}{\delta g_I(x)}$
    - Necessary for consistent space-time dependent dilatations (Weyl variations)

$$g(\mu) \to g(e^{-\sigma(x)} \mu)$$

- Extend renormalized Lagrangian by all possible counterterms (operators of dimension 4) constructed out of metric and couplings, and their derivatives (consistent with diff invariance)
- Then repeat argument presented in "Trace Anomaly I" but with space-time dependent couplings

$$T^{\mu}_{\mu} = \beta_I(g)[\mathcal{O}_I] + \nabla_\mu J^\mu + \cdots$$

(... = stuff that vanishes in flat space, constant couplings)

Bonus: JO consistency conditions (our version)

$$e^{i\tilde{W}} = \int [d\phi] e^{i(S_0+S_{\rm c.t.})} \qquad \tilde{W} = W + W_{\rm c.t.} = W + S_{\rm c.t.}$$

$S_{\rm c.t.}$ contains field independent counterterms, dim-4 operators constructed of $g_{\mu\nu}(x)$ and $g_I(x)$

$$W_{\rm c.t.} = -\int \sqrt{-\gamma}\,\mu^{-\epsilon}\lambda\cdot\mathscr{R},$$

with

$$\lambda\cdot\mathscr{R} = \lambda_a F + \lambda_b G + \lambda_c H^2 + \mathscr{E}_i \partial_\mu g^i \partial^\mu H + \tfrac{1}{2}\mathscr{F}_{ij}\partial_\mu g^i \partial^\mu g^j H + \tfrac{1}{2}\mathscr{G}_{ij}\partial_\mu g^i \partial_\nu g^j G^{\mu\nu}$$
$$+ \tfrac{1}{2}\mathscr{A}_{ij}\nabla^2 g^i \nabla^2 g^j + \tfrac{1}{2}\mathscr{B}_{ijk}\partial_\mu g^i \partial^\mu g^j \nabla^2 g^k + \tfrac{1}{4}\mathscr{C}_{ijkl}\partial_\mu g^i \partial^\mu g^j \partial_\nu g^k \partial^\nu g^l.$$

where $\lambda_a = \lambda_a(g) = \dfrac{\lambda_a^1}{\epsilon} + \dfrac{\lambda_a^2}{\epsilon^2} + \cdots$ etc

As usual, for each coupling we can define a beta-function (describing the response of the coupling to changes in the arbitrary renormalization scale)

$$\left(\varepsilon - \hat{\beta}^i \frac{\partial}{\partial g^i}\right)\lambda\cdot\mathscr{R} = \beta_\lambda\cdot\mathscr{R}$$

For example: $\chi^a_{ij} = (\epsilon - \hat{\beta}^k\partial_k)\mathscr{A}_{ij} - \mathscr{A}_{ik}\partial_j\hat{\beta}^k - \mathscr{A}_{jk}\partial_i\hat{\beta}^k$

By changing variables in functional integral

$$W[e^{-2\tau(x)}\gamma_{\mu\nu}(x), g^i(e^{-\tau(x)}\mu)] = W[\gamma_{\mu\nu}(x), g^i(\mu)]$$

so under infinitesimal $\tau \to \tau + \sigma$

$$\Delta_\sigma \tilde{W} = \Delta_\sigma W + \Delta_\sigma W_{\text{c.t.}} = \Delta_\sigma W_{\text{c.t.}} = \text{finite}$$

where $\quad \Delta_\sigma W_{\text{c.t.}} = W_{\text{c.t.}}[(1-2\sigma)\gamma_{\mu\nu}, g^i - \sigma\mu\, dg^i/d\mu] - W_{\text{c.t.}}[\gamma_{\mu\nu}, g^i],$

This imposes relations among the counterterms ("consistency conditions") and therefore among the corresponding "beta-functions."

For example, $\quad \frac{1}{2}\mathscr{A}_{ij}\nabla^2 g^i \nabla^2 g^j = \frac{1}{2}\mathscr{A}_{ij}\hat{\beta}^i\hat{\beta}^j(\nabla^2\tau)^2 + \cdots$

and $\quad \sqrt{-\gamma}\,(\lambda_a F + \lambda_b G + \lambda_c H^2) = 8\lambda_b\left[(\nabla^2\tau)^2 - (\partial_\mu\partial_\nu\tau)^2 + \cdots\right] + 4\lambda_c\left[(\nabla^2\tau)^2 + \cdots\right]$

gives $\quad 4\lambda_c \sim \frac{1}{2}\mathscr{A}_{ij}\hat{\beta}^i\hat{\beta}^j, \quad$ where ~ means up to finite terms, say X

Then for beta functions one obtains one of the JO consistency conditions:

$$8\beta_c = \chi^a_{ij}\beta^i\beta^j - \beta^i\partial_i X,$$

Where do N_I and S arise in this language? Ans: Renormalized Lagrangian needs additional terms

