Recent progress in lattice fermion formulations

Tatsuhiro MISUMI (*Keio U.*)

07/17/2013 JICFuS seminar@KMI Nagoya U.

Lattice fermions

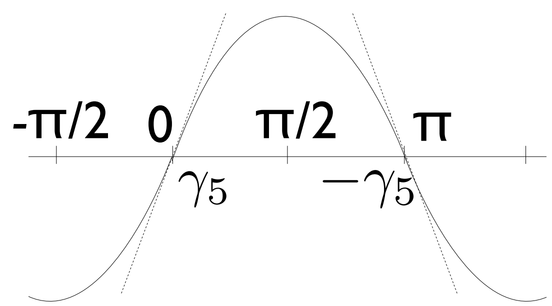
- Doubling problem : Naive chiral&local fermion → 16 species

$$S_{\mathrm{N}} = \sum_n \left[\frac{a^3}{2} \bar{\psi}_n \gamma_\mu (U_{n,\mu}\psi_{n+\mu} - U^\dagger_{n-\mu,\mu}\psi_{n-\mu}) + a^4 m \bar{\psi}_n \psi_n \right]$$

Free propagator

$$D^{-1}(pa) = \frac{-i\gamma_\mu \sin ap_\mu + am}{\sin^2 ap_\mu + a^2 m^2} \rightarrow \frac{1}{a} \underline{\sum_{p_\mu = 0, \pi/a}} \frac{-i(-1)^{\delta_\mu}\gamma_\mu \hat{p}_\mu + m}{\hat{p}^2_\mu + m^2}$$

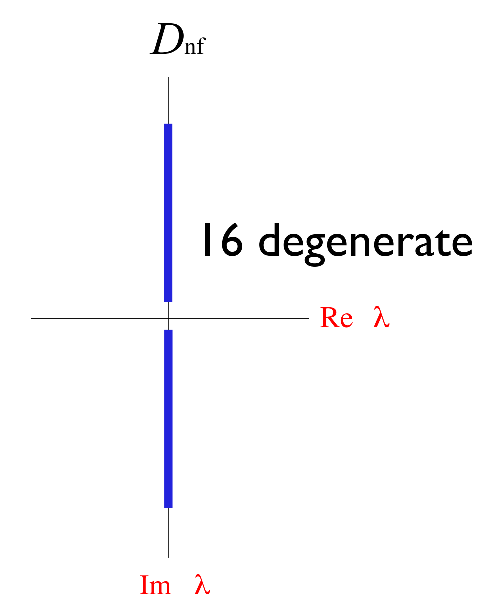
2 poles per dim. → 16 doublers in 4d



Nielsen-Ninomiya

Chiral symmetry v.s. desirable flavor number

	flavors	chiral	tuning	artifact
Wilson:	1	0	severe	O(a)
Staggered:	4	1	N/A	O(a^2)
Domain-wall	1	(1)	easy	O(a^2)
Overlap	1	1	N/A	O(a^2)



Dirac spectrum

Why lattice fermions?

(1) Radical improvement of lattice simulations

Wilson : O(a) errors & bad chiral → Symanzik:O(a)-improving, Smearing:UV-filter

Staggered : taste breaking at O(a^2) → HISQ : O(a^2)-improving, UV-filter

Domain-wall, Overlap : Numerical cost & more → Fixed topology, Reweighting...

New formulations have possibility to eliminate them.
Even if not, one can enjoy feedbacks.

(2) Further understanding on lattice field theory

Further variety of Ginsparg-Wilson fermions ?

Other ways of keeping chiral symmetry on the lattice ?

Possible new setups

1. Flavored mass

New Wilson and overlap fermions → O(a) error reduction

CPU time reduction

2. Central branch

Wilson w/o additive renorm. → Chiral symmetry (No fine-tuning)

O(a) improved

3. Two-flavor chiral fermion

Chiral two-flavor w/ ultra locality, based on 6D clifford algebra

→ Chiral symmetry

4D-Rotational, C, P, T invariance

1. Flavored mass

◆Wilson fermion : species-splitting by mass

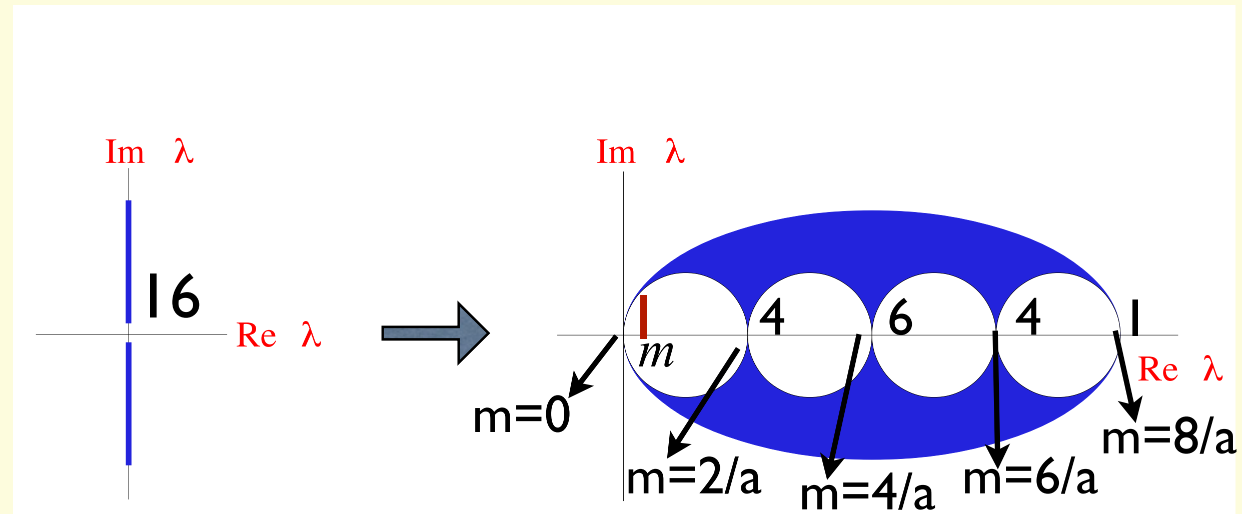
$$S_{\mathrm{N}} \;+\; S_W = \frac{a^5}{2}\bar{\psi}_n(2\psi_n - \psi_{n+\mu} - \psi_{n-\mu})$$

$$\Rightarrow \; D_W(p) = \frac{1}{a}\sum_\mu [i\gamma_\mu \sin ap_\mu \;+\; \underline{(1-\cos ap_\mu)}]$$ Flavored mass

Physical (0,0,0,0) : $D_W(p) = i\gamma_\mu p_\mu \;+\; O(a)$

Doubler(π/*a*,0,0,0) : $D_W(p) = i\gamma_\mu p_\mu \;+\; \frac{2}{a} \;+\; O(a)$

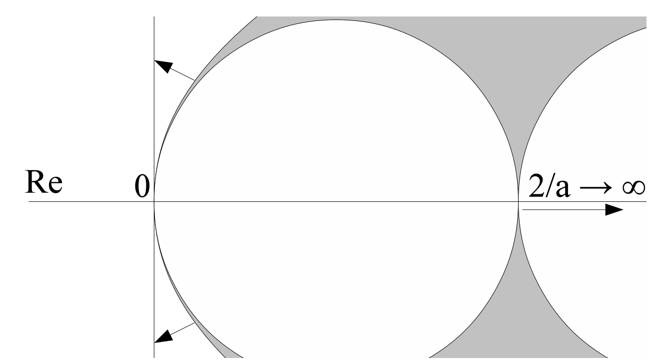
Only one flavor is massless, while others have 1/a mass.



◆ 15 species are decoupled → doubler-less

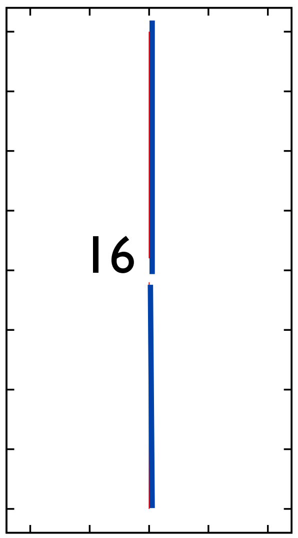
◆ 1/*a* additive mass renormalization → Fine-tune

◆ Overlap formula and GW symmetry → costs

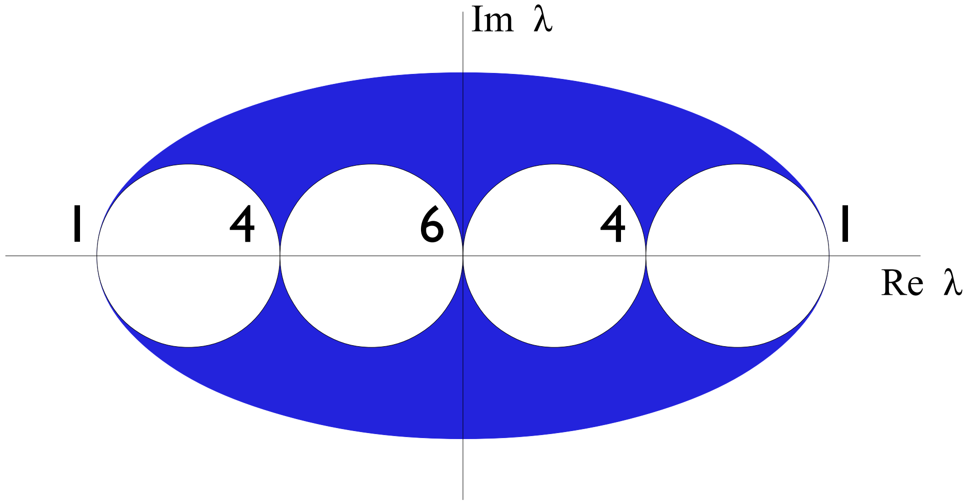


The only way of species-splitting ?

Naive



Wilson



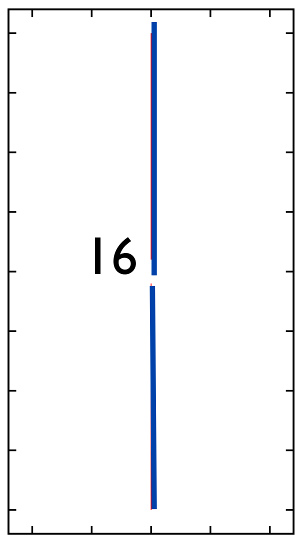
$$\sum_{\mu} C_{\mu}$$

$$C_{\mu} = (T_{+\mu} + T_{-\mu})/2$$
$$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$$

U(4)×U(4)

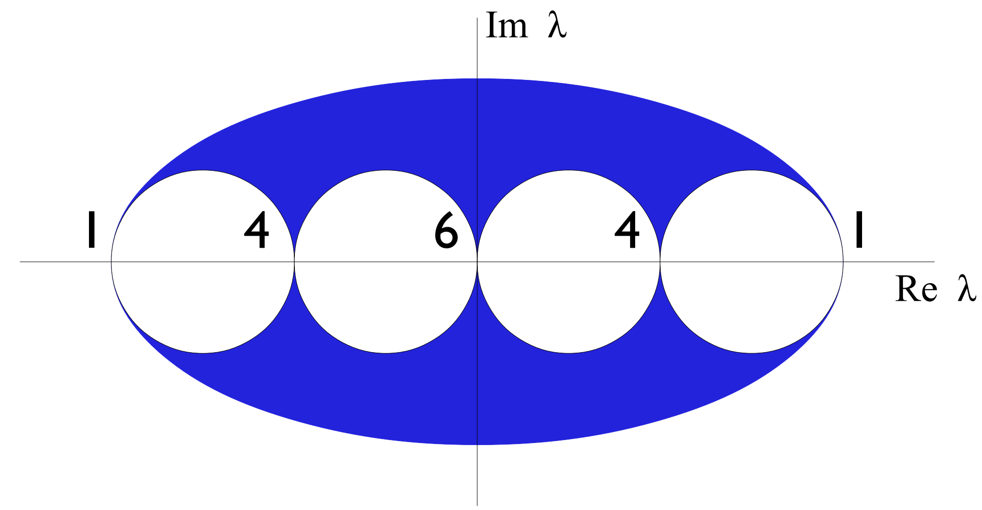
Naive

Wilson



$$\sum_\mu C_\mu$$

$$C_\mu = (T_{+\mu} + T_{-\mu})/2$$
$$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$$

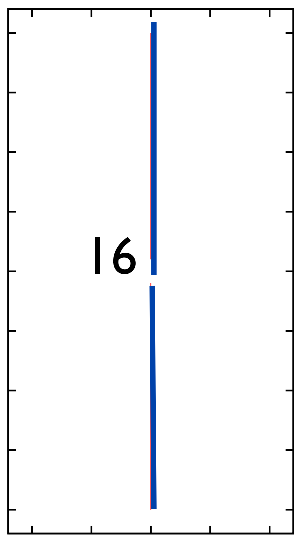


U(4)×U(4)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\,,\ (-1)^{n_\mu} i\gamma_\mu\gamma_5\,,\ (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2} \right\}$$
$$\Gamma_X^{(-)} \in \left\{ (-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,,\ \gamma_5\,,\ (-1)^{n_\mu}\gamma_\mu\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\gamma_5\,,\ (-1)^{\check{n}_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2} \right\}$$

$$\psi_n \to \psi'_n = \exp\left[\mathrm{i}\sum_X \left(\theta_X^{(+)}\Gamma_X^{(+)} + \theta_X^{(-)}\Gamma_X^{(-)}\right)\right]\psi_n\,, \quad \bar\psi_n \to \bar\psi'_n = \bar\psi_n \exp\left[\mathrm{i}\sum_X\left(-\theta_X^{(+)}\Gamma_X^{(+)} + \theta_X^{(-)}\Gamma_X^{(-)}\right)\right]$$

