“AdS/CFT and local renormalization group”,

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Aim: to explain

- **What is local renormalization group(RG)?**
- **How is that described in the AdS/CFT correspondence?**

References

K.Kikuchi and T.S., PTEP(2016) 033B02, arXiv:1511.00403,
and related papers

KMI topics.
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§1. local RG

What is conformal perturbation theory?

- $S_* = \int d^4x\, \mathcal{L}_*$: fixed point Lagrangian, 4d conformal field theory(CFT)
- $\mathcal{O}_i(x)$: a set of scalar primaries with $\Delta = 4$. It follows from conformal symmetry that

$$\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle_{S_*} = \frac{\delta_{ij}}{|x|^8} \ .$$

- perturbed Lagrangian

$$S_\lambda = S_* + \int d^4x\, \lambda^i \mathcal{O}_i(x) \ .$$

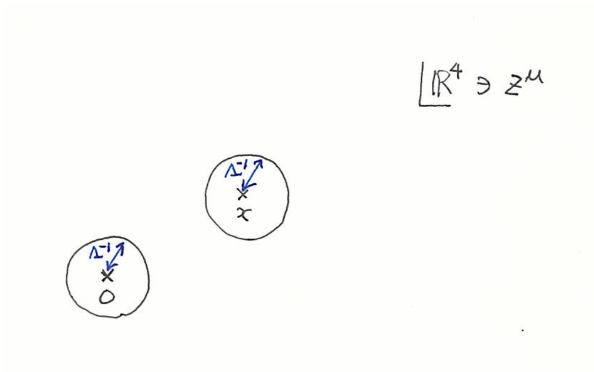
- The perturbed two-point function becomes

$$\begin{aligned} \langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle_{S_\lambda} &= \left\langle \mathcal{O}_i(x) \mathcal{O}_j(0)\, e^{-\int d^4z\, \lambda^k \mathcal{O}_k} \right\rangle_{S_*} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \mathcal{O}_i(x) \mathcal{O}_j(0) \left(\int d^4z\, \lambda^k \mathcal{O}_k \right)^n \right\rangle_{S_*} \ . \end{aligned}$$

- $\int d^4z_1 \cdots d^4z_n$ gives rise to UV divergences due to short distance singularities of OPE

$$\mathcal{O}_i(x)\mathcal{O}_k(z) \sim \sum_l \frac{C_{ikl}}{|x-z|^{\Delta_i+\Delta_k-\Delta_l}}\mathcal{O}_l(z) \ .$$

- UV cut-off Λ is necessary to remove a tiny disc around $z = x, 0$.



- The perturbed two-point function becomes

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle_{S_\lambda} = \frac{1}{|x|^8}\left(\delta_{ij} - 2C_{ijk}\lambda^k \log(\Lambda^2|x|^2) + \mathcal{O}(\lambda^2)\right) \ .$$

$\mathcal{O}_i$ get renormalized as

$$\mathcal{O}_i \to Z_i{}^j \mathcal{O}_j = \left(\delta_i^j + C_{ijk}\lambda^k \log \Lambda^2 + \mathcal{O}(\lambda^2)\right) \mathcal{O}_j \ .$$

This yields the anomalous dimension of $\mathcal{O}_i$

$$\gamma_i{}^j = (Z^{-1})_i{}^k \frac{d}{d \log \Lambda} Z_k{}^j = 2C_{ijk}\lambda^k + \mathcal{O}(\lambda^2) \ .$$

- renormalization of λ^i

$$\lambda_i \to \lambda_i + C_{ijk}\lambda^j \lambda^k \log \Lambda + \mathcal{O}(\lambda^3) \ .$$

This gives the β-function of λ^i

$$\beta^i = \frac{d\lambda_i}{d \log \Lambda} = C_{ijk}\lambda^j \lambda^k + \mathcal{O}(\lambda^3) \ .$$

local RG:

- λ^i are promoted to spacetime-dependent function $\lambda^i(x)$.
  - ◇ enabling one to study RG flows of the generating functional of correlators
- The perturbed theory remains renormalizable.
  - ◇ UV divergences come from short distance singularities of OPE even in the presence of spacetime-dependent couplings $\lambda^i(x)$.
  - ◇ The UV divergences are thus local, being proportional to local functional of $\lambda^i(x)$.
  - ◇ They can be removed by a finite number of local counter terms.
- renormalized generating functional $\Gamma[\lambda^i(x), \mu]$

$$\Gamma_0[\lambda^i_{(0)}(x), \Lambda] = \Gamma[\lambda^i(x), \mu] + S_{\mathrm{div}}[\lambda^i(x), \frac{\mu}{\Lambda}] \ .$$

