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Hawking Radiation, Tunneling Method and Quantum Thermometers

Contents:

The Unknown Nature of Gravity

Four known elementary Interactions
Electromagnetic, Weak and Strong Interactions are unified
at quantum level:
Standard Model of elementary particles well understood

The fourth fundamental interactions, Gravity is the oldest one(Newton)
but also the only one not completely understood.
Question:
Has the Gravity to be quantized ?
Probably yes (Covariant and Path Integral approaches,
SuperGR, String Theory, Loop Gravity) **unsolved till now**
Other questions (Jacobson, Panamandhan and many others)
Is Gravity an emergent phenomena?
Is the Gravity the thermodynamics of the space-time ?
Not easy to answer

Aim of this talk: making use of

Semiclassical methods: classical gravity and quantum matter

Try to answer to specific the question:

How hot is our (expanding) universe?

main idea: Universe as Dynamical Black Hole

For the FRW de Sitter universe:

$$ds^2 = -dt^2 + e^{2H_0 t} d^2\vec{x}$$

the answer is well know:

$$\text{Temperature} \quad T = \frac{H_0}{2\pi} \quad \text{(Gibbon-Hawking 77)}$$

$H_0 = \sqrt{\frac{\Lambda}{3}}$, Hubble parameter, cosmological constant $\Lambda > 0$.

We will re-derive this famous result. The key point is the existence also of the static dS patch:

$$ds^2 = -(1 - H_0^2 r^2)dt^2 + \frac{dr^2}{1 - H_0^2 r^2)} + r^2 d\Omega^2$$

a Static Spherical Symmetric BH space-time.

Static Hawking effect well understood.

Recall dS space-time is particularly important in modern Cosmology:

Inflation and Dark Energy

What about FRW dS (Dynamical situation)?

Necessity to discuss on general ground:

Hawking effect, and Hawking temperature for Spherically Symmetric Dynamical BHs.

Hawking radiation

Hawking radiation is one of the most important predictions of quantum field theory in curved space-time.
The Hawking effect is associated with the presence of Black Holes
Classically BHs are compact gravitational objects from which matter (massless and massive) enter but not escape.

If quantum theory enters the game, something can escape:
Thermal Hawking radiation.
It turns out that (stationary) BHs are thermodinamical objects, and at least three fundamental issues have to be addressed:

i. BH entropy issue

ii. BH energy or mass issue

iii. BH temperature issue

We will mainly concentrate on the temperature issue.

Within covariant gravitational Lagrangian theories, the Entropy issue is well understood by the Wald method:

a robust recipe for evalutating BH entropy starting from the Lagrangian of the model.

For Ex: in $f(R)$ modified gravity, Wald method leads to

Generalized Area Law $S_{BH} = \frac{A_H f'_H}{4G}$

The energy issue is more problematic, we only recall :

ADM mass in asymptoticaly flat BH space-times.

For spherically symmetric BHs in Lovelock theories:

($d = 4$, $R - 2\Lambda$, $d = 5$, $R - 2\Lambda + \alpha G$, G Gauss-Bonnet)

there is the Misner-Sharp quasilocal energy. Others quasilocal energies can be defined (Brown-York, Bartnich,..).

The Hawking effect is kinematics: dynamics plays no role

Several derivations have been proposed,

among them the most popular:

i. Bogoliubov transformation method Hawking 75

ii. Damour-Ruffini method 1976

iii. Path-Integral in Kruskal gauge Hartle-Hawking 76

iv. Tunneling method, Parikh and Wilczek 2000

In the following, we will review a variant of tunneling method:

v. Hamilton-Jacobi Method Padmanabahn,our group 03-05

covariant and extendable to the dynamical case

(Hayward, Di Criscienzo, Nadalini, Vanzo, S.Z.)