Naive

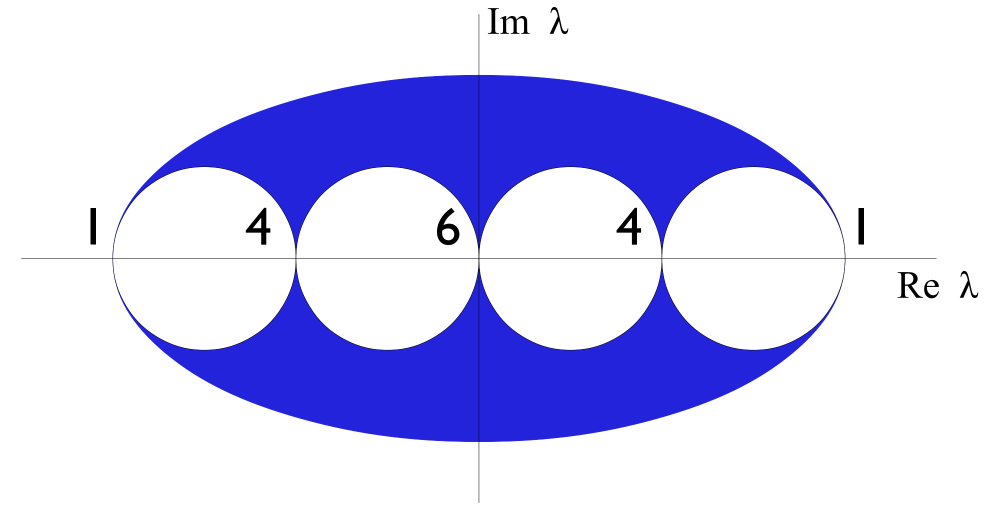


$$\sum_{\mu} C_\mu$$

$$C_\mu = (T_{+\mu} + T_{-\mu})/2$$

$$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$$

Wilson



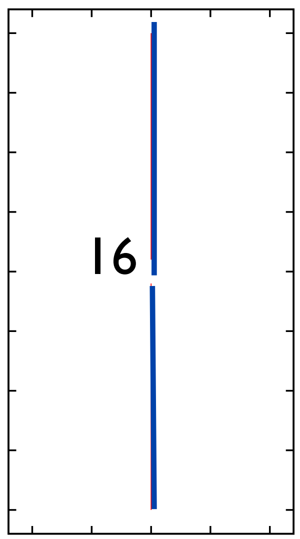
U(4)×U(4) U(1)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4\,, \qquad \right\}$$

$$\Gamma_X^{(-)} \in \left\{ \qquad \right\}$$

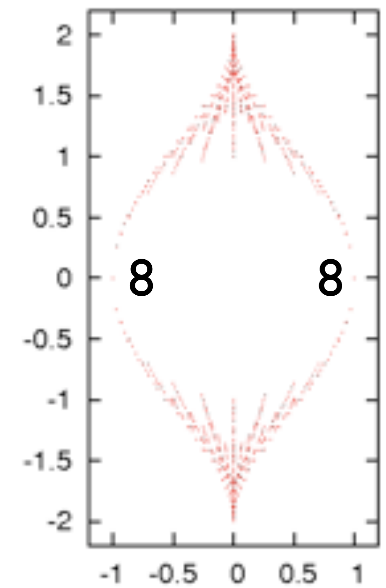
Naive

Wilson'



$$\sum_{sym.} C_1 C_2 C_3 C_4$$

$$C_\mu = (T_{+\mu} + T_{-\mu})/2$$
$$T_{\pm\mu}\psi_n = U_{n,\pm\mu}\psi_{n\pm\mu}$$



U(4)×U(4) U(2)×U(2)

$$\Gamma_X^{(+)} \in \left\{ \mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,, \qquad (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2} \right\}$$

$$\Gamma_X^{(-)} \in \left\{ \qquad \right\}$$

◆ Naive flavored mass

Creutz, Kimura, TM (10)

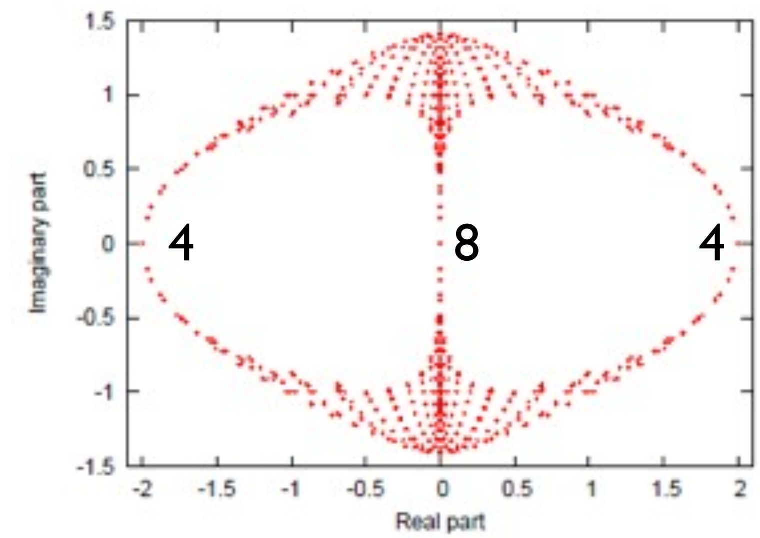
$$M_{\rm V} = \sum_{\mu} C_{\mu},$$ *Vector (1-link)*

$$M_{\rm T} = \sum_{perm.} \sum_{sym.} C_{\mu} C_{\nu},$$ *Tensor (2-link)*

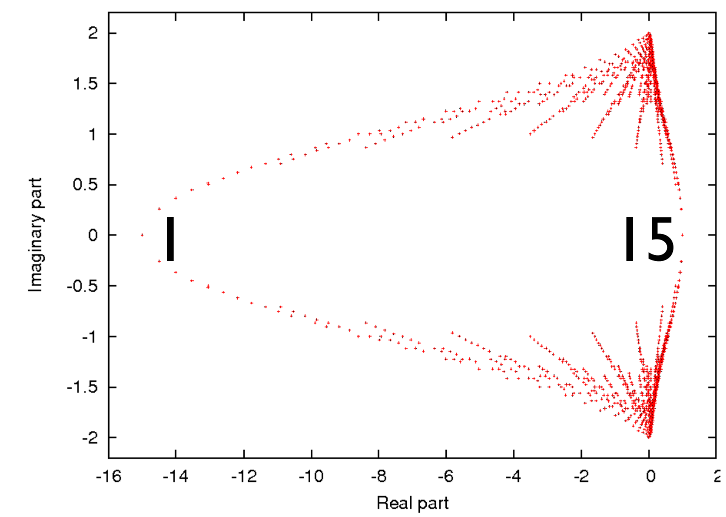
$$M_{\rm A} = \sum_{perm.} \sum_{sym.} \prod_{\nu} C_{\nu},$$ *Axial-V (3-link)*

$$M_{\rm P} = \sum_{sym.} \prod_{\mu=1}^{4} C_{\mu},$$ *Pseudo-S (4-link)*

Dirac op. eigenvalues



$M_{\rm T}$



$M_{\rm V}+M_{\rm T}+M_{\rm A}+M_{\rm P}$

- *Hypercubic sym, gamma5-hermiticity, C, P, T...*
- *Lattice laplacian* $\sum_{n} \bar{\psi}_n (M_P - 1)\psi_n \;\to\; -a\int d^4x \bar{\psi}(x) D_{\mu}^2 \psi(x) + O(a^2)$
- *Generalized Wilson fermions*

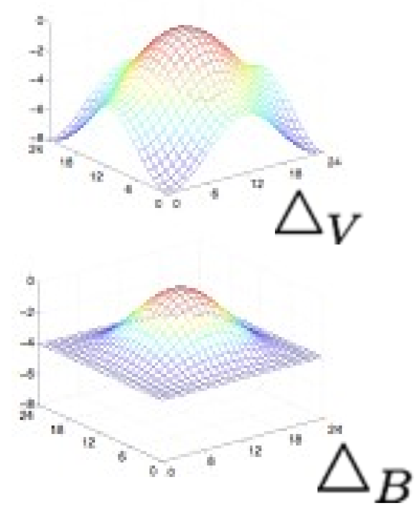
§ Brillouin fermion

Durr, Koutsou (11)(12) Creutz, Kimura, TM (10)

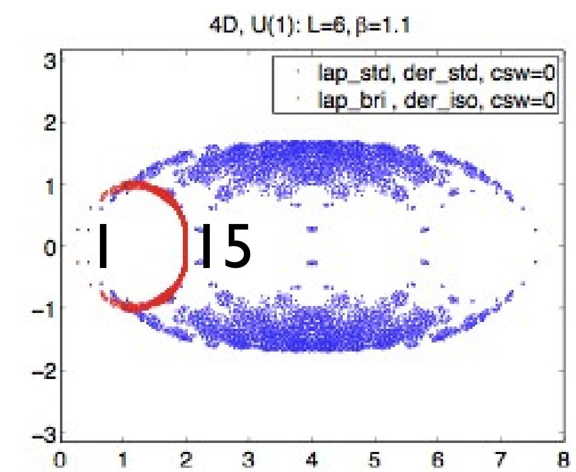
$$M_{\rm V} \;\rightarrow\; M_{\rm B} = M_{\rm V} + M_{\rm T} + M_{\rm A} + M_{\rm P}$$

$$\triangle_V \;:\; 4 - \cos p_1 - \cos p_2 - \cos p_3 - \cos p_4$$

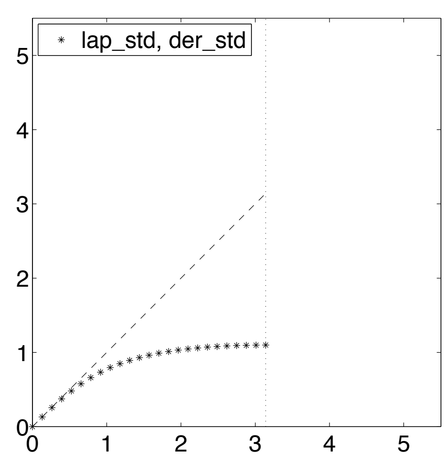
$$\triangle_B \;:\; 4 - 4\cos^2\frac{p_1}{2}\cos^2\frac{p_2}{2}\cos^2\frac{p_3}{2}\cos^2\frac{p_4}{2}$$



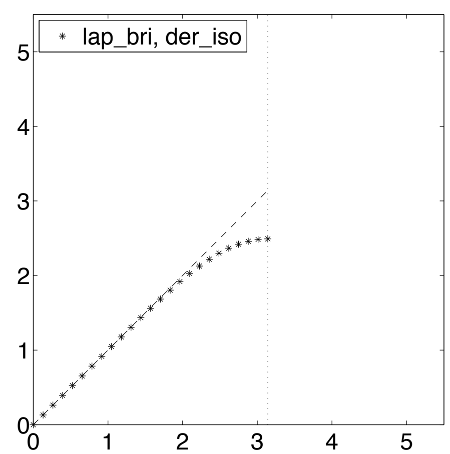
Dirac op. eigenvalues



- Better dispersion relation

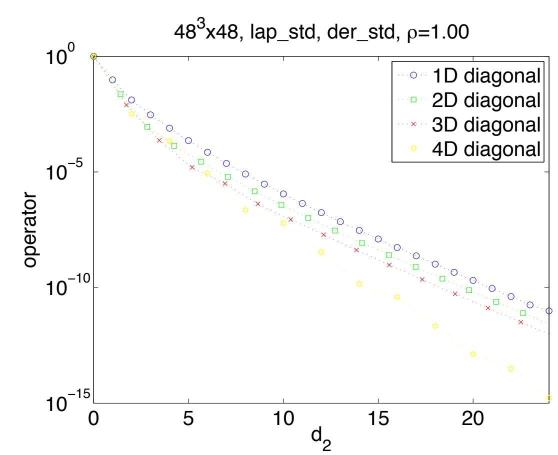


Wilson

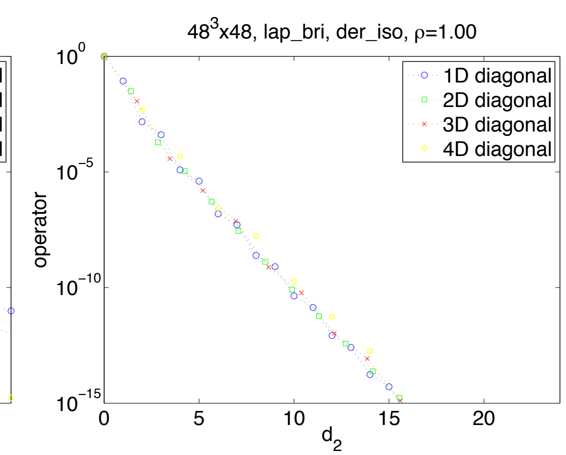


Brillouin

- Locality of overlap operator



Wilson kernel

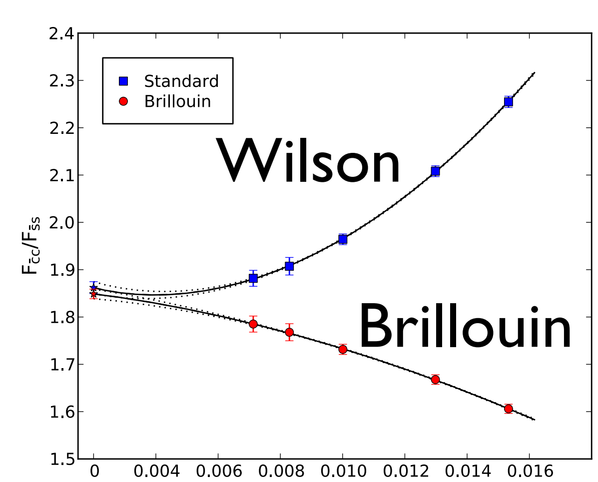


Brillouin kernel

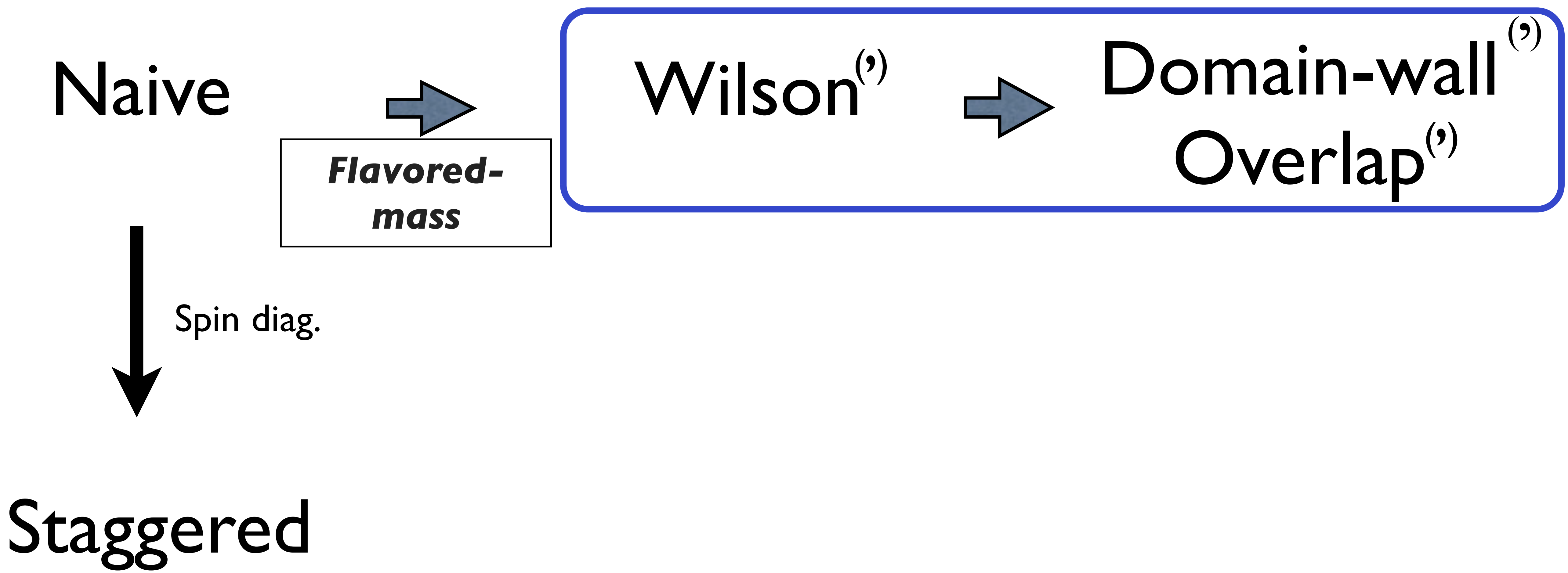
- Good scaling in heavy mass region

$$F_{\bar{c}c}/F_{\bar{s}s} = d_0 + \underbrace{d_1}_{\sim 0}\alpha(a)a + \underbrace{d_2}_{\sim 0}a^2$$