The renormalized correlators are computed as

$$\frac{\delta}{\delta\lambda^{i_1}(x_1)}\frac{\delta}{\delta\lambda^{i_2}(x_2)}\cdots\frac{\delta}{\delta\lambda^{i_n}(x_n)}\Gamma[\lambda^i(x), \mu] = \langle \mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\cdots\mathcal{O}_{i_n}(x_n)\rangle_\lambda \ .$$

- local RG equation

$$
\begin{aligned}
0 &= \left(\mu\frac{\partial}{\partial\mu}\right)_0 \Gamma_0[\lambda^i_{(0)}(x), \Lambda] \\
&= \left(\mu\frac{\partial}{\partial\mu} + \int d^4x \left(\mu\frac{\partial\lambda^i(x)}{\partial\mu}\right)_0 \frac{\delta}{\delta\lambda^i(x)}\right)\Gamma[\lambda^i(x), \mu] - \int d^4x\mathcal{A}[\lambda(x)] \ .
\end{aligned}
$$

Here $\mathcal{A}$ is a local functional of $\lambda^i(x)$. The local RG equation of the correlators reads

$$
\begin{aligned}
&\left(\mu\frac{\partial}{\partial\mu} + \int d^4x\, \beta^i[\lambda(x)]\frac{\delta}{\delta\lambda^i(x)}\right)\langle\mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\cdots\mathcal{O}_{i_n}(x_n)\rangle_\lambda \\
&\qquad + \sum_k \gamma_{i_k}{}^l[\lambda(x_k)]\langle\mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_l(x_k)\cdots\mathcal{O}_{i_n}(x_n)\rangle_\lambda = 0 \ .
\end{aligned}
$$

Here,

$$
\beta^i = \left(\mu\frac{\partial}{\partial\mu}\right)_0 \lambda^i(x) \ , \quad \gamma_i{}^j = \frac{\partial\beta^j}{\partial\lambda^i} \ ,
$$

denote the β-function and the anomalous dimension matrix, respectively.

- Local RG equation can be generalized by considering spacetime-dependent RG scale
  - define a field theory on a curved background of metric $\gamma^{(0)}_{\mu\nu}(x)$.
  - The resultant theory is left renormalizable.
  - Renormalized metric $\gamma_{\mu\nu}(x)$ differs from $\gamma^{(0)}_{\mu\nu}(x)$ only by a Weyl factor that is equal to the spacetime-dependent renormalization scale.
  - The unintegrated form of local RG equation follows

$$\left(2\gamma_{\mu\nu}(x)\frac{\delta}{\delta\gamma_{\mu\nu}(x)}+\beta^i[\lambda(x)]\,\frac{\delta}{\delta\lambda^i(x)}\right)\Gamma[\gamma_{\mu\nu}(x),\lambda^i(x)]=\sqrt{\gamma}\,\mathcal{A}[\gamma_{\mu\nu}(x),\lambda^i(x)]\ .$$

  - global scale transformation is reproduced as

$$\mu\frac{\partial}{\partial\mu}=\int d^4x\,2\gamma_{\mu\nu}(x)\frac{\delta}{\delta\gamma_{\mu\nu}(x)}\ .$$

- Using

$$\langle T^{\mu\nu}(x)\rangle_{\gamma,\lambda} = \frac{2}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma_{\mu\nu}(x)}\Gamma[\gamma_{\mu\nu}(x),\lambda^i(x)] \ ,$$

we obtain the trace anomaly

$$\begin{aligned}\gamma_{\mu\nu}\langle T^{\mu\nu}(x)\rangle_{\gamma,\lambda} &= -\beta^i\,\frac{1}{\sqrt{\gamma}}\,\frac{\delta\Gamma}{\delta\lambda^i(x)} + \mathcal{A}[\gamma_{\mu\nu}(x),\lambda^i(x)] \\ &= -\beta^i\,\langle\mathcal{O}_i(x)\rangle_{\gamma,\lambda} + \mathcal{A}[\gamma_{\mu\nu}(x),\lambda^i(x)] \ .\end{aligned}$$

- β^i and $\mathcal{A}$ are constrained by the Wess-Zumino consistency condition

Osborn '91, Jack-Osborn '13

$$0 = [\delta_x,\delta_y]\Gamma = \delta_x(\sqrt{\gamma}\mathcal{A}(y)) - \delta_y(\sqrt{\gamma}\mathcal{A}(x)) \ .$$

Here

$$\delta_x = 2\gamma_{\mu\nu}(x)\frac{\delta}{\delta\gamma_{\mu\nu}(x)} + \beta^i[\lambda(x)]\,\frac{\delta}{\delta\lambda^i(x)} \ ,$$

is the local RG generator.