H-J tunneling method: introduction

The H-J Tunneling method is **reasonably simple**:
Gravity at classical level, QM of particle enters in WKB or Eikonal approximation:

$$\text{Amplitude} \propto e^{i\frac{I}{\hbar}} \quad (c = 1)$$

Then **computation** of the classical action I along a trajectory in curved space-time **which includes the horizon**
: First key point: **the presence of Horizons leads to** $\text{Im}\, I \neq 0$ and semiclassical emission rate,

$$\Gamma \propto |\text{Amplitude}|^2 \propto e^{-2\frac{\text{Im}\, I}{\hbar}}$$

Second key point: Appearance of Boltzmann factor with linear dependence on energy ω

$$\Gamma \propto e^{-2\frac{\mathrm{Im}\, I}{\hbar}} \propto e^{-\frac{\beta}{\hbar}\omega}$$

In the static case, the Boltzmann factor suggests
a radiation in thermal equilibrium, and
it is reasonable that $T = \frac{\hbar}{\beta}$ is the BH temperature.
From now on $\hbar = 1$
What about the dynamical case?
Remark: In the dynamical case it is crucial that the argument of the exponent be a coordinate scalar (invariant quantity) otherwise no physical meaning can be given to Γ: Life is simple with **Spherical Symmetry** and related **Hayward Covar. Formalism**

Kodama-Hayward formalism

Generic spherically symmetric (SS) space-time metric

$$ds^2 = \gamma_{ij}(x^i)dx^i dx^j + R^2(x^i)dS^2\,, \qquad i,j = 0,1\,,$$

First ingredient: 2-dim normal metric, x^i related coordinates

$$d\gamma^2 = \gamma_{ij}(x^i)dx^i dx^j$$

Second ingredient: the scalar quantity in γ $R(x^i)$ called areal radius of sphere S^2.
Third ingredient: the further scalar in γ

$$\chi(x) = \gamma^{ij}(x)\partial_i R \partial_j R\,.$$

One has three cases: the two sphere with areal radius R is called
i. untrapped if $\chi_R < 0$,
ii. marginal if $\chi_R = 0$,
iii. trapped if $\chi_R > 0$.
A hypersurface foliated by marginal spheres is called trapping horizon, and defined by

$$\chi(x_H) = 0\,, \quad \partial_i \chi_H \neq 0\,.$$

Trapping horizons:
i time-like in evaporating BH and cosmology,
ii. null in Static (eternal) BH,
iii. space-like (accreting BH).

Fourth ingredient: scalar evaluated on the dynamical horizon

$$\kappa_H = \frac{1}{2}\Box_\gamma R_H = \frac{1}{2\sqrt{-\gamma}}\partial_i(\sqrt{-\gamma}\gamma^{ij}\partial_j R)_H \,.$$

called Hayward dynamical surface gravity.

One has three cases: a trapping horizon is said to be

i. outer if $\kappa_H > 0$,

ii. degenerate if $\kappa_H = 0$,

iii. inner if $\kappa_H < 0$.

Fifth ingredient: conserved Kodama vector

$$K^i(x) = \frac{1}{\sqrt{-\gamma}}\varepsilon^{ij}\partial_j R\,, \qquad \text{and} \qquad K^\theta = 0 = K^\varphi$$

which gives a preferred flow of time, generalizing the flow of time given by Killing vector ∂_t in static cases.

Ex. 4-dim static Schwarzschild BH $x = (t, r)$, $R = r$

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 dS^2$$

with $V(r) = 1 - \frac{2M}{r}$. Here

$$\chi(r) = V(r) = 1 - \frac{2M}{r}, \quad \chi(r_H) = 0 \rightarrow r_H = 2M$$

example of singular gauge, not defined on the horizon.
The surface gravity $k_H = \frac{V'_H}{2} = \frac{1}{2M}$. In GR, for spherical horizons, the Misner-Sharp quasi-local gravitational energy is defined by

$$E_{MS}(x) = \frac{R(x)}{2}\left(1 - \chi(x)\right) .$$

This is an invariant quantity on the normal space. Note that on the horizon $E_{MS}(R_H) = \frac{R_H}{2} = M$, typically in GR the BH mass.