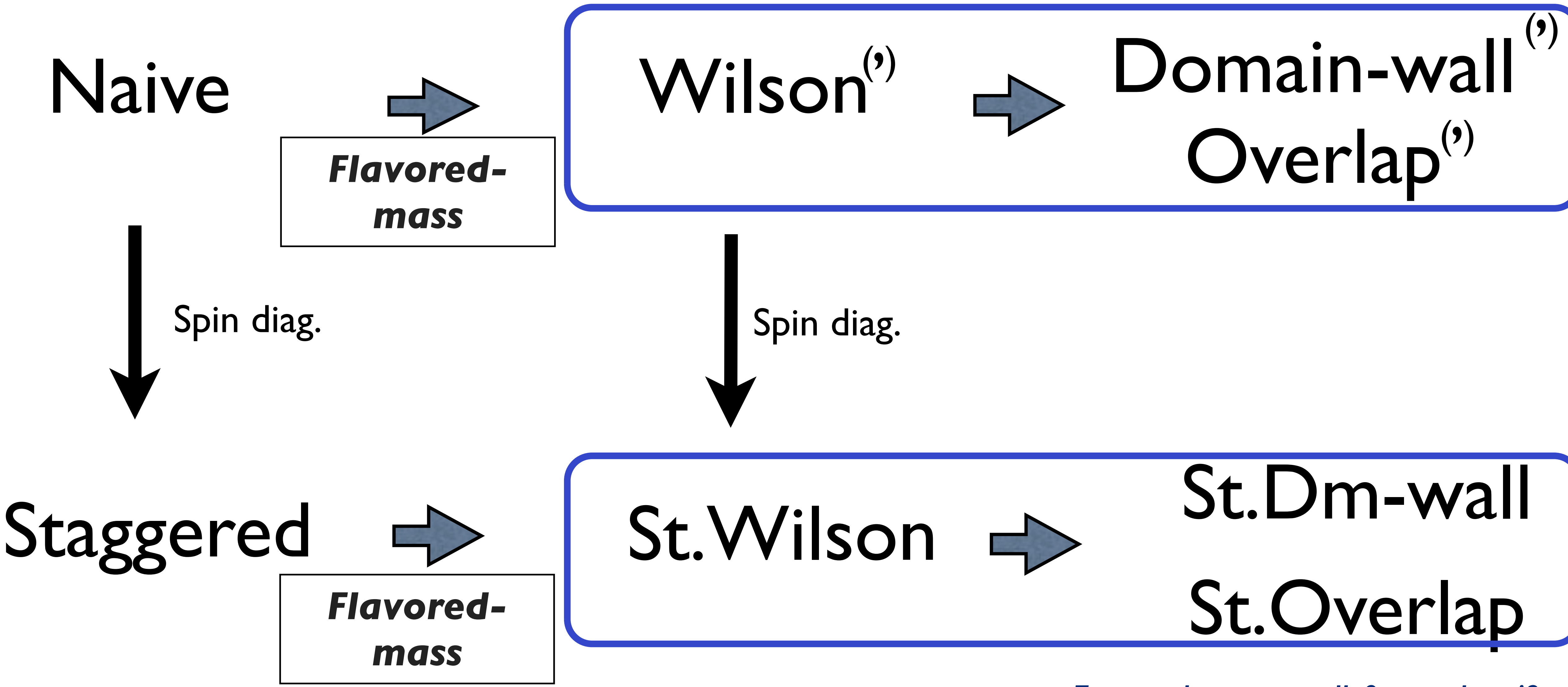
⇨ would suit heavy-quark system



○ Heavy PS meson decay constant
- Quenched LQCD
- Wilson gauge action
- Clover action csw=1
- APE-smeared



Naive
Wilson(')
Domain-wall (')
Overlap(')
Flavored-
mass
Spin diag.
Staggered



Faster domain-wall & overlap !?

◆Staggered fermion

Spin diagonalization : $\psi_n = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\chi_n, \quad \bar{\psi}_n = \bar{\chi}_n\gamma_4^{n_4}\gamma_3^{n_3}\gamma_2^{n_2}\gamma_1^{n_1}$

$$S_N \rightarrow S_{st} = \frac{1}{2}\sum_{n,\mu}\eta_\mu(n)\bar{\chi}_n(\chi_{n+\hat{\mu}} - \chi_{n-\hat{\mu}}) \quad \sim \quad i(\gamma_\mu \otimes \mathbf{1})\sin p_\mu$$

- Spin-taste structure

Spin Taste

$$\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu} \qquad \eta_\mu(n)\delta_{n,n+\hat{\mu}} \rightarrow (\gamma_\mu \otimes \mathbf{1})$$

$$\zeta_\mu(n) = (-1)^{\sum_{\nu>\mu} n_\nu} \qquad \zeta_\mu(n)\delta_{n,n+\hat{\mu}} \rightarrow (\mathbf{1} \otimes \xi_\mu)$$

$$\epsilon(n) = (-1)^{\sum n} \qquad \epsilon(n)\delta_{n,n} \rightarrow (\gamma_5 \otimes \xi_5)$$ Staggered chiral symmetry

- Symmetries

Shift symmetry $\Xi_\mu : \chi_x \rightarrow \zeta_\mu(x)\chi_{x+\hat{\mu}} \quad \phi(p) \rightarrow \exp(ip_\mu)(\mathbf{1} \otimes \xi_\mu)\phi(p)$

Axis reversal $I_\mu : \chi_x \rightarrow (-1)^{x_\mu}\chi_{Ix} \quad \phi(p) \rightarrow (\gamma_\mu\gamma_5 \otimes \xi_5\xi_\mu)\phi(Ip)$

Rotation $R_{\mu\nu} : \chi_x \rightarrow S_R(R^{-1}x)\chi_{R^{-1}x} \quad \phi(p) \rightarrow \exp(\gamma_\mu\gamma_\nu \otimes \xi_\nu\xi_\mu)\phi(R^{-1}p)$

Conjugation $C_0 : \chi_x \rightarrow \epsilon_x\bar{\chi}_x^T \quad \phi(p) \rightarrow \bar{\phi}(-p)^T$

◆Staggered fermion

Spin diagonalization : $\psi_n = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\chi_n, \quad \bar{\psi}_n = \bar{\chi}_n \gamma_4^{n_4}\gamma_3^{n_3}\gamma_2^{n_2}\gamma_1^{n_1}$

$$S_N \;\rightarrow\; S_{st} = \frac{1}{2}\sum_{n,\mu} \eta_\mu(n)\bar{\chi}_n(\chi_{n+\hat{\mu}} - \chi_{n-\hat{\mu}}) \quad \sim \quad i(\gamma_\mu \otimes \mathbf{1})\sin p_\mu$$

- Spin-taste structure

Spin Taste

$\eta_\mu(n) = (-1)^{\sum_{\nu<\mu} n_\nu}$ $\quad \eta_\mu(n)\delta_{n,n+\hat{\mu}} \;\rightarrow\; (\gamma_\mu \otimes \mathbf{1})$

$\zeta_\mu(n) = (-1)^{\sum_{\nu>\mu} n_\nu}$ $\quad \zeta_\mu(n)\delta_{n,n+\hat{\mu}} \;\rightarrow\; (\mathbf{1} \otimes \xi_\mu)$

$\epsilon(n) = (-1)^{\sum n}$ $\quad \epsilon(n)\delta_{n,n} \;\rightarrow\; (\gamma_5 \otimes \xi_5)$ Staggered chiral symmetry

- Symmetries

P

Shift symmetry $\quad \Xi_\mu : \chi_x \rightarrow \zeta_\mu(x)\chi_{x+\hat{\mu}} \quad \phi(p) \;\rightarrow\; \exp(ip_\mu)(\mathbf{1}\otimes\xi_\mu)\phi(p)$

Axis reversal $\quad I_\mu : \chi_x \rightarrow (-1)^{x_\mu}\chi_{Ix} \quad \phi(p) \;\rightarrow\; (\gamma_\mu\gamma_5 \otimes \xi_5\xi_\mu)\phi(Ip)$

$SW4$

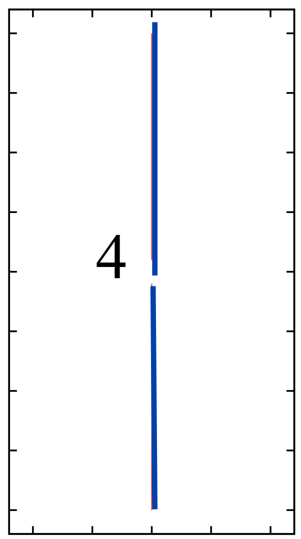
Rotation $\quad R_{\mu\nu} : \chi_x \rightarrow S_R(R^{-1}x)\chi_{R^{-1}x} \quad \phi(p) \;\rightarrow\; \exp(\gamma_\mu\gamma_\nu \otimes \xi_\nu\xi_\mu)\phi(R^{-1}p)$

C — *Conjugation* $\quad C_0 : \chi_x \rightarrow \epsilon_x\bar{\chi}_x^T \quad \phi(p) \;\rightarrow\; \bar{\phi}(-p)^T$

◆ Staggered flavored mass

Golterman, Smit (1984) Adams(2010)

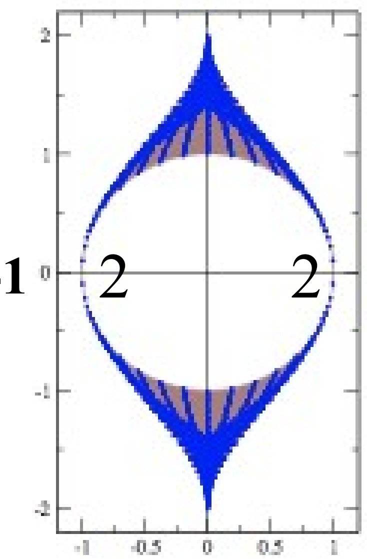
Staggered



$$\sum_{sym.} \zeta_1\zeta_2\zeta_3\zeta_4 C_1 C_2 C_3 C_4$$

$$\sim (\mathbf{1} \otimes \xi_5)$$

St. Wilson



de Forcrand, Kurkela, Panero(2012)

$$\{C_0,\, \Xi_\mu,\, I_s,\, R_{\mu\nu}\} \;\times\; \{U^\epsilon(1)\}_{m=0} \;\Rightarrow\; \{C_0, \underline{\Xi'_\mu, R_{\mu\nu}}\}$$

TM, Sharpe(2012)

- Action & Physical mode

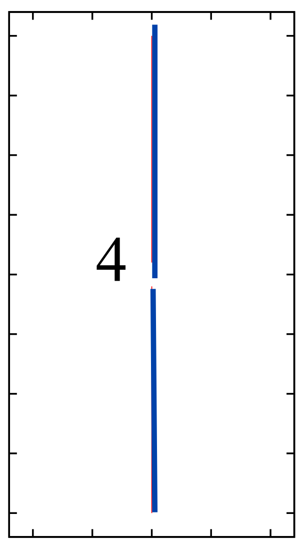
$$D_{\rm sw} = \eta_\mu D_\mu + \underbrace{r(1+M_{\mathcal{A}})}_{\text{Wilson-like term}} + \underbrace{m}_{\text{mass parameter}} \qquad M_{\mathcal{A}} = \epsilon_x \sum_{sym.} \eta_1\eta_2\eta_3\eta_4 C_1 C_2 C_3 C_4$$

ξ_5=-1 → physical sector : ℓ ξ_5=+1 → decoupled sector : h

◆ Staggered flavored mass

Golterman, Smit (1984)   Adams(2010)

Staggered



$$\sum_{sym.} \zeta_1\zeta_2\zeta_3\zeta_4 C_1C_2C_3C_4$$

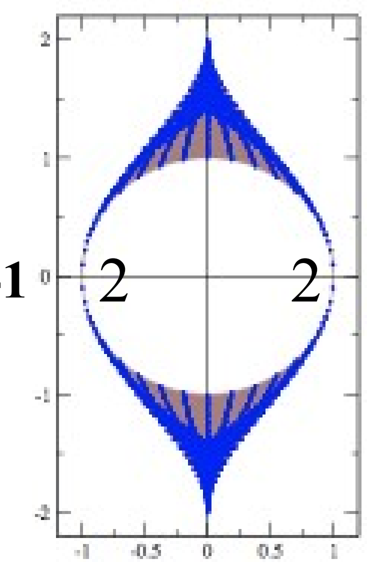
$$\Rightarrow$$

$$\sim (\mathbf{1} \otimes \xi_5)$$

St. Wilson

de Forcrand, Kurkela, Panero(2012)

ξ_5=-1



ξ_5=+1

$$\{C_0,\, \Xi_\mu,\, I_s,\, R_{\mu\nu}\} \times \{U^\epsilon(1)\}_{m=0} \;\Rightarrow\; \{C_0, \Xi'_\mu, \underline{R_{\mu\nu}}\}$$

TM, Sharpe(2012)

C P SW_4

Hypercubic symmetry, C, P, T !