- For $d = 2$, the WZ condition is powerful enough to give another proof of Zamolodchikov's c-theorem.
  - ◇ Osborn constructed a function $c(\mu)$ such that

$$\mu \frac{d}{d\mu} c(\mu) = 3 G_{ij}(\lambda) \beta^i \beta^j \ ,$$

  - ◇ G_{ij} is defined as

$$G_{ij}(\lambda) = \langle \mathcal{O}_i(1) \mathcal{O}_j(0) \rangle_\lambda \ .$$

    This is positive-definite thank to unitarity of CFT. Thus

$$\mu \frac{d}{d\mu} c(\mu) \geq 0 \ .$$

- Unfortunately, the same technique cannot be employed to prove c-theorem in $d \geq 3$ dimensions.

§2. Local RG flows in AdS/CFT

AdS/CFT correspondence

$$\text{classical gravity in AdS}_5 \quad \Longleftrightarrow \quad \text{CFT in } d=4$$

- GKP-Witten allows one to compute the CFT correlators from classical AdS_5 gravity.
- Local RG flows from a UV fixed point can be described in 5d gravity; **holographic RG**

Consider a $d=5$ gravity coupled to scalar fields

$$2\kappa_5^2\,\mathcal{L}_{5\mathrm{d}} = -R + \frac{1}{2}G^{MN}\partial_M\phi^i\partial_N\phi^j\,L_{ij}(\phi) + V(\phi)\ .$$

- 5d metric can be gauge-fixed to be

$$ds^2 = d\tau^2 + g_{\mu\nu}(x,\tau)dx^\mu dx^\nu\ .$$

Example: AdS_5 metric is given by setting $g_{\mu\nu}(x,\tau) = e^{-2\tau/l}\eta_{\mu\nu}$ with the AdS boundary at $\tau = -\infty$.

- Solve the classical EOM under the boundary condition

$$g_{\mu\nu}(x,\tau=-\infty)=\gamma^{(0)}_{\mu\nu}(x)\ ,\quad \phi^i(x,\tau=-\infty)=\lambda^i_{(0)}(x)\ ,$$

  and then substitute them into the 5d action to obtain the generating functional of CFT:

$$S_0[\gamma^{(0)}_{\mu\nu},\lambda^i_{(0)}(x)]=\int d^4x\int_{-\infty}^{\infty}d\tau\sqrt{G}\,\mathcal{L}_{5\mathrm{d}}\big|_{\mathrm{EOM}}\ .$$

5d gravity		4d CFT
$\gamma^{(0)}_{\mu\nu}(x)$	$\Longleftrightarrow$	background metric
$\lambda^i_{(0)}(x)$	$\Longleftrightarrow$	coupling to $\mathcal{O}_i$
		$\lambda^i_{(0)}(x)\mathcal{O}^{(0)}_i$

- The τ integral diverges at the boundary $\tau=-\infty$. A regularization must be made

$$S_0[\gamma^{(0)}_{\mu\nu},\lambda^i_{(0)}(x)]=\int d^4x\int_{\log\epsilon}^{\infty}d\tau\sqrt{G}\,\mathcal{L}_{5\mathrm{d}}\big|_{\mathrm{EOM}}\ .\quad \text{(cut-off } \epsilon\ll 1)$$

The boundary conditions are imposed at $\tau = \log\epsilon$ instead

$$g_{\mu\nu}(x, \tau = \log\epsilon) = \gamma^{(0)}_{\mu\nu}(x) \ , \quad \phi^i(x, \tau = \log\epsilon) = \lambda^i_{(0)}(x) \ .$$

ϵ is regarded as a UV cut-off in CFT, and , $\gamma^{(0)}_{\mu\nu}(x)$ and $\lambda^i_{(0)}(x)$ as bare couplings.

- Divide the bare S_0 into two terms as

$$\begin{aligned} S_0[\gamma^{(0)}_{\mu\nu}, \lambda^i_{(0)}(x)] &= \int d^4x \int_{\log\epsilon}^{\infty} d\tau' \sqrt{G}\,\mathcal{L}_{5\mathrm{d}}\big|_{\mathrm{EOM}} \\ &= \int d^4x \left(\int_{\log\epsilon}^{\tau} + \int_{\tau}^{\infty}\right) d\tau' \sqrt{G}\,\mathcal{L}_{5\mathrm{d}}\big|_{\mathrm{EOM}} \\ &= S_{\mathrm{div}} + S[g_{\mu\nu}(x,\tau), \phi^i(x,\tau)] \ . \end{aligned}$$

- $S[g_{\mu\nu}(x,\tau), \phi^i(x,\tau)]$ is the renormalized generating functional with τ being the renormalization scale. It is evident that S_0 is independent of τ.
- $g_{\mu\nu}(x,\tau), \phi^i(x,\tau)$ are the solutions of EOM and identified with the running couplings

$$\gamma_{\mu\nu}(x,\tau) = g_{\mu\nu}(x,\tau) \ , \quad \lambda^i(x,\tau) = \phi^i(x,\tau) \ ,$$