Say, if $$\mathcal{L}_0 = \tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi_{0a} \partial_\nu \phi_{0a} - \frac{1}{4!} g^0_{abcd} \phi_{0a}\phi_{0b}\phi_{0c}\phi_{0d}$$

need to add a counter-term $g^{\mu\nu}(N_I)_{ab}\partial_\nu g_I(\phi_{0b}\partial_\mu\phi_{0a})$ $(= \partial^\mu \phi_0^T N_I \partial_\mu g_I \phi_0$ in compact form)

In fact JO take $$\mathscr{L}_0 = \tfrac{1}{2}\gamma^{\mu\nu} D_{0\mu}\phi_{0a} D_{0\nu}\phi_{0a} + \tfrac{1}{8}(d-2)\phi_{0a}\phi_{0a} H - \tfrac{1}{4!} g^0_{abcd}\phi_{0a}\phi_{0b}\phi_{0c}\phi_{0d}$$

$$D_{0\mu}\phi_0 = (\partial_\mu + A_{0\mu})\phi_0, \qquad A_{0\mu} = A_\mu + N_I (D_\mu g)_I, \qquad D_\mu = \partial_\mu + A_\mu.$$

The background gauge field:

- Inserts N_I counterterm
- Elevates G_F symmetry (with action on spurions) to local symmetry
- Plays the role of a source for a *finite* operator, a scalar current

$$\begin{aligned}[D_\mu\phi\,\omega\,\phi] &= \omega_{ab}\frac{\delta \tilde{S}_0}{\delta A^\mu_{ab}} \\ &= D_\mu \phi_0^T(\omega + N_I(\omega g)_I)\phi_0\end{aligned}$$

Is B_I the new beta function?

$$\mu \frac{\partial g_I}{\partial \mu} = \beta_I(g)$$ gives the response of coupling to renormalization re-scaling

B_I does not *generally* satisfy this. It does satisfy, however

$$T^\mu_\mu = \beta_I(g)[\mathcal{O}_I] + \nabla_\mu J^\mu = B_I(g)[\mathcal{O}_I] \quad \text{up to EOMs}$$

Moreover, G_F symmetry of the effective action

$$\left[(\omega g)_I \frac{\delta}{\delta g_I} + (\omega \phi) \cdot \frac{\delta}{\delta \phi} \right] \Gamma = 0$$

can be combined with the RGE to give a new RGE but with

$$\beta_I \to \beta_I - (\omega g)_I, \qquad \gamma \to \gamma + \omega$$

$$\left[\mu \frac{\partial}{\partial \mu} + (\beta_I - (\omega g)_I) \frac{\delta}{\delta g_I} - ((\gamma + \omega)\phi) \cdot \frac{\delta}{\delta \phi} \right] \Gamma = 0$$

This is not a new beta function. It is a trick for solving equations.

Scheme change

Recall, w.f. renormalization determines only Z $\qquad \frac{1}{2}\partial^\mu \phi^T Z \partial_\mu \phi$

Square root is ambiguous: $\quad Z^{\frac{1}{2}} = O\tilde{Z}^{\frac{1}{2}}, \quad O^T O = 1 \qquad Z^{\frac{T}{2}} Z^{\frac{1}{2}} = \tilde{Z}^{\frac{T}{2}} \tilde{Z}^{\frac{1}{2}}$

This allows for different infinite subtractions: $O = 1 + \frac{O^1}{\epsilon} + \cdots \qquad (O^1)^T = -O^1$

This induces

$$\hat{\gamma} \to \hat{\gamma} - \omega, \qquad \hat{\beta}_I \to \hat{\beta}_I + (\omega g)_I, \qquad S \to S + \omega,$$

where $\omega = g_I \partial_I O^1$

Note that $\quad B_I = \beta_I - (Sg)_I, \quad \Gamma = \gamma + S$

are unambiguous ("gauge invariant").

There is a gauge in which $S = 0$ and B_I is a beta function.

SUSY

$$\mathscr{L} = \int d^4\theta\, \Phi_a^\dagger \Phi_a + \left(\int d^2\theta\, \frac{1}{3!} y_{abc} \Phi_a \Phi_b \Phi_c + \text{h.c.} \right)$$

Promote coupling to super-space dependent:

$$Y_{abc}(z) = y_{abc}(z) + \sqrt{2}\theta y^{\psi}_{abc}(z) + \theta^2 y^F_{abc}(z),$$

$$z^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$$

Only possible counterterm: $\mathscr{L}_{\text{c.t.}} = \int d^4\theta\, \Phi_a^\dagger F_{ab} \Phi_b,$ $\quad F_{ab}(Y, \overline{Y}) = F_{ba}(\overline{Y}, Y) = F^*_{ba}(Y, \overline{Y}).$