- Action & Physical mode

$$D_{\rm sw} = \eta_\mu D_\mu + \underbrace{r(1+M_{\mathcal{A}})}_{\text{Wilson-like term}} + \underbrace{m}_{\text{mass parameter}} \qquad M_{\mathcal{A}} = \epsilon_x \sum_{sym.} \eta_1\eta_2\eta_3\eta_4 C_1C_2C_3C_4$$

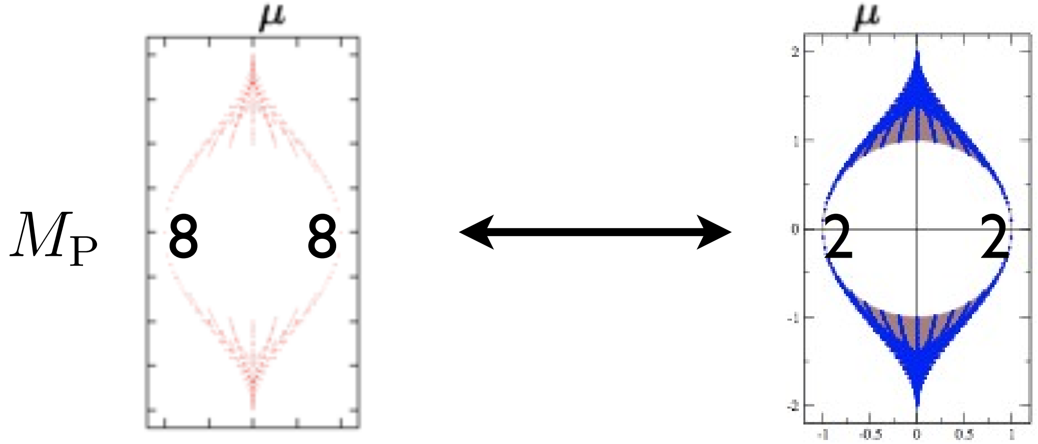
ξ_5=-1 → physical sector : ℓ ξ_5=+1 → decoupled sector : h

◆Staggered-Wilson (Domain-wall, Overlap)

Application	:	As Wilson	→	Mass parameter tuning required
		As Domain-wall	→	5th dimension introduced
		As Overlap	→	Overlap formula with StWil kernel

- Spin diagonalization   Creutz, Kimura, TM (10)

$$\bar{\psi}_x \prod_\mu C_\mu \psi_x \quad \rightarrow \quad \bar{\chi}_x \Big(\prod_\mu \zeta_\mu C_\mu\Big) \chi_x$$



Staggered flavored-mass

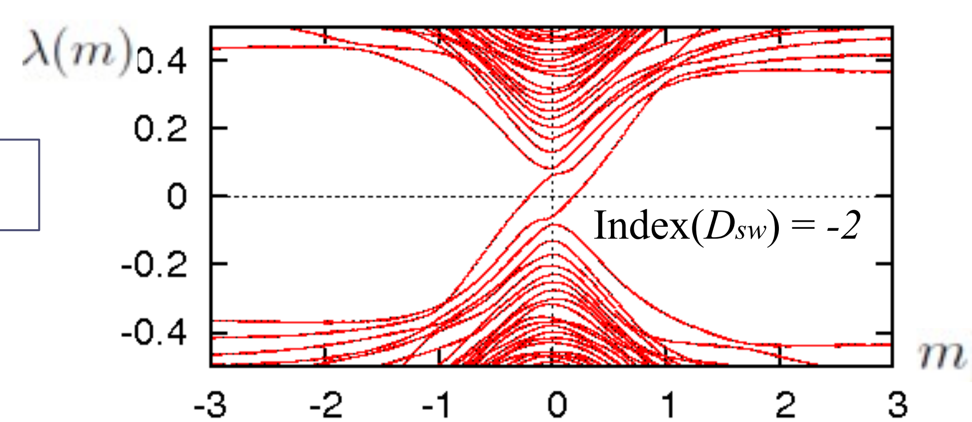
- Index theorem (spectral flow)   Adams(2009)   Creutz, Kimura, TM (2010)

$$H_{\mathrm{sw}} = \epsilon(D_{\mathrm{sw}} - m)$$
$$= (\gamma_5 \otimes \xi_5)(D_{\mathrm{sw}} - m)$$

Index(D_{sw}) = - Spectral flow(H_{sw})

$$\mathrm{Index}(D_{\mathrm{sw}}) = 2^{d/2-1}(-1)^{d/2} Q$$

36×36 lattice, randomness *δ=0.25, Q=1*

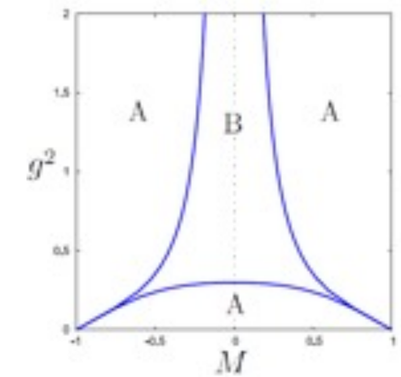


- Aoki phase   Creutz, Kimura, TM(11) TM, Nakano, Kimura, Ohnishi(12)

Strong-coupling LQCD & 2d Gross-Neveu

→ Implies parity-flavor broken phase

ChPT analysis required → 1st or 2nd order ?

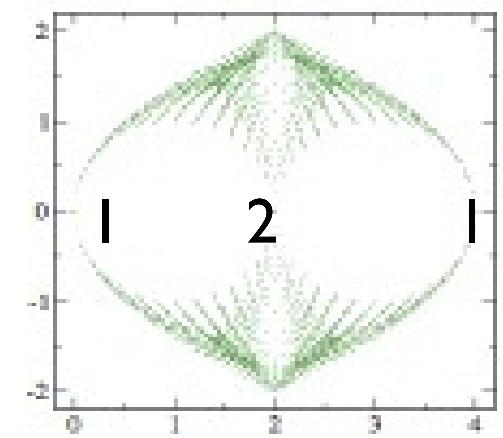


2d Aoki phase

- Another type (Hoelbling type)   Hoelbling (10), de Forcrand, Kurkela, Panero (10)

$$i\eta_\mu\eta_\nu\epsilon_{\mu\nu}(C_\mu C_\nu + C_\nu C_\mu)$$

$$\sim (\mathbf{1}\otimes i(\sigma_{\mu\nu}+\sigma_{\nu\mu})) + O(a)$$



Hoelbling (10)

$$\{C_0,\, \Xi_\mu,\, I_s,\, R_{\mu\nu}\} \;\times\; \{U^\epsilon(1)\}_{m=0} \;\Rightarrow\; \{C_T,\, \Xi'_\mu,\, \underline{R_{12}, R_{34}, R_{13}R_{42}}\}$$

C P

Rotation sym. broken !

→ Requires fine-tuning of parameters for continuum   TM, Sharpe (12)

→ Numerical tests indicate no problematic signature   Durr (13)

§ Potential advantages and problems of $D_{\mathrm{sw}} = \eta_\mu D_\mu + r(1 + M_{\mathcal{A}}) + m$

1. Less numerical costs for overlap? de Forcrand, Kurkela, Panero(2012)

One component (small matrix) vs (i) 24 terms (ii) 4 transporters

2. Wilson improvement works better ? Durr(2013)

Clover term + HEX smearing

3. Less taste-breaking for 2 flavors ? TM, Sharpe(2012)

Staggered sym. for 4 tastes vs Halved staggered sym. for 2 tastes

1. Less numerical costs ? de Forcrand, Kurkela, Panero(2011)

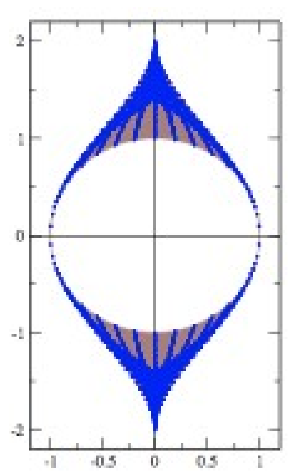
Staggered-Overlap Dirac propagator

◆Small matrix size

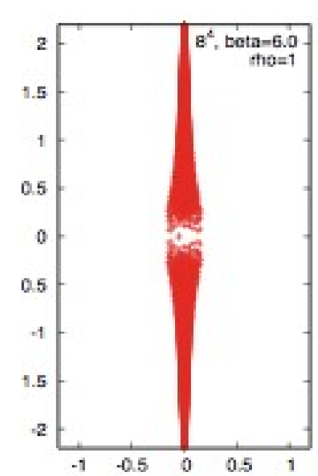
Fewer Matrix-Vector multiplications
for overlap sign function !

◆4-link hopping terms

Gauge fluctuation raised to 4th power !
→ Gap of two branches reduced

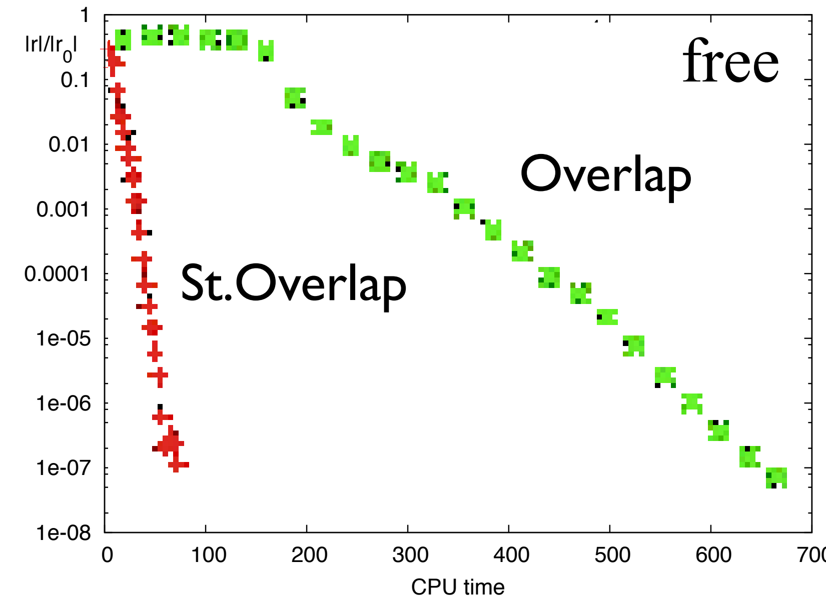


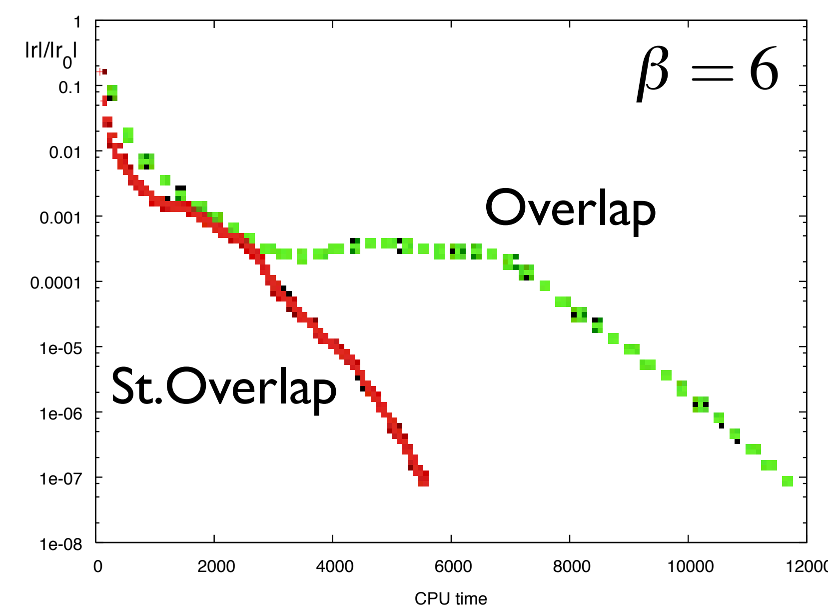
free



β = 6

CG solver (12^4, m=0.1)





CPU

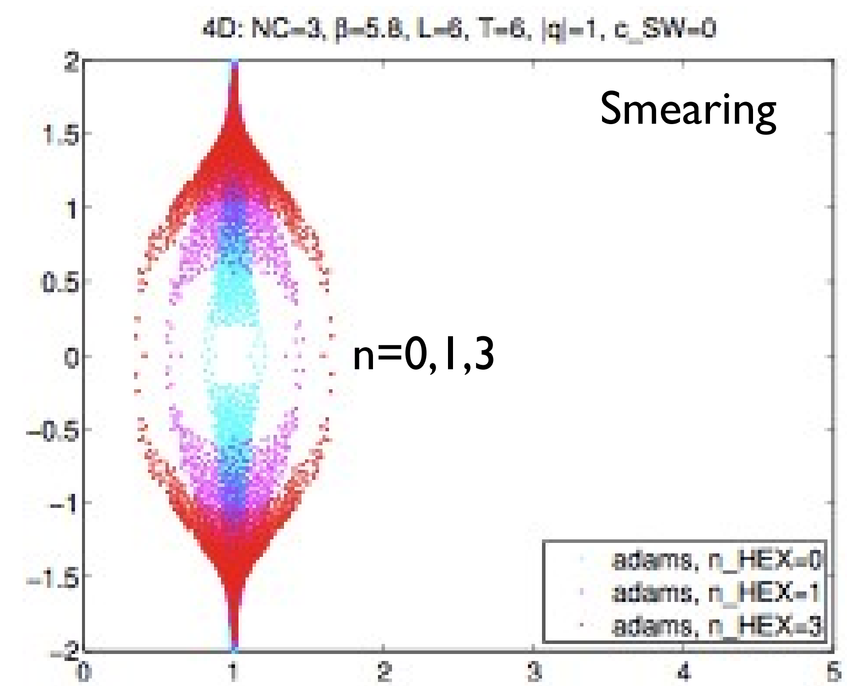
Staggered-Wilson kernel is better than Wilson kernel, but not much better.

2. Wilson improvement works better ? Durr(2013)

◆Smearing makes the gap larger

・The gap gets wider as the HEX smearing level goes up.

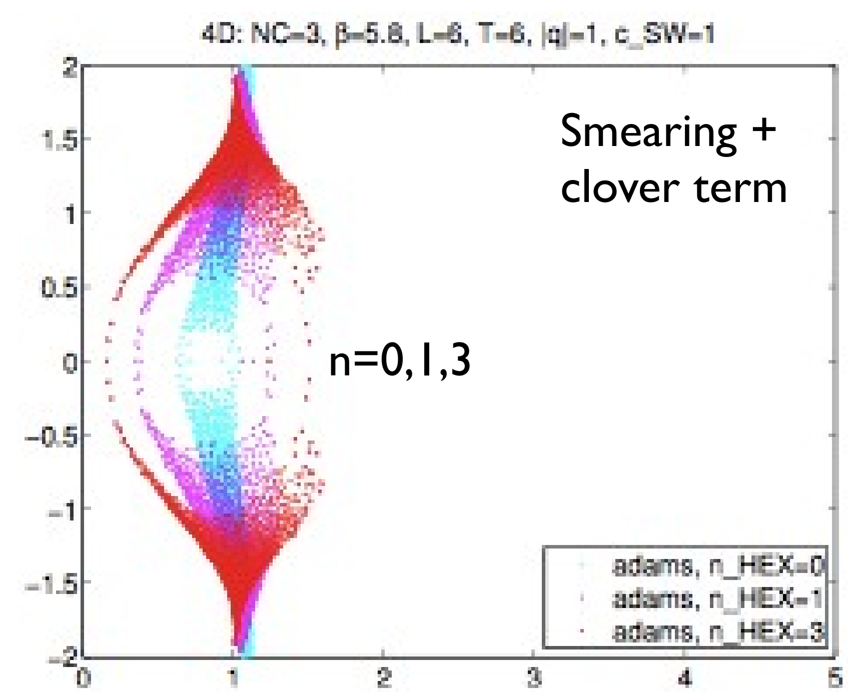
・Gauge fluctuation due to 4-hopping is compensated.



◆Clover (Symanzik) improvement

$$c_{\mathrm{SW}} \sum_{\mu<\nu} (\gamma_\mu \gamma_\nu \otimes \mathbf{1}) F_{\mu\nu}$$

・The physical branch gets close to the origin.