Vaidya black hole

As a simple example of dynamical BH:
the Vaidya black hole

$$ds^2 = -(1 - \frac{2m(v)}{r})dv^2 + 2dv\,dr + r^2 d\Omega^2$$

where $m(v)$ is the mass function of advanced time v.
The invariant $\chi(v)$

$$\chi(v) = 1 - \frac{2m(v)}{r}$$

Trapping horizon is $r_H(v) = 2m(v)$ and Hayward surface gravity

$$k_H(v) = \frac{1}{2m(v)}.$$

Intermezzo: The First Law in GR

The Kodama vector and κ_H are geometric kinematical quantities . If the dynamics enters, one has

Lemma: within GR and on spherical dynamical horizon

$$\kappa_H = \frac{1}{2R_H} + 2\pi R_H T^{(2)}_H \,,$$

where the reduce stess-tensor trace scalar in the normal space

$$T^{(2)} = \gamma^{ij} T_{ij} \,.$$

Proof: Use the Einstein Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}$$

and evaluated them on the spherical dynamical horizon.

Consequence: Introduce

$$\mathcal{A}_H = 4\pi R_H^2\,, \quad \text{horizon area}$$

$$V_H = \frac{4}{3}\pi R_H^3\,, \quad \text{horizon volume}$$

Then,

$$\frac{\kappa_H}{8\pi} d\mathcal{A}_H = d\left(\frac{R_H}{2}\right) + T_H^{(2)} dV_H\,.$$

Since $M = \frac{R_H}{2}$, the Lemma gives: The First Law (Hayward)

$$dM = \frac{\kappa_H}{2\pi} d\left(\frac{\mathcal{A}_H}{4}\right) - T_H^{(2)} dV_H$$

The viceversa is very interesting:

Assuming the First law and existence of horizons, Einstein equations follow: gravity emergent phenomena? (Jacobson, Padmanabhan,..)

H-J tunneling method in action

Several ingredients:

i. Invariant definition of energy of the particle via Kodama vector

$$\omega = -K^{\mu} p_{\mu} \, ,$$

p_{μ} is the 4-momentum of the particle: Working in SS spaces independence of a specific choice of spacetime coordinates.

ii. Relativistic Hamilton-Jacobi equation (radial trajectories)

Recall $p_{\mu} = \partial_{\mu} I$ where I classical action

$$\gamma^{ij} \partial_i I \partial_j I + m^2 = 0 \, .$$

The effective mass m here defines two important and complementary energy scales:

a. In the Hawking effect as horizon tunnelling we may neglect the mass,

$$\gamma^{ij}\partial_i I \partial_j I = 0\,.$$

Physics comes from horizon singularities.
b. Effects in the bulk away from the horizon, ex decay rate of composite particles, the effective mass becomes relevant as the energy of the particle can be smaller than the energy scale settled by m, and branch cut singularity is present.
Then one make two other key assumptions:
a. The near horizon approximation.
b. The null expansion assumption.
This last requirement force us to make use of regular gauges on the horizon. For example, the usual static Schwarzschild gauge is singular on the horizon.

iii. Reconstruction of the action

$$I = \int_\gamma dx^i \, \partial_i I$$

solving the HJ equation and with γ being a suitable path in the normal space, which includes the horizon.

iv. Feynman prescription

The classical action is divergent on the horizon and the imaginary part comes from Feynman prescription (or contour deformation):

$$\int_\gamma \frac{f(r)}{r - r_H} dr \rightarrow \int_\gamma \frac{f(r)}{r - r_H - i0} dx = i\pi f(r_H) + \text{real part}$$

A generic spherically symmetric dynamical BH

Starting point: **any** spherically symmetric metric can be rewritten in the Eddington-Filkenstein-Bardeen regular gauge

$$ds^2 = -e^{2\Psi} C dv^2 + 2e^{\Psi} dv dr + r^2 d\Omega^2 \,,$$

where $x^i = (v, r)$ as coordinates, and $C = C(v, r)$, $\Psi = \Psi(v, r)$ smooth functions. Here simply $R = r$ and $\chi = C$,
thus the dynamical horizon $C(r_H, v) = 0$.
The Kodama vector $K = (e^{-\Psi}, 0)$, and Kodama energy $\omega = e^{-\Psi} \partial_v I$
Hayward invariant surface gravity $\kappa_H = \frac{\partial_r C_H}{2}$.
First step: expansion on a null direction in the neighbour of the horizon gives

$$0 = e^{\Psi_H} dv dr \,.$$

No temporal contribution to the imaginary part of the action.