- A sketch of how to solve 5d EOM when λ^i are independent of x^μ:
  5d metric is taken to be

$$ds^2 = d\tau^2 + (\mu(\tau))^2 \eta_{\mu\nu} dx^\mu dx^\nu \ .$$

  $\mu = \mu(\tau)$ is identified with an RG scale.
  With an ansatz $\phi^i = \phi^i(\mu(\tau))$, EOM of ϕ can be integrated to be

$$\mu \frac{d}{d\mu} \phi^i = -\frac{6}{W(\phi)} L^{ij}(\phi) \frac{\partial W(\phi)}{\partial \phi^j} \equiv \beta^i \ ,$$

  if the scalar potential $V(\phi)$ is written in terms of a "superpotential" $W(\phi)$

$$V(\phi) = -\frac{1}{3} W(\phi)^2 + \frac{1}{2} L^{ij} \frac{\partial W}{\partial \phi^i} \frac{\partial W}{\partial \phi^j} \ .$$

- To summarize,

4d CFT		5d gravity
RG scale	$\Longleftrightarrow$	5th coordinate
RG flow	$\Longleftrightarrow$	profile of solutions of EOM

Derivation of local RG equation in AdS/CFT

Hamilton-Jacobi formulation of 5d gravity is useful

dVV '99
Fukuma-Matsuura-T.S. '00

- ADM decomposition of 5d metric with τ regarded as Eucledean time

$$ds^2 = N^2 d\tau^2 + g_{\mu\nu}(x,\tau)(dx^\mu + \lambda^\mu d\tau)(dx^\nu + \lambda^\nu d\tau) \ .$$

- Hamiltonian and momentum constraint

$$\mathcal{H} = \mathcal{P}^\mu = 0 \ .$$

- The renormalized generating functional S is an on-shell action of 5d gravity, obeying the HJ equation

$$\frac{\partial S}{\partial \tau} = 0 \ , \quad \frac{1}{\sqrt{\gamma}} \frac{\delta S}{\delta \gamma_{\mu\nu}(x,\tau)} = -\Pi^{\mu\nu}(x,r) \ , \quad \frac{1}{\sqrt{\gamma}} \frac{\delta S}{\delta \phi^i(x,\tau)} = -\Pi_i(x,r) \ .$$

- Substituting these into the Hamiltonian constraint gives the flow equation

$$\{S, S\} = \mathcal{L}_4 \ ,$$

from which we obtain the local RG equation

$$\left(2\gamma_{\mu\nu}(x) \frac{\delta}{\delta\gamma_{\mu\nu}(x)} + \beta^i[\lambda(x)] \, \frac{\delta}{\delta\lambda^i(x)} \right) \Gamma = \sqrt{\gamma} \, \mathcal{A} \ .$$

Here

$$S = \Gamma + S_{\rm loc} \ , \quad \beta^i = -\frac{6}{W} \, L^{ij} \frac{\partial W}{\partial \phi^j} \ .$$

- The relation

$$V(\phi) = -\frac{1}{3} W(\phi)^2 + \frac{1}{2} L^{ij} \, \frac{\partial W}{\partial \phi^i} \frac{\partial W}{\partial \phi^j} \ .$$

follows immediately from the flow equation.
- $\mathcal{A}$ can be fixed from $\mathcal{L}_5$; a most efficient framework for computing a holographic trace anomaly.

work of Kikuchi and T.S.

extension of holographic RG by incorporating 5d gauge fields $A_M^a(X)$ in HJ formulation

5d gravity		4d CFT
gauge group G	$\Longleftrightarrow$	flavor group G
boundary value of $A_\mu^a(x, \tau = \log \epsilon)$	$\Longleftrightarrow$	background gauge potential $A_\mu^a J^{\mu a}$

- A simple derivation of Ward-Takahashi identity

$$\boldsymbol{\nabla}_\mu \langle J^{\mu a} \rangle_{\gamma,\phi,A} - i(T^a \phi)^i \langle \mathcal{O}_i \rangle_{\gamma,\phi,A} = 0 \ ,$$

from the Gauss law constraint

$$\frac{1}{\gamma} \frac{\delta S}{\delta A_\tau^a} = \boldsymbol{\nabla}_\mu \Pi^{\mu a} - i(T^a \phi)^i \Pi_i = 0 \ .$$

- derivation of vector β-function of A_μ^a

$$\left(\mu\frac{\partial}{\partial\mu}\right)_0 A_\mu^a(x) = \rho_i^a\,\boldsymbol{\nabla}_\mu\phi^i\ .$$

  RG flows of A_μ^a driven only by flavor nonsinglet perturbations
- a proof of holographic c-theorem in $d=4$ on the basis of the trace anomaly coefficients