3. Less taste-breaking for 2 flavors ? TM, Sharpe(2012)

Pion spectrum based on symmetry

§ Staggered $\{C_0,\, \Xi_\mu,\, I_s,\, R_{\mu\nu}\} \;\times\; \{U^\epsilon(1)\}_{m=0}$ ➡ $\{C_0, \Xi_j, I_s, R_{ij}\}$
Transfer-matrix sym.

➡ classify 15 pseudoscalar operators

$\mathbf{1} \;:\; \xi_4,\, \xi_{45},\, \xi_5,$

$\mathbf{3} \;:\; \xi_i,\, \xi_{i5},\, \xi_{ij}\; \xi_{i4}$

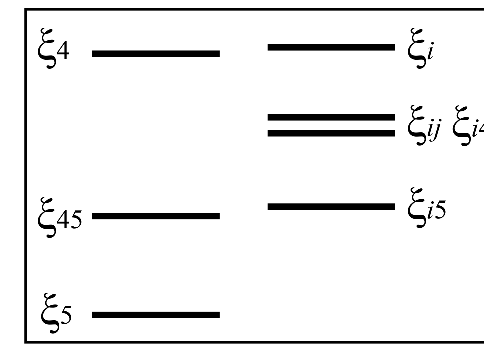
7 irreps

cf.) ChPT → SO(4) upto $O(a^4),\, O(a^2 m)\;\; O(a^2p^2)$

$\mathbf{1} \;:\; \xi_5,$

$\mathbf{4} \;:\; \xi_\mu,\, \xi_{\mu 5},$

$\mathbf{6} \;:\; \xi_{\mu\nu}.$



§ Staggered-Wilson $\boxed{\{C_0, \Xi'_\mu, R_{\mu\nu}\}}$ ➡ $\{C_0, \Xi'_j, R_{ij}\}$

Irreps mix in ξ_5 pairs

Physical sector

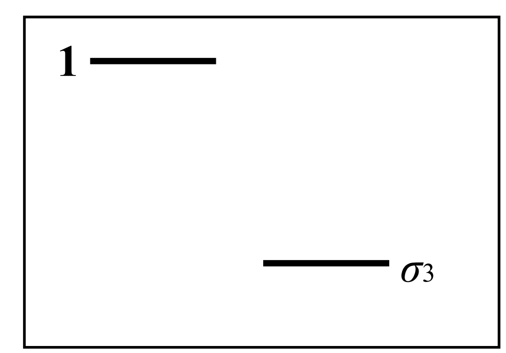
$$\mathbf{1} \;\&\; \xi_5 \;\rightarrow\; \bar\ell\ell \;,\; \bar h h \;\longrightarrow\; \boxed{\bar\ell(\gamma_5\otimes\mathbf{1})\ell} \;\; \eta'$$

$$\xi_4 \;\&\; \xi_{45} \;\rightarrow\; \bar\ell h \;,\; \bar h\ell$$

$$\xi_{i4} \;\&\; \xi_{i45} \;\rightarrow\; \bar\ell\sigma_j\ell \;,\; \bar h\sigma_j h \;\longrightarrow\; \boxed{\bar\ell(\gamma_5\otimes\sigma_i)\ell} \;\; \pi_0,\, \pi_\pm$$

$$\xi_i \;\&\; \xi_{i5} \;\rightarrow\; \bar\ell\sigma_j h \;,\; \bar h\sigma_j\ell$$

States in 3d irrep !



Discrete symmetries are sufficient for degenerate pion triplet!

Lattice ChPT potential → classify operators allowed by sym. $\boxed{\{C_0, \Xi'_\mu, R_{\mu\nu}\}}$

- Dim3, 4 : $\bar{Q}(1 \otimes \xi_F)Q \quad \bar{Q}(\gamma_\mu \otimes \xi_F)D_\mu Q$ for $\xi_F = 1$ or ξ_5
- Dim5 O(*a*): $\bar{Q}(i\sigma_{\mu\nu}F_{\mu\nu} \otimes \xi_F)Q$ for $\xi_F = 1$ or ξ_5

⇒ $\bar{\ell}\ell, \ \bar{\ell}\gamma_\mu D_\mu \ell \quad \bar{\ell}i\sigma_{\mu\nu}F_{\mu\nu}\ell$

No taste-breaking

- Dim6 O(*a*²) : 2 types of four-fermi operators $\mathcal{L}_6^{FF(A)}$ and $\mathcal{L}_6^{FF(B)}$

In $\mathcal{L}_6^{FF(A)}$ the spin and flavor independently forms scalar e.g.) $TV \equiv \bar{Q}(\gamma_{\mu\nu} \otimes \xi_\rho)Q\bar{Q}(\gamma_{\nu\mu} \otimes \xi_\rho)Q$

⇒ *SA, SV, AS, VS, PV, PA, VP, AP, TV, TA, VT, AT, AA, PP , SP, PS, ST, PT, TS, TP, VV, AA, VA, AV, TT* $SU(2) \times SO(4)$

→ No taste-breaking. No derivative in ChPT. Contribute to χ-effective potential $\boxed{\mathcal{V}_6^{FF(A)}}$

In $\mathcal{L}_6^{FF(B)}$ the spin and flavor are not independent e.g.) $TV \equiv \bar{Q}(\gamma_{\mu\nu} \otimes \xi_\mu)Q\bar{Q}(\gamma_{\nu\mu} \otimes \xi_\mu)Q$

⇒ *TV, TA, VT, AT, VV, AA, VA, AV, TT+, TT-* $\Gamma_{2,1} \rtimes SW_{4,\mathrm{diag}}/Z_2$

→ Taste-breaking. Derivative in ChPT. No contribution to χ-effective potential $\boxed{\mathcal{V}_6^{FF(B)}}$ ×

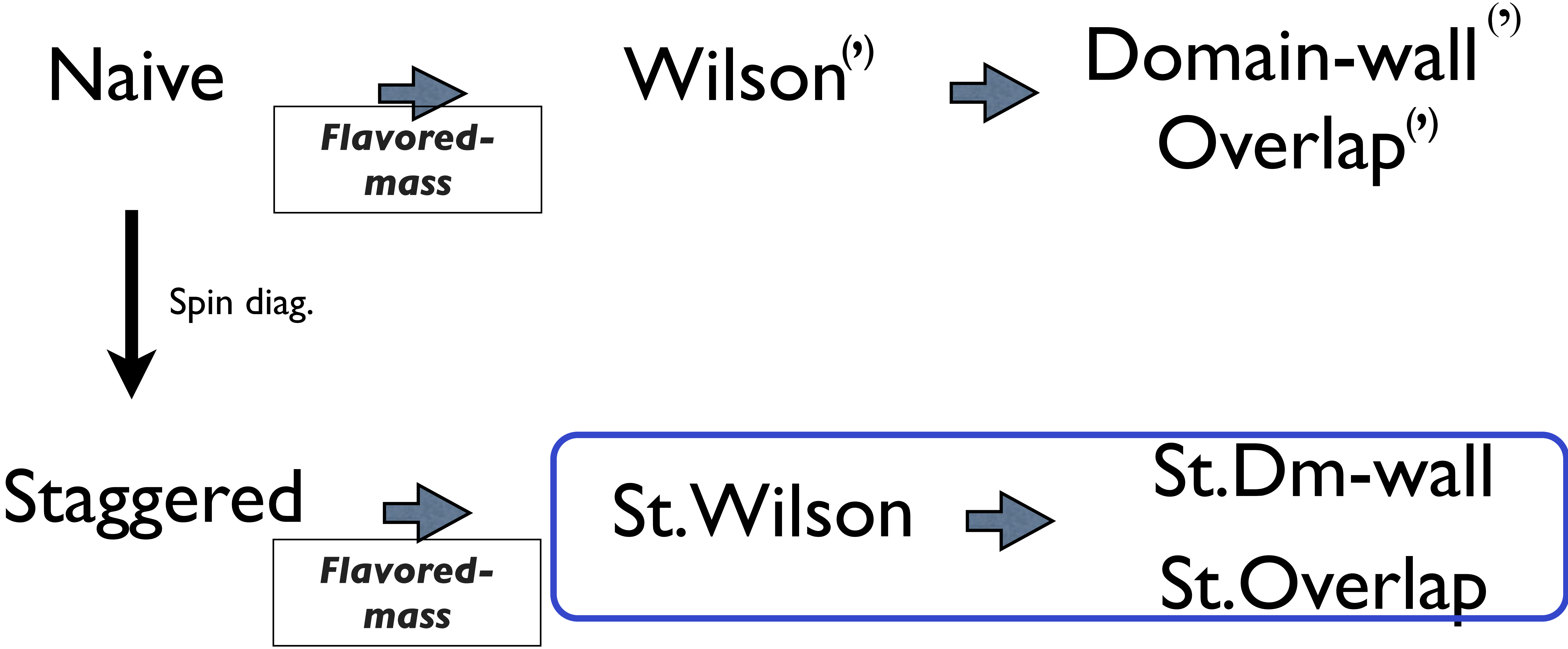
No taste-breaking in ChPT potential up to O(a^4): exact SU(2)

◆ Short summary

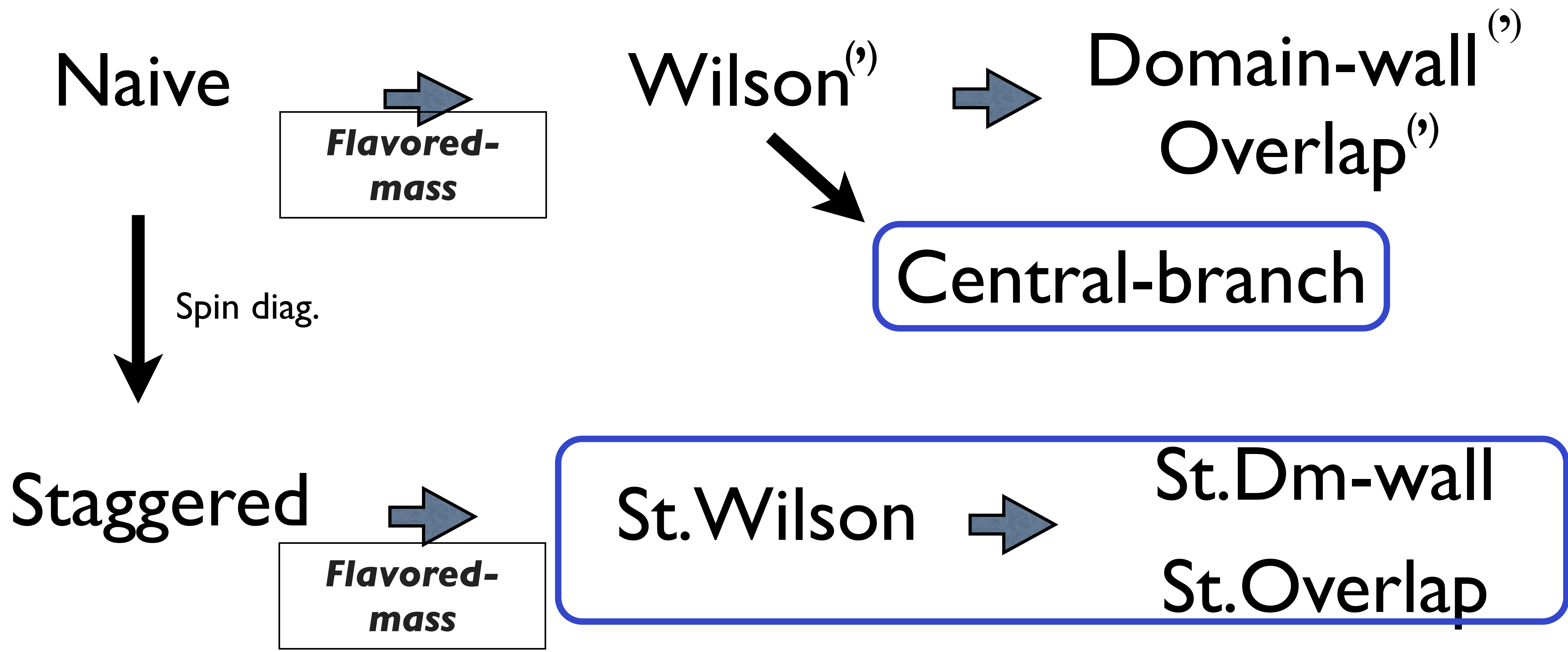
- Flavored mass leads to generalization of Wilson, Domain-wall and Overlap fermions.

- Brillouin fermion → less O(a) error
  → better kernel for overlap

- Staggered-Wilson → less costs for overlap
  → better improving effects
  → less taste breaking



Naive
Flavored-mass
Wilson(')
Domain-wall (')
Overlap(')
Spin diag.
Staggered
Flavored-mass
St.Wilson
St.Dm-wall
St.Overlap



Naive
Flavored-mass
Wilson(')
Domain-wall (')
Overlap(')
Central-branch
Spin diag.
Staggered
Flavored-mass
St.Wilson
St.Dm-wall
St.Overlap

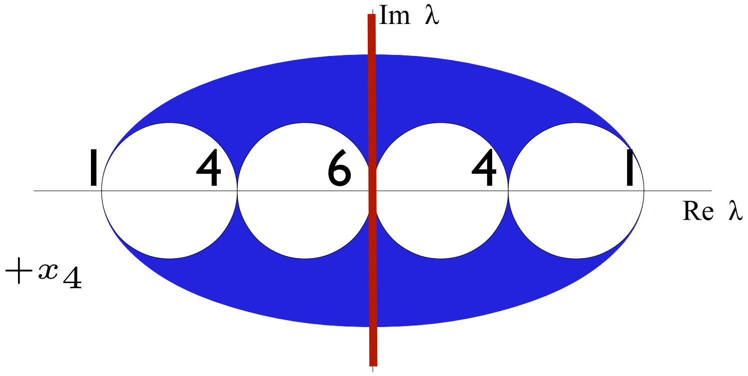
2. Central-branch

2. Central-branch

Creutz, Kimura, TM (11)
Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

◆Wilson w/o onsite $M_W \equiv m + 4r = 0$

$$S = \frac{1}{2}\sum_{x,\mu} \bar{\psi}_x[\gamma_\mu(U_{x,\mu}\psi_{x+\mu} - U_{x,-\mu}\psi_{x-\mu}) - (U_{x,\mu}\psi_{x+\mu} + U_{x,-\mu}\psi_{x-\mu})]$$



⇨ Extra U(1) with 6 flavors!

$$\psi_x \rightarrow e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}\psi_x,\ \ \bar{\psi}_x \rightarrow \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}$$

- Flavor-chiral symmetry

$$\Gamma_X^{(+)} \in \left\{\mathbf{1}_4\,,\ (-1)^{n_1+\ldots+n_4}\gamma_5\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\,,\ (-1)^{n_\mu}i\gamma_\mu\gamma_5\,,\ (-1)^{n_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2}\right\}$$

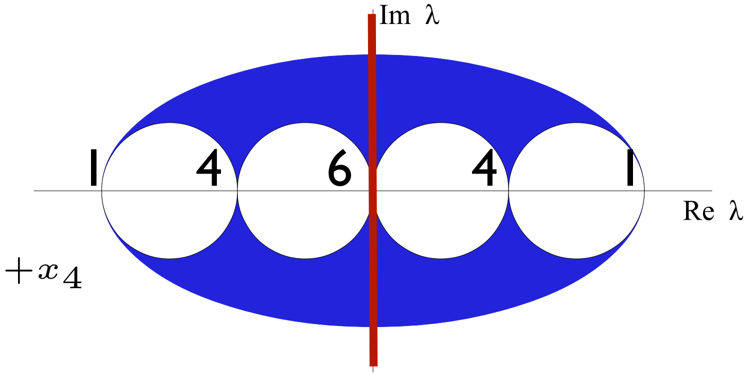
$$\Gamma_X^{(-)} \in \left\{(-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,,\ \gamma_5\,,\ (-1)^{n_\mu}\gamma_\mu\,,\ (-1)^{\check{n}_\mu}\gamma_\mu\gamma_5\,,\ (-1)^{\check{n}_{\mu,\nu}}\frac{[\gamma_\mu\,,\gamma_\nu]}{2}\right\}$$

2. Central-branch

Creutz, Kimura, TM (11)
Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

◆ Wilson w/o onsite $M_W \equiv m + 4r = 0$

$$S = \frac{1}{2}\sum_{x,\mu}\bar{\psi}_x[\gamma_\mu(U_{x,\mu}\psi_{x+\mu} - U_{x,-\mu}\psi_{x-\mu}) - (U_{x,\mu}\psi_{x+\mu} + U_{x,-\mu}\psi_{x-\mu})]$$



⇨ Extra U(1) with 6 flavors!

$$\psi_x \to e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}\psi_x,\ \ \bar{\psi}_x \to \bar{\psi}_x e^{i\theta(-1)^{x_1+x_2+x_3+x_4}}$$

- Flavor-chiral symmetry

$$\Gamma_X^{(+)} \in \left\{\mathbf{1}_4\,, \right\}$$

$$\Gamma_X^{(-)} \in \left\{(-1)^{n_1+\ldots+n_4}\mathbf{1}_4\,, \right\}$$

$$\gamma_5 \otimes \underbrace{\gamma_5 \otimes \mathbf{1}}_{\text{flavor}}$$

spin

Prohibits additive mass renormalization !
No fine-tuning !