Expanding $\mathscr{L}_{\text{c.t.}} \supset \left((N_I)_{ab} \partial^\mu y_I - (N_I)^*_{ba} \partial^\mu y^*_I \right) \left(\phi^*_a \partial_\mu \phi_b - \partial_\mu \phi^*_a\, \phi_b \right)$

where $(N_I)_{ab} = \dfrac{\partial F_{ab}(y, y^*)}{\partial y_I}, \qquad (N_I)^*_{ba} = \dfrac{\partial F_{ab}(y, y^*)}{\partial y^*_I}$

Finally compute S:

$$S_{ab} \equiv -\tfrac{1}{2} (N^1_I)_{ab} y_I - \text{h.c.},$$

$$= -\frac{1}{2} \left(y_I \frac{\partial F^1_{ab}(y, y^*)}{\partial y_I} - y^*_I \frac{\partial F^1_{ab}(y, y^*)}{\partial y^*_I} \right)$$

$$F_{ab}(Y, \overline{Y}) = F_{ab}(e^{-i\alpha} Y, e^{i\alpha} \overline{Y}) \quad \Rightarrow \quad S = 0$$

THEN: We can have

$$\partial_\mu D^\mu = T^\mu_\mu - \partial_\mu V^\mu = 0 \qquad \text{scale invariance}$$

while

$$T^\mu_\mu = \partial_\mu V^\mu \neq 0 \qquad \text{no conformal symmetry}$$

A scale transformation together with a U(1) rotation is still a symmetry.

Now use trace anomaly.

Ignore for now divergence of *S*-current, fix later.

Ignore EOM (extra slide if needed)

Then the condition for scale invariance but not conformal can be casted as an algebraic condition on the beta function,

$$\beta_I = (Q\lambda)_I \neq 0$$

Same general model: $$V = \tfrac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d + (\tfrac{1}{2}y_{a|ij}\phi_a\psi_i\psi_j + \text{h.c.}).$$

Only candidate for virial current:

$$V^\mu = R_{ab}\partial^\mu\phi_a\phi_b + iP_{ij}\bar{\psi}_i\bar{\sigma}^\mu\psi_j \qquad Q^T = -Q, \quad P^\dagger = -P$$

Trace anomaly: $$T^\mu_\mu = -\frac{1}{4!}\big(\beta_{abcd} - (S\lambda)_{abcd}\big)\phi_a\phi_b\phi_c\phi_d - \frac{1}{2}\beta_{a|ij}\phi_a\psi_i\psi_j + \text{h.c.}$$

recall: $(S\lambda)_{abcd} = S_{ae}\lambda_{ebcd} + \cdots + S_{de}\lambda_{abce}$

Div of virial: $$\partial_\mu V^\mu = R_{ab}\partial^2\phi_a\phi_b - iP^\dagger_{ij}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j + iP_{ij}\partial_\mu\bar{\psi}_i\bar{\sigma}^\mu\psi_j$$

Using EOM and $T^\mu_\mu = \partial_\mu V^\mu \quad \Rightarrow \quad \beta_{abcd} - (S\lambda)_{abcd} = (R\lambda)_{abcd}$

$$\beta_{abcd} = -Q_{a'a}\lambda_{a'bcd} - Q_{b'b}\lambda_{ab'cd} - Q_{c'c}\lambda_{abc'd} - Q_{d'd}\lambda_{abcd'}$$

$$\beta_{a|ij} = -Q_{a'a}y_{a'|ij} - P_{i'i}y_{a|i'j} - P_{j'j}y_{a|ij'}$$

For short, 1st equation: $\beta_I = (Q\lambda)_I$

Scale IS Conformal

Legal Disclosure: In $d = 4$, unitary, local, renormalizable, perturbative, interacting QFT with well defined correlators of stress-energy tensor

Recall: scale invariance $\partial_\mu D^\mu = T^\mu_\mu - \partial_\mu V^\mu = 0$

no conformal symmetry $T^\mu_\mu = \partial_\mu V^\mu \neq 0$

Use now correct form of Trace Anomaly

$$T^\mu_\mu = \beta_I(g)[\mathcal{O}_I] + \partial_\mu J^\mu \qquad \Big(= (\beta_I - (Sg)_I)[\mathcal{O}_I] = B_I[\mathcal{O}_I] \Big)$$

If the virial is $V^\mu = \partial^\mu \phi^T R\, \phi$ (we reserve Q for solutions to $\beta_I = (Q\lambda)_I$)

the condition for scale but not conformal is

$$B_I = (Rg)_I \neq 0$$

But this gives B_I in the Lie algebra of G_F and by the preceding proof $B_I = 0$ and hence $(Rg)_I = 0$

(R may not vanish but then it generates a symmetry)

Hence: scale + Poincare + unitarity + no-nonsense ⇒ conformal