Second step: from the H-J equation and Kodama energy

$$\partial_r I = 2\frac{\omega}{C}$$

Thus, within a neighborhood of γ containing the horizon where $C(r_H, v) = 0$

$$\text{Im}\, I = \text{Im} \int_\gamma dr \partial_r I = 2 \int_\gamma dr \frac{\omega}{C} \simeq 2 \int_\gamma dr \frac{\omega}{\partial_r C} \frac{1}{(r - r_H - i0)} = \frac{\pi \omega_H}{\kappa_H},$$

where $C = \frac{k_H}{2}(r - r_H)$ around the horizon along the null direction and the Feynman prescription have been used. Thus

$$\Gamma \simeq e^{-\frac{2\pi}{\kappa_H}\omega_H}$$

with κ_H and ω_H scalars on the normal space: Γ invariant. This result is valid for generic SS space-time, in particular for a static BH space-time.

Here we present the operational interpretation.
Static observers in static BH become in the dynamical case Kodama observers whose velocity

$$v_K^i = \frac{K^i}{\sqrt{\chi}}, \quad \gamma_{ij} v_K^i v_K^j = -1$$

Lemma:
Kodama observers are such that $R = R_0$, constant areal radius
The energy measured by this Kodama observer at fixed R_0 is

$$E = -v_K^i \partial_i I = -\frac{K^i \partial_i I}{\sqrt{\chi_0}} = \frac{\omega}{\sqrt{\chi_0}}$$

The tunneling rate can be rewritten as

$$\Gamma \simeq e^{-\frac{2\pi}{\kappa_H}\sqrt{\chi_0}E} \simeq e^{-\frac{E}{T_0}}$$

and the local quantity T_0 at radial radius R_0 is also invariant: it contains an invariant factor $\sqrt{\chi}$

$$T_0 = \frac{T_H}{\sqrt{\chi_0}}, \quad T_H = \frac{k_H}{2\pi}$$

In the static case $\chi = g^{rr} = -g_{00}$ and recalling Tolman's theorem:
For a gravitational system at thermal equilibrium, $T\sqrt{-g_{00}} =$ constant,
it follows $T_H = \frac{\kappa_H}{2\pi}$ is the intrisic temperature of the BH:
the Hawking temperature.
In the dynamical case and for slow changes in the geometry, the question is :
Is still $T_H = \frac{\kappa_H}{2\pi}$ the dynamical Hawking temperature ?
It is an unsolved issue till now.

However, in a generic SS space-time and in GR, Hayward surface gravity κ_H and ω_H are invariant quantities and recalling the Area Law $S_H = \frac{A_H}{4G}$ (Bekenstein-Hawking Entropy), the First Law can be rewritten as

$$dM = T_H dS_H - T_H^{(2)} dV_H \,.$$

A further hint that $T_H = \frac{\kappa_H}{2\pi}$ could be a sort of temperature.

General static black hole space-time

The starting point may be a BH metric in the Schwarzschild static singular gauge

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 dS^2$$

with $V(r_H) = 0$, $V'_H \neq 0$.
Horizon located at $r = r_H$, and the Kodama vector coincides with the usual Killing vector $(1, 0, 0, 0)$, Hayward surface gravity is the Killing surface gravity, namely $\kappa_H = \kappa = \frac{V'_H}{2}$.
One can use of regular Painleve' gauge (Parikh-Wilczek)

$$ds^2 = -V(r)dt^2 - \sqrt{1 - V(r)}dt\, dr + dr^2 + r^2 dS^2$$

But, the final result has to be gauge-invariant, and we use the complete and regular time dependent Kruskal gauge.
Introduce the tortoise coordinate