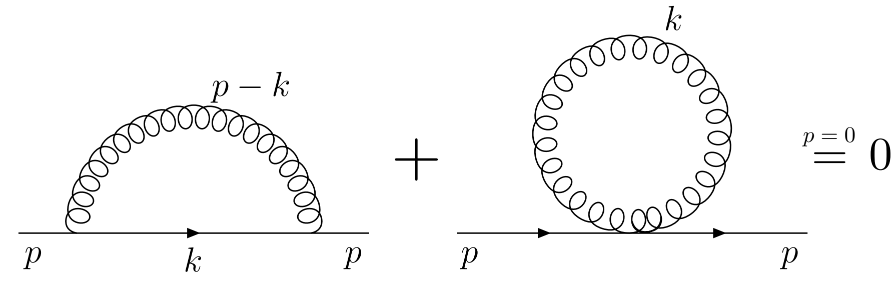
- <u>Lattice perturbation</u> TM (12) Chowdhury, et.al. (13)

quark self-energy

$$\frac{g_0^2}{16\pi^2}\left(\frac{\Sigma_0}{a} + i\gamma_\mu p_\mu \Sigma_1 + m_0 \Sigma_2\right)$$

$\downarrow$ additive mass

$$\Sigma_0^{(\alpha)} = \Sigma_0^{(\alpha)}(\mathrm{sun}) + \Sigma_0^{(\alpha)}(\mathrm{tad}) = 0$$

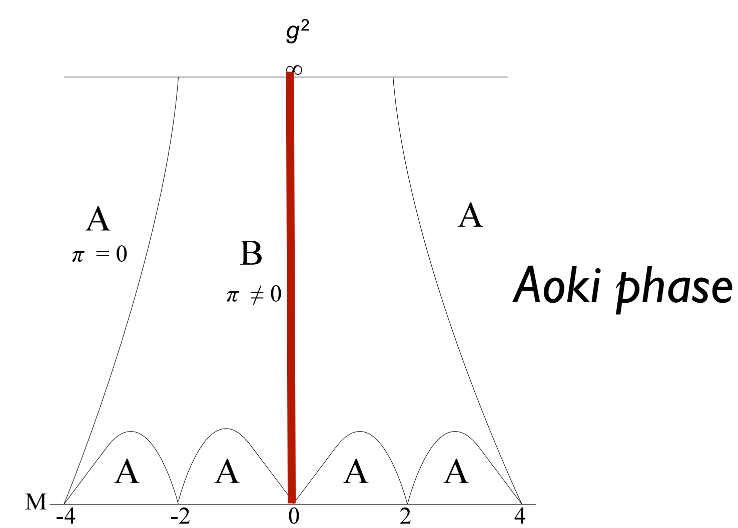


No additive mass renormalization

- <u>Strong-coupling QCD</u> Kimura, Komatsu, TM, Noumi, Torii, Aoki (11)

$$\pi \sim \bar{\psi}(-1)^{\sum n}\psi$$ Meson operator a/w the U(1)

$$\cosh(m_\pi) = 1 + \frac{2M_W^2(16+M_W^2)}{16-15M_W^2} \to m_\pi = 0$$



※Special condensate

$$\langle\bar{\psi}\gamma_5\psi\rangle \neq 0 \quad \langle\bar{\psi}\psi\rangle = 0$$

⇨ Twisted-mass basis $\bar{\psi}\psi \leftrightarrow \bar{\psi}\gamma_5\psi$

NG boson emerges
Spontaneous breaking of the U(1)

§ Advantages

- No additive mass renormalization (no fine-tuning)
- SSB of U(1) and massless NG boson
- No O(*a*) errors   cf.) Twisted-mass Wilson $m_3 \bar{\psi} i \tau_3 \gamma_5 \psi$

⇨ $\bar{\psi}\psi \quad \leftrightarrow \quad \bar{\psi}\gamma_5\psi$

change of mass basis

$\boxed{\rightarrow \textit{6-flavor QCD}}$

§ Potential drawbacks

- Negative quark determinant (odd negative zero modes)
- Sign problem for different topological sector

⇨ Two sets of Wilson CB

$\det D = \det D_1 \det D_2$

$\boxed{\rightarrow \textit{12-flavor QCD}}$

◆ CB for other flavored masses

M_T : $U(2)$ restored

M_P : $U(2)\times U(2)$ restored

M_{Adams} : $C_T'\Xi$, $C_T'I$ restored

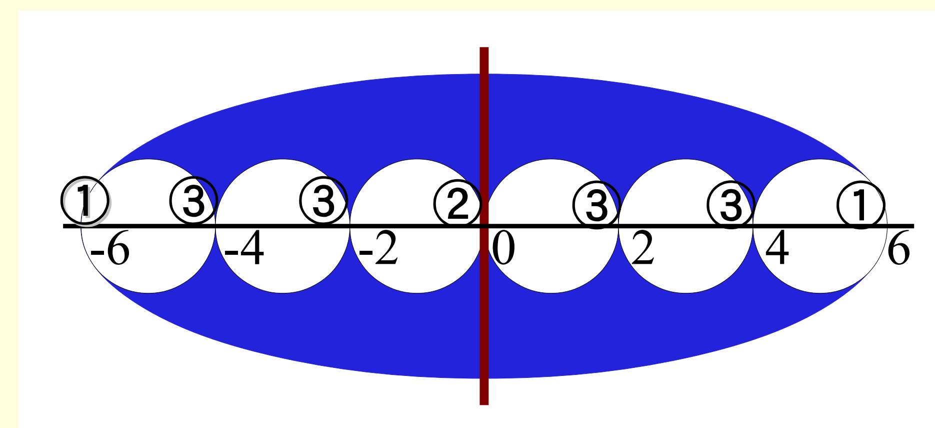
M_{Hoel} : C_T' restored

→ No additive mass

$$C_T' \, : \, \chi_x \to \bar{\chi}_x^T, \quad \bar{\chi}_x \to \chi_x^T, \quad U_{x,\mu} \to U_{x,\mu}^*$$

◆ 2-flavor CB

§ 4D $\displaystyle\sum_{\mu=1}^{4} C_\mu \quad \to \quad \sum_{j=1}^{3} C_j + 3C_4.$



(1,3,3,2,3,3,1) splitting

・No additive mass for 2-flavor fermion

・Hypercubic symmetry → Cubic symmetry

§ 5D $\displaystyle\sum_{\mu=1}^{5} C_\mu \quad \to \quad \sum_{j=1}^{4} C_j + 4C_5$

(1,4,6,4,2,4,6,4,1) splitting

・No additive mass

・5D hypercubic → 4D hyperubic

(4+1)D extra-dim setup !

3. Two-flavor chiral fermion

3. Two-flavor chiral fermion TM (2013)

SUSY lattice : doubling problem is more harmful

1. #boson = #fermion
2. R symmetry ~ chiral symmetry

"Well-defined SUSY lattice" $\leftrightarrows$ "Successful doubling bypass"

◆2D N=(2,2) SUSY lattice Sugino (2003)

- 4D N=1 → 2D N=(2,2)
- 4 SUSY $Q_{\pm}$ $\bar{Q}_{\pm}$ 4 real spinor $\lambda_{\pm}$ $\bar{\lambda}_{\pm}$ 2 U(1) R-sym.
- Topological twist → one scalar supercharge (BRST charge)

$$\underbrace{SO(2)_T = SO(2)_R}_{\text{R-Flavor}} \oplus \underbrace{SO(2)}_{\text{Lorentz}} \longrightarrow Q = Q_+ + Q_- \longrightarrow Q^2 = 0$$

→Scalar SUSY can survive on the lattice $S = QV(U, \phi, \psi)$

2D N=(2,2) fermion part

$$S_f^{(2)} = \frac{a^4}{2g_0^2}\sum_{x,\mu}\mathrm{tr}\left[\underbrace{-\frac{1}{2}\Psi(x)^T\gamma_\mu(\Delta_\mu+\Delta_\mu^*)\Psi(x)}_{\text{Kinetic term}} \underbrace{- a\frac{1}{2}\Psi(x)^T P_\mu\Delta_\mu\Delta_\mu^*\Psi(x)}_{\text{Wilson-like term}}\right]$$

- 2-flavor × 2-spinor → 4 spinor $\Psi^T = \left(\psi_1, \psi_2, \chi, \frac{1}{2}\eta\right)$

$$\gamma_1 = -i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \gamma_2 = i\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad P_2 = -i\begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}$$

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}, \quad \{P_\mu, P_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_\mu, P_\nu\} = 0.$$ 4D clifford algebra

1. No more species doubling (never conflicts with no-go theorem)

$$D = \sum_{\mu=1}^{2}\left[-i\gamma_\mu\frac{1}{a}\sin(q_\mu a) + 2P_\mu\frac{1}{a}\sin^2\left(\frac{q_\mu a}{2}\right)\right] \Rightarrow D^2 = \frac{1}{a^2}\sum_{\mu=1}^{2}\left[\sin^2(q_\mu a) + 4\sin^4\left(\frac{q_\mu a}{2}\right)\right]$$

with only zero at $p = (0,0,0,0)$

2. U(1) R-invariance (chiral invariance)

$$\bar\gamma_1\bar\gamma_2 P_1 P_2 = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix}$$

→ prohibits mass term

Main points

1. D-dim Two-flavor → (D+2)-dim fermion

2. D+2 dimensional clifford algebra

   (Two sets of D-dim gamma matrices)

   → No further doubling

3. D+2 chiral symmetry

   → R invariance (chiral invariance)

Let us construct chiral 2-flavor setup inspired by SUSY !

◆New 2D two-flavor setup $\Psi=(\psi_A,\ \psi_B)^T$ TM (2013)

$$D=i\bar{\gamma}_\mu \sin p_\mu+\sum_\mu iP_\mu(1-\cos p_\mu)$$

- 4D gamma matrix

$$\bar{\gamma}_1=\mathbf{1}\otimes\sigma_1=\begin{pmatrix}\sigma_1 & \\ & \sigma_1\end{pmatrix} \qquad \bar{\gamma}_2=\mathbf{1}\otimes\sigma_2=\begin{pmatrix}\sigma_2 & \\ & \sigma_2\end{pmatrix}$$

$$P_1=-\sigma_2\otimes\sigma_3=\begin{pmatrix} & i\sigma_3 \\ -i\sigma_3 & \end{pmatrix} \qquad P_2=\sigma_1\otimes\sigma_3=\begin{pmatrix} & \sigma_3 \\ \sigma_3 & \end{pmatrix}$$

$\Rightarrow$

$$\{\bar{\gamma}_\mu,\bar{\gamma}_\nu\}=2\delta_{\mu\nu}$$
$$\{P_\mu,P_\nu\}=2\delta_{\mu\nu}$$
$$\{\bar{\gamma}_\mu,P_\nu\}=0$$

1. No more species doubling $D^2=\sum_{\mu=1}^{2}\left[\sin^2 p_\mu+(1-\cos p_\mu)^2\right]$ $p=(0,0,0,0)$

2. Flavored-chiral invariance $\Gamma_5=\bar{\gamma}_1\bar{\gamma}_2P_1P_2=(\sigma_3\otimes\sigma_3)=\begin{pmatrix}\gamma_5 & \\ & -\gamma_5\end{pmatrix}$

3. O(a) SU(2) flavor symmetry breaking Wilson-like term → Flavor-Lorentz mixing

cf.)staggered-like $D=i(\mathbf{1}\otimes\sigma_\mu)\sin p_\mu+(\sigma_\mu\sigma_3\otimes\sigma_3)(1-\cos p_\mu)$

◆New 4D two-flavor setup

TM (2013)

6D clifford algebra : only 6 gamma matrices

$\{\bar\gamma_\mu, \bar\gamma_\nu\} = 2\delta_{\mu\nu}$

$\cancel{\{P_\mu, P_\nu\} = 2\delta_{\mu\nu}}$ cf.) 4D N=2

$\{\bar\gamma_\mu, P_\nu\} = 0$

1. Common P for 4 directions

$$D = i\bar\gamma_\mu \sin p_\mu + iP\sum_\mu (1-\cos p_\mu) \quad \Rightarrow \quad D^2 = \sum_{\mu=1}^{4} \sin^2 p_\mu + \left[\sum_\mu^4 (1-\cos p_\mu)\right]^2$$

2. P respecting hypercubic symmetry

i) Failed case

$$\bar\gamma_j = \mathbf{1}\otimes\sigma_1\otimes\sigma_j = \begin{pmatrix} \gamma_j & \\ & \gamma_j \end{pmatrix}$$

$$\bar\gamma_4 = \sigma_3\otimes\sigma_2\otimes\mathbf{1} = \begin{pmatrix} \gamma_4 & \\ & -\gamma_4 \end{pmatrix}$$

$$P = \sigma_1\otimes\sigma_2\otimes\mathbf{1} = \begin{pmatrix} & \gamma_4 \\ \gamma_4 & \end{pmatrix}$$

- No more doubling
- Chiral symmetry $\bar\gamma_5 = \mathbf{1}\otimes\sigma_3\otimes\mathbf{1} = \begin{pmatrix} \gamma_5 & \\ & \gamma_5 \end{pmatrix}$
- 4th-dim specified → hypercubic broken

cf.) Minimal-doubling

$$D = i\gamma_\mu \sin p_\mu + i\gamma_4 \sum_j (1-\cos p_j)$$

ii) Successful case

$$\bar{\gamma}_j = \sigma_3 \otimes \sigma_1 \otimes \sigma_j = \begin{pmatrix} \gamma_j & \\ & -\gamma_j \end{pmatrix}$$

$$\bar{\gamma}_4 = \sigma_3 \otimes \sigma_2 \otimes \mathbf{1} = \begin{pmatrix} \gamma_4 & \\ & -\gamma_4 \end{pmatrix}$$

$$\Rightarrow \quad D = i\bar{\gamma}_\mu \sin p_\mu + iP \sum_\mu (1 - \cos p_\mu)$$

No hypercubic breaking

$$P = \sigma_1 \otimes \mathbf{1} \otimes \mathbf{1} = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix}$$

1. No more species doubling $\quad D^2 = \sum_{\mu=1}^{2} \left[\sin^2 p_\mu + (1 - \cos p_\mu)^2\right]$
2. Flavored-chiral invariance $\quad \bar{\gamma}_5 = \sigma_3 \otimes \sigma_3 \otimes \mathbf{1} = \begin{pmatrix} \gamma_5 & \\ & -\gamma_5 \end{pmatrix}$
3. Hypercubic and C, P, T invariance
4. Flavor symmetry breaking ?

◆ Short summary

- Two-flavor chiral and hypercubic-symmetric fermion is constructed by using 6D gamma matrix.
- Gamma-5 hermiticity and C,P,T invariance.
- Further study will uncover flavor symmetry breaking.
- Relation to twisted-mass Wilson?

	flavors	chiral	tuning	artifact	SW4
Wilson:	1	0	severe	O(a)	○
Staggered:	4	1	N/A	O(a^2)	○*
Domain-wall	1	(1)	easy	O(a^2)	○
Overlap	1	1	N/A	O(a^2)	○
Brillouin	1	0	severe	O(a)*	○○
Br-Overlap	1	1	N/A	O(a^2)	○○
St-Wil	2	0	severe	O(a)	○*
St-Overlap	2	1	N/A	O(a^2)	○*
6-f CB	6	1	N/A	O(a^2)	○
2-f CB	2	1	N/A	O(a^2)	×
2-f chiral	2	1	N/A	O(a^2)	○

Summary

1. Flavored-mass terms give us new types of Wilson and overlap fermions.

2. Central-branch fermion is a new possibility of use of Wilson for many-flavor QCD without fine-tuning of parameters.

3. Two-flavor chiral fermion is constructed based on 6-dim clifford algebra.

Back-up slides

Minimal-doubling

Karsten(81) Wilczek(87)
Creutz(07) Borici(87) Creutz,TM(10)

Flavored imaginary chemical potential term lifts species degeneracy.

cf.) Flavored mass in Wilson

Wilson	Flavored chemical-pot.
① ④ ⑥ ④ ① → m	② ⑥ ⑥ ② → μ
0 2 4 6 8	0 2 4 6
$\sum_{\mu}(1-\cos p_{\mu})$	$(\mathrm{i})\,\gamma_4\sum_{j=1}^{3}(1-\cos p_j)$ → *keeping one chiral sym.*

$\boxed{\text{Finite-mass system(Wil)} \leftrightarrows \text{Finite-density system(FCP)}}$

◆ Advantage

- U(1) chiral symmetry
- Ultra-local
- 2 flavor possible

◆ Drawbacks

- Hypercubic symmetry breaking
- Tuning parameters for a correct continuum limit

Bedaque, Buchoff, Tiburzi, Walker-Loud(08)
Capitani, Creutz, Weber, Wittig (09)(10)

◆Symmetries

1. U(1) chiral
2. P
3. CT
4. Cubic

In a continuum limit

1. SU(2) chiral
2. P
3. CT
4. Spatial rotation

Symmetries of finite-density systems

Application to Finite-(T, μ) QCD TM, Kimura, Ohnishi (2012)

Additive chemical potential v.s Additive mass renormalization

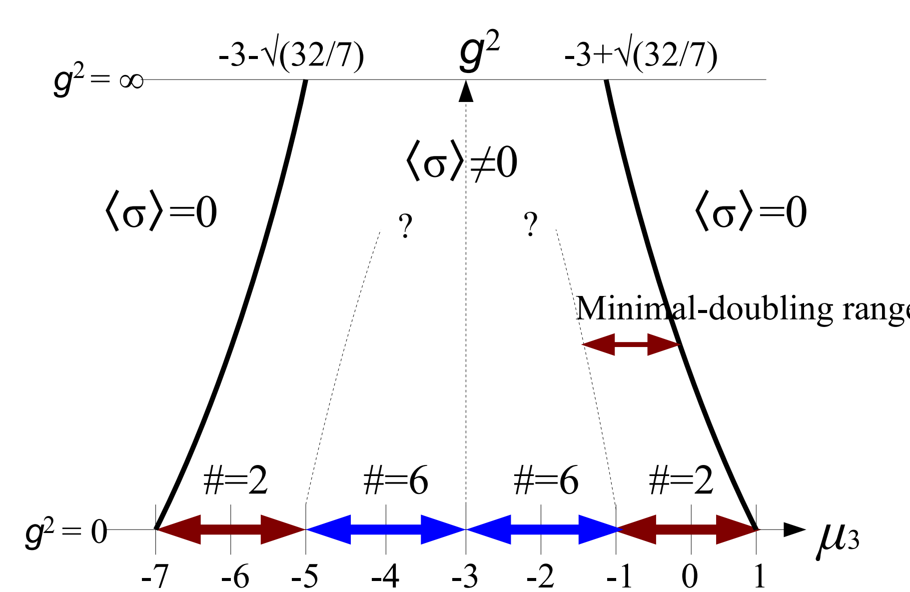
◆Chiral phase structure TM (12)

The renormalization leads to different flavor number.

Nontrivial chiral phase diagram

(Need to tune the parameter)

Parameter phase structure



◆ Strong-coupling lattice QCD

TM, Kimura, Ohnishi (2012)

1. Link variable integral
2. Bosonization
3. Determine the vacuum from the effective potential

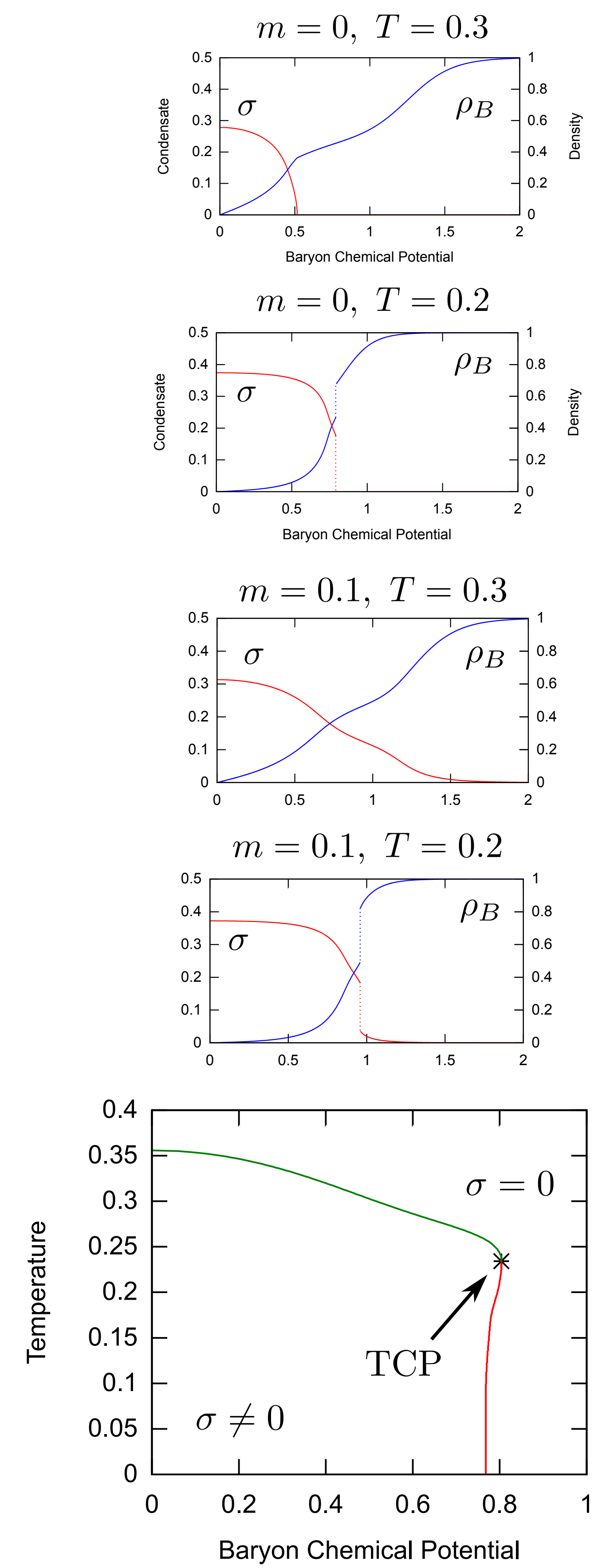
$$\mathcal{F}_{\rm eff}(\sigma, \pi_4; T, \mu_B, \mu_3, d_4) = \frac{9}{2}\sigma^2 - \frac{3}{2}\log\left((1+d_4)^2 + (\mu_3+3)^2\right) - \max\left\{3\,\mathrm{arcsinh}\left(\frac{3\sigma}{\sqrt{(1+d_4)^2+(\mu_3+3)^2}}\right), \mu_B\right\}$$

Effective potential of σ as a function of T, μ and μ3

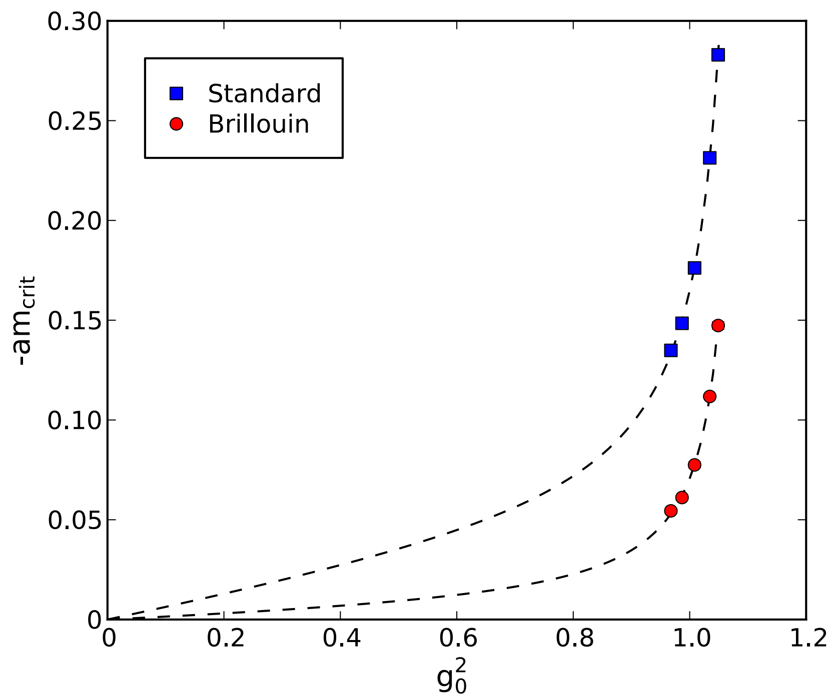
Chiral phase structure

- 1st and 2nd phase transition (m=0)
- 1st, critical point and crossover (m≠0)

New possibility of (T,μ) lattice QCD !



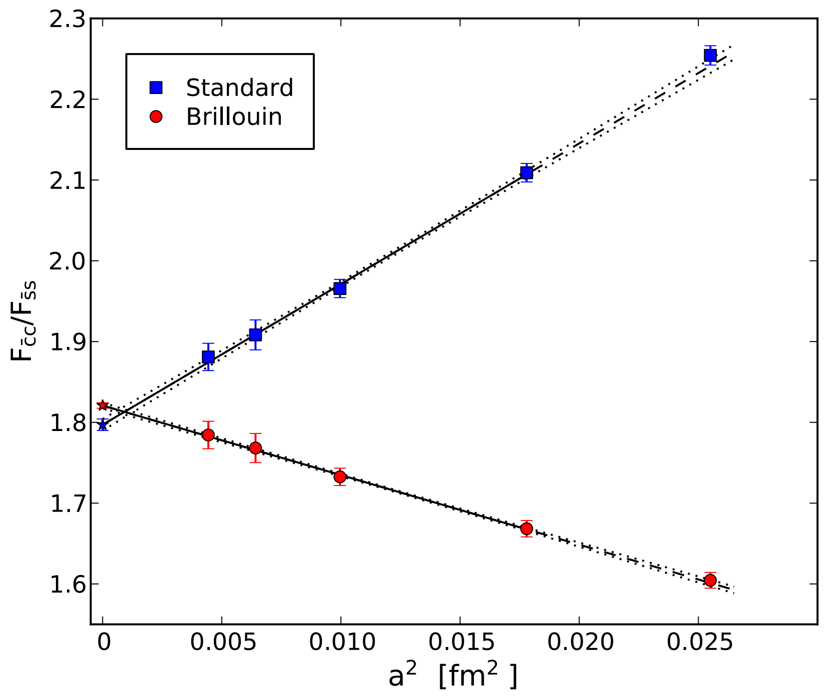
β	geom.	a^{-1}[GeV]	$\kappa_{\rm crit}^{\rm std/std}$ ("Standard")		$\kappa_{\rm crit}^{\rm iso/bri}$ ("Brillouin")	
5.72	$10^3\times20$	1.236	0.134516(65)	0.134533	0.129780(64)	0.129798
5.80	$12^3\times24$	1.479	0.132673(47)	0.132650	0.128594(30)	0.128582
5.95	$16^3\times32$	1.978	0.130760(59)	0.130769	0.127469(48)	0.127471
6.08	$20^3\times40$	2.463	0.129818(45)	0.129864	0.126940(30)	0.126973
6.20	$24^3\times48$	2.964	0.129362(57)	0.129303	0.126725(42)	0.126676



$$A_\mu^{\rm ren} = Z_A(1+b_A a m^{\rm W})(A_\mu + a c_A \bar{\partial}_\mu P)$$

$$F_{\bar{c}c}/F_{\bar{s}s} = d_0 + d_1\alpha(a)a + d_2 a^2$$

$$(r_0 M_\pi)^2 = 1.56,\ (r_0 M_{\bar{s}s})^2 = 4.56 \text{ and } (r_0 M_{\bar{c}c})^2 = 46.5.$$