$$dr^* = \frac{dr}{V(r)}$$

$-\infty < r^* < \infty$. Thus

$$ds^2 = V(r^*)(-dt^2 + (dr^*)^2) + r^2(r^*)dS^2$$

Then Kruskal-like coordinates

$$X = \frac{1}{\kappa} e^{\kappa r^*} \cosh \kappa t \,, \quad T = \frac{1}{\kappa} e^{\kappa r^*} \sinh \kappa t$$

with

$$-T^2 + X^2 = \frac{1}{\kappa^2} e^{2\kappa k r^*}$$

and a conformally flat normal metric appears

$$ds^2 = V(r^*) e^{-2\kappa r^*}(-dT^2 + dX^2) + r^2(T,X)dS^2$$

new coordinates being T and X while $r^* = r^*(T,X)$.

In this gauge, the metric is still spherically symmetric, regular on the horizon: the conformal factor $e^{\Psi} = V(r*)e^{-2\kappa r^*}$ well defined for $r = r_H$, but time dependent. The trapping horizon

$$(\partial_T r)_H = (\partial_X r)_H \,,$$

equivalent to $T = \pm X$ and $r^* \to -\infty$. The Killing-Kodama vector

$$K = e^{-\Psi(r^*)}(\partial_X r, -\partial_T r)$$

Hayward surface gravity is the Killing (recall κ_H is invariant):

$$\kappa_H = \frac{e^{-\Psi_H}}{2}\left(-\partial_T^2 r + \partial_X^2 r\right)_H = \frac{V'_H}{2}$$

Thus

$$\Gamma \simeq e^{-\frac{4\pi}{V'(r_H)}\omega_H} \simeq e^{-\frac{E}{T_0}}\,, \quad T_0 = \frac{V'_H}{4\pi\sqrt{V_0}}\,, \quad T_0 = \frac{T_H}{\sqrt{V_0}}$$

Important check of the H-J dynamical tunneling method.

The FRW Spacetime

FRW space-times are important in cosmology.
The flat FRW space as Spherical Symm. Dynamical BH

$$ds^2 = -dt^2 + a^2(t)dr^2 + [a(t)r]^2 dS^2$$

The normal reduced metric is diagonal, areal radius $R = a(t)r$ and the invariant χ

$$\chi(t,r) = 1 - a(t)r^2 H^2(t)\,, \qquad H(t) = \frac{\dot{a}(t)}{a(t)}$$

The dynamical horizon is implicitly given by $\chi_H = 0$,

$$R_H := a(t)r_H = \frac{1}{H(t)}$$

Hayward surface gravity is

$$\kappa_H = -\left(H^2(t) + \frac{1}{2}\dot{H}(t)\right) R_H(t)\,,$$

and the minus sign refers to the fact the Hubble horizon is, in Hayward's terminology, of the inner type.

In the flat case $R_H = \frac{1}{H(t)}$, Hubble radius, and we rewrite

$$\kappa_H = -\left(H(t) + \frac{\dot{H}(t)}{2H(t)}\right)$$

Thus the rate has the exponential form

$$\Gamma \simeq e^{-\frac{2\pi}{(\kappa_H)}\omega_H} \simeq e^{-\frac{E}{T_0(t)}}$$

the local time dependent "factor" at R_0

$$T_0(t) = \frac{T_H}{\sqrt{1 - R_0^2 H^2}}.$$

For slowly metric changing, it seems suggestive to interpret $T_H = -\frac{\kappa_H}{2\pi}$ as dynamical temperature associated with FRW space-times, explicitely

$$T_H = \frac{1}{2\pi}\left(H(t) + \frac{\dot{H}(t)}{2H(t)}\right)$$

For dS space $T_0 = \frac{T_H}{\sqrt{1-R_0^2 H_0^2}}$, $T_H = \frac{H_0}{2\pi}$, real temperatures.

It should be important to have a QFT confirmation of the H-J tunnelling result:

Quantum Field Theory in action.

Quantum field theory in conformally flat space-times

A quantum field theoretical approach, with a quantum probe is used to unveil the intrinsic temperature associated with static space-times with horizons.
The probe: a conformally coupled massless scalar field Φ
We work on conformally FRW flat space-time, conformal time η $d\eta = \frac{dt}{a}$,

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2)\,, \qquad x = (\eta, \vec{x})$$

The knowledge of Wightman function $W(x,x')$ is crucial

$$W(x,x') =< \Phi(x)\Phi(x') >= \sum_{\vec{k}} f_{\vec{k}}(x) f^*_{\vec{k}}(x')\,, \quad \Phi(x) = \sum_{\vec{k}} f_{\vec{k}} a_{\vec{k}} + h.c.$$

The modes functions $f_{\vec{k}}(x)$ satisfy the conformally invariant equation ($\mathcal{R}$ being the curvature scalar) sastisfied by $\Phi(x)$

$$\left(\Box - \frac{\mathcal{R}}{6}\right) f_{\vec{k}}(x) = 0\,.$$

The solution of this equation is a vacuum choice

$$f_{\vec{k}}(x) = \frac{e^{-i\eta k}}{2\sqrt{k}a(\eta)} e^{-i\vec{k}\cdot\vec{x}} \quad k = |\vec{k}|$$

$W(x, x')$ can be computed in an exact way

$$W_\varepsilon(x, x') = \frac{1}{4\pi^2 a(\eta)a(\eta')} \frac{1}{|\vec{x} - \vec{x}'|^2 - |\eta - \eta' - i\epsilon|^2}.$$

As is usual in distribution theory we shall leave understood the limit as $\epsilon \to 0^+$. However, it has been shown by Takagi and Schlicht that it is necessary the covariant form

$$W_\varepsilon(x, x') = \frac{1}{4\pi^2 a(\eta)a(\eta')} \frac{1}{[(x - x') - i\epsilon(\dot{x} + \dot{x}')]^2}.$$

where an over dot stands for derivative with respect to proper time

The Unruh-De Witt detector

The Unruh-De Witt detector approach is a well known and used technique for exploring quantum field theoretical aspects in curved space-time.
The transition probability per unit proper time of the detector depends on the response function per unit proper time which, for radial trajectories, at finite proper time τ, and this depends on Wightman function

$$\dot{F}(E,\tau) = \frac{1}{2\pi^2}\mathrm{Re}\int_0^{\tau-\tau_0} ds e^{-iEs} W_\varepsilon(x(\tau), x(\tau - s))$$

where τ_0 is the initial detector proper time, E is two-level positive detector transition energy. The Wightman function $W(x(\tau), x(\tau - s))$ is singular at $s = 0$ because $x \to x'$ is singular and $\epsilon \to 0^+$ cures the singularity.

Subtracting the leading divergence at $s = 0$ and using the normalisation condition

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \equiv [a(\tau)\dot{x}(\tau)]^2 = -1\,,$$

introducing the geodesic distance

$$\sigma^2(\tau, s) \equiv a(\tau)a(\tau - s)[x(\tau) - x(\tau - s)]^2\,,$$

which for small s

$$\sigma^2(\tau, s) = -s^2[1 + s^2 d(\tau, s)]\,.$$

one may take the limit $\epsilon \to 0$ (Louko, Satz 08)

Final expression

$$\dot{F}(E,\tau) = \frac{1}{2\pi^2}\int_0^\infty ds\, \cos(Es)\left(\frac{1}{\sigma^2(\tau,s)} + \frac{1}{s^2}\right) + J_\tau(E)\,,$$

where the "tail" or finite time fluctuating term is

$$J_\tau(E) := -\frac{1}{2\pi^2}\int_{\Delta\tau}^\infty ds\, \frac{\cos(E\, s)}{\sigma^2(\tau,s)}\,.$$

In the important stationary cases:

$$\sigma(\tau,s)^2 = \sigma^2(s) = \sigma^2(-s)$$

$$\dot{F}(E,\tau) = \frac{1}{4\pi^2}\int_{-\infty}^\infty ds\, e^{-iEs}\left(\frac{1}{\sigma^2(s)} + \frac{1}{s^2}\right) + J_\tau = \dot{F}(E) + J_\tau(E)\,.$$

The first term is τ independent, and all the time dependence is contained only in the fluctuating tail, and $J_\tau(E) \to 0$ for large τ

Quantum thermometers in static and stationary spaces

Applications:

Rindler space (Unruh effect)

Generic static black hole (Hawking effect)

I will talk only on Hawking effect. The important case of the de Sitter space, independence of the coordinates choice (gauge independence) will be explicitly verified.