◆ Spectral flow

(i) Hermitian operator

$$H(m) = \gamma_5(D - m) \qquad (H^2 = D^\dagger D + m^2 \geq 0)$$

(ii) Eigenvalue flow $\lambda_i(m)$

$$\lambda_0(m) = \mp m \quad \text{only for zero modes}$$

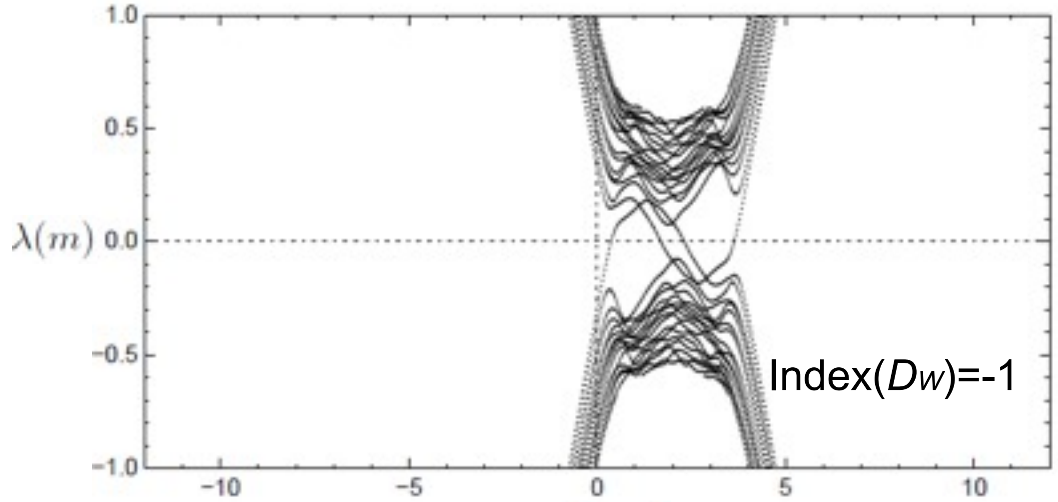
zero mode : low-lying crossing

chirality : minus the sign of slope

⇒ Index theorem

2d Wilson fermion

$H_W(m) = \gamma_5(D_W - m)$



$Q = 1$

R.Edwards, U.Heller, R.Narayanan (1998)

・lattice theory (Wilson fermion)

(i) Hermitian operator

$$H_W(m) = \gamma_5(D_W - m)$$

(ii) Eigenvalue flow

would-be zero modes : low-lying real crossing

approximate chirality : $\lambda'(m) = -\psi(m)^\dagger \gamma_5 \psi(m)$

$$\boxed{\text{Index}(D_W) \;=\; -\,\text{Spectral flow}(H_W)}$$

$$\Rightarrow \quad \text{Index}(D_W) = (-1)^{d/2} Q$$

※ Spectral flow :
Crossings counted with ± slopes

- Index theorem (spectral flow) Adams(2009) M. Creutz, T. Kimura, TM (2010)

Generalized Wilson

$$H_{gw} = \gamma_5(D_{nf} - M_P)$$

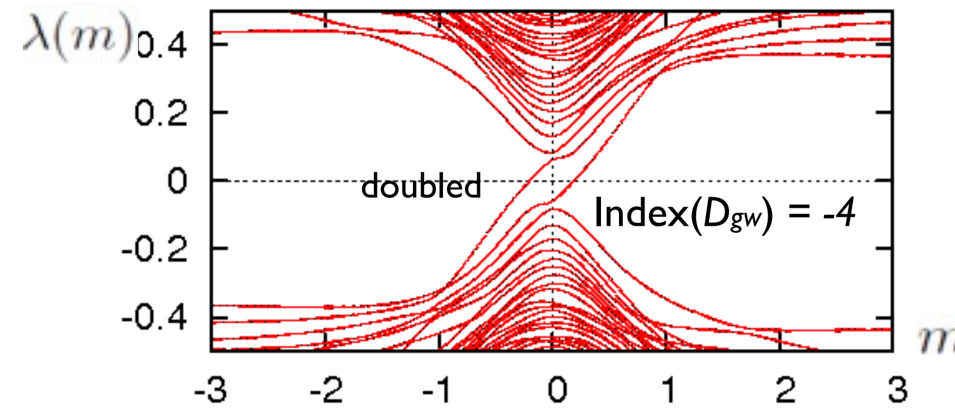
Index(D_{gw}) = - Spectral flow(H_{gw})

$$\mathrm{Index}(D_{gw}) = 2^d(-1)^{d/2}Q$$

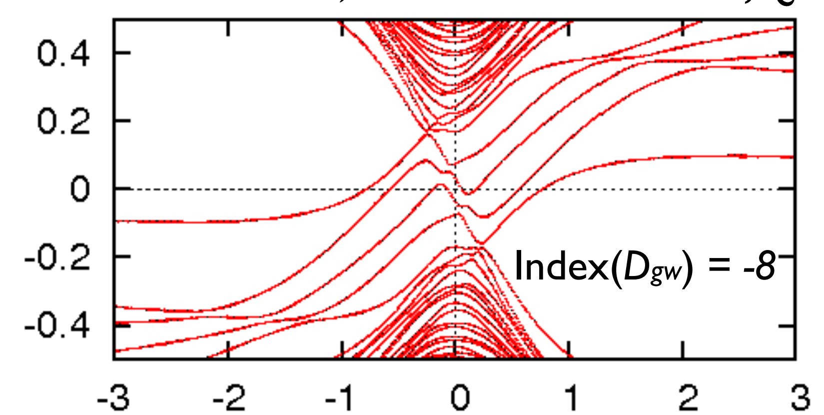
※ gauge configuration :

$$U_{x,\hat{1}} = e^{i\omega x_2}, \qquad U_{x,\hat{2}} = \begin{cases} 1 & (x_2 = 1, 2, \cdots, L-1) \\ e^{i\omega L x_1} & (x_2 = L) \end{cases} \qquad \omega = 2\pi Q$$

36×36 lattice, randomness δ=0.25, Q=1



36×36 lattice, randomness δ=0.25, Q=2



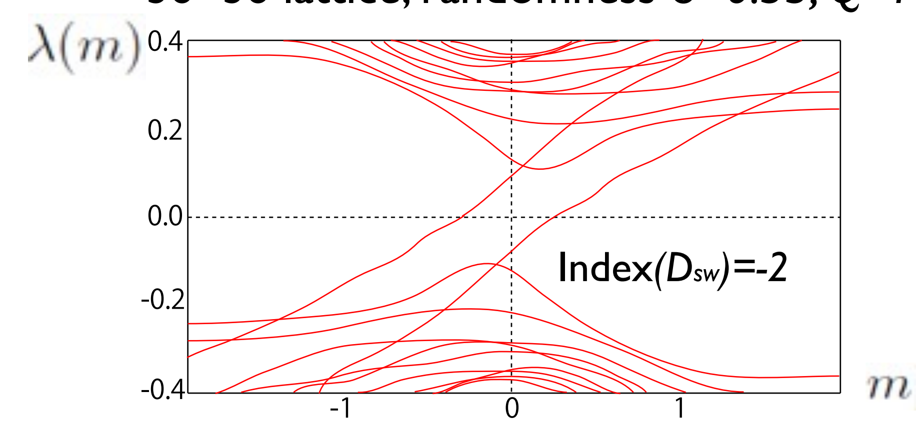
Staggered-Wilson

$$H_{sw} = \epsilon(D_{st} - M_f^{(A)}) = \Gamma_{55}(D_{st} - M_f^{(A)})$$

Index(D_{sw}) = - Spectral flow(H_{sw})

$$\mathrm{Index}(D_{sw}) = 2^{d/2}(-1)^{d/2}Q$$

36×36 lattice, randomness δ=0.33, Q=1

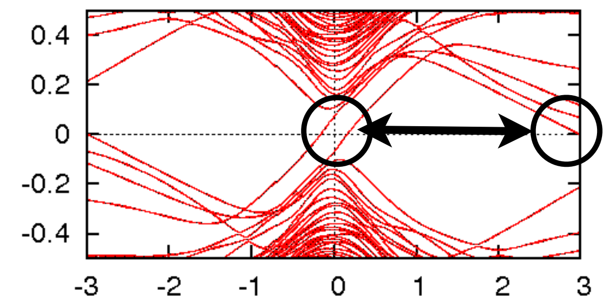


Index theorem holds for them.

◆Overlap formulation

negative-mass mode in D_W → massless mode in D_{ov}

Low-lying crossings are far from high-lying ones



- Generalized overlap

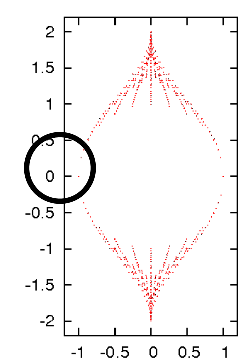
$$D_{go} = 1 + \gamma_5 \frac{H_{gw}(m)}{\sqrt{H_{gw}^2(m)}}$$

Any-flavor (1~15) overlap is possible!

cf.) 2 or 3-flavor overlap → lattice QCD

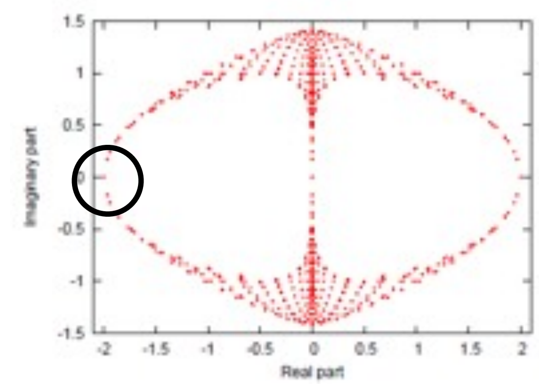
12-flavor overlap → conformal window

(8,8)



8-flavor overlp

(4,8,4)



4-flavor overlap

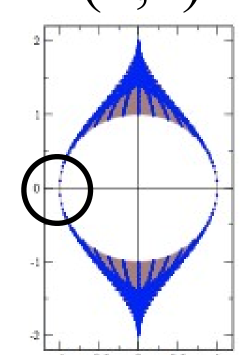
- Staggered-overlap

$$D_{so} = 1 + \Gamma_{55} \frac{H_{sw}(m)}{\sqrt{H_{sw}^2(m)}}$$

Less expensive overlap!

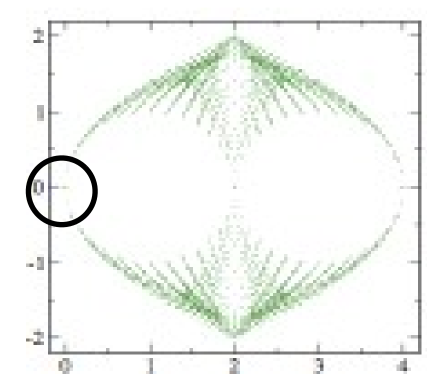
cf.) 1/4 matrix size → less CPU cost for Lanczos process

(2,2)



2-flavor overlap

(1,2,1)



1-flavor overlap

Details of StWil symmetries

$$\{\Xi_\mu, I_s, R_{\mu\nu}\} \quad \longrightarrow \quad \Gamma_4 \rtimes SW_4$$

$$\{\Xi'_\mu, R_{\mu\nu}\} \quad \longrightarrow \quad \Gamma_3 \rtimes SW_4$$

Physical-sector symmetry

$$\Xi'_j \Xi'_4 R^2_{j4} = \Xi_j \Xi_4 \sim (1 \otimes \sigma_j)$$

$$\Xi'_4 R^2_{34} R^2_{12} = \Xi_4 I_s \sim (\gamma_4 \otimes \mathbf{1})$$

$$C_0 \Xi'_2 \Xi'_4 R^2_{24} \sim C$$

Details of timeslice symmetries

staggered sym : $\{C_0, \Xi_\mu, I_s, R_{\mu\nu}, T_\mu^{1/2}\}$ $\Xi_\mu^2 = 1$

$$\longrightarrow T_\mu^{1/2} \rtimes [\{C_0, \Xi_\mu\} \rtimes \{R_{\mu\nu}, I_s\}] = (\otimes_j Z_{N_\mu}) \rtimes [\Gamma_{4,1} \rtimes W_4]$$

Timeslice sym : $T_\mu^{1/2} \rtimes [\{C_0, \Xi_\mu\} \rtimes \{R_{ij}, I_s\}] = (\otimes_j Z_{N_j}) \rtimes \underline{[\Gamma_{4,1} \rtimes W_3]}$

Relevant group at rest

$$\begin{aligned}
\Gamma_{4,1} \rtimes W_3 &\sim [\underline{\{R_{ij}, \Xi_{ij}\}} \times \{C_0, \Xi_4, \Xi_{123}, I_s\}]/Z_2 \\
&= [\underline{\{R_{ij}, \tilde{R}_{4i} \equiv \epsilon_{ijk} R_{jk} \Xi_{kj}\}} \times \{C_0, \Xi_4, \Xi_{123}, C_0 \Xi_4 I_s\}]/Z_2 \\
&= [\underline{SW_4} \times \Gamma_{2,2}]/Z_2
\end{aligned}$$

Staggered-Wilson

$\{C_0, \Xi'_\mu, R_{\mu\nu}, T'^{1/2}_\mu\}$

$$\begin{aligned}
&\sim [\{R_{ij}, \Xi'_{ij}\} \times \{C_0, \Xi'_4, \Xi'_{123}, I_s\}]/Z_2 \\
&= [\{R_{ij}, \tilde{R}_{4i} \equiv \epsilon_{ijk} R_{jk} \Xi'_{kj}\} \times \{C_0, \Xi'_4, \Xi'_{123}]/Z_2 \\
&= [\underline{SW_4 \times \Gamma_{1,2}}]/Z_2
\end{aligned}$$

$$\bar{\gamma}_1 = \sigma_3 \otimes \sigma_1 = \begin{pmatrix} \sigma_1 & \\ & -\sigma_1 \end{pmatrix} \quad \bar{\gamma}_2 = \sigma_3 \otimes \sigma_2 = \begin{pmatrix} \sigma_2 & \\ & -\sigma_2 \end{pmatrix}$$

$$P_1 = \sigma_1 \otimes \mathbf{1} = \begin{pmatrix} & \mathbf{1} \\ \mathbf{1} & \end{pmatrix} \qquad P_2 = \sigma_2 \otimes \mathbf{1} = \begin{pmatrix} & -i\mathbf{1} \\ i\mathbf{1} & \end{pmatrix}$$

$$\begin{aligned} S^{\mathrm{LAT}}_{\mathcal{N}=2} = Q \frac{1}{2g_0^2} \sum_x \operatorname{tr} \Bigg[& \frac{1}{4}\eta(x)\left[\phi(x),\ \bar{\phi}(x)\right] - i\chi(x)\Phi(x) + \chi(x)H(x) \\ & + i\sum_{\mu=1}^{2} \psi_\mu(x)\left(\bar{\phi}(x) - U_\mu(x)\bar{\phi}(x+\hat{\mu})U_\mu(x)^\dagger\right) \Bigg] \end{aligned}$$