The generic static black hole

Recall for a static black hole reads

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 dS^2 \,,$$

the horizon is located at $r = r_H$
the surface gravity, is $\kappa_H = \kappa = V'_H/2$.
Recall the Kruskal gauge

$$ds^2 = e^{-2\kappa_H r^*} V(r^*)[-dT^2 + dX^2] + r^2(T,X)dS^2 \,,$$

coordinates are T and X, $r^* = r^*(T,X)$, and the normal metric turns out to be conformally flat. For Kodama observers the areal radius $r(T,X)$ and r^* are **constant**, say $r = r_0$.

the proper time along Kodama trajectories reads

$$d\tau^2 = V_0\, dt^2 = V_0 e^{-2\kappa r_0^*}(dT^2 - dX^2) = a^2(r_0)(dT^2 - dX^2)\,.$$

Thus $t = \tau/\sqrt{V_0}$ and, as a function of proper-time τ

$$\begin{aligned} X(\tau) &= \frac{1}{\kappa_H} e^{\kappa_H r_0^*} \cosh\left(\kappa_H \frac{\tau}{\sqrt{V_0}}\right) \\ T(\tau) &= \frac{1}{\kappa_H} e^{\kappa_H r_0^*} \sinh\left(\kappa_H \frac{\tau}{\sqrt{V_0}}\right)\,. \end{aligned}$$

The geodesic distance reads

$$\sigma^2(\tau,s) = V_0 e^{-2\kappa r_0^*}\left[-\left(T(\tau)-T(\tau-s)\right)^2+\left(X(\tau)-X(\tau-s)\right)^2\right]\,,$$

and one gets

$$\sigma^2(\tau,s) = -\frac{4V_0}{\kappa_H^2}\sinh^2\left(\frac{\kappa_H\, s}{2\sqrt{V_0}}\right)\,.$$

Stationarity: $\sigma^2(\tau,s)=\sigma^2(s)=\sigma^2(-s)$, $\dot{F}$ can be exactly computed

$$\dot{F}(E) = \frac{1}{2\pi}\,\frac{E}{\exp\left(\frac{2\pi\sqrt{V_0}E}{\kappa_H}\right)-1}\,.$$

Planck distribution: Unruh-DeWitt detector in the generic SSS BH space-time detects a quantum system in thermal equilibrium at the local temperature

$$T_0 = \frac{\kappa_H}{2\pi\sqrt{V_0}}\,.$$

Again Tolman's theorem: $T\sqrt{-g_{00}} =$ constant.
Generic static black hole has intrinsic constant temperature, the Hawking temperature, i.e.

$$T_H = \frac{\kappa}{2\pi} = \frac{V'_H}{4\pi}\,.$$

Another form of Tolmann factor: the Killing-Kodama observers with $r = r_0$ constant, have an invariant acceleration

$$a_0^2 = \frac{V_0'^2}{4V_0}\,,$$

where $a^\mu = u^\nu \nabla_\nu u^\mu$, u^μ being the observer's four-velocity. Thus, the local equilibrium temperature can be rewritten in the form

$$T_0 = T_H \frac{2a_0}{V'_0}\,.$$

de Sitter space in the static patch

The static de Sitter metric is a particular case with

$$V(r) = 1 - H_0^2 r^2 , \qquad H_0^2 = \frac{\Lambda}{3} .$$

The horizon is located at $r_H = H_0^{-1}$ and the surface gravity is $k_H = H_0$

The acceleration at fixed r_0 reads $a_0^2 = \frac{H_0^4 r_0^2}{1 - H_0^2 r_0^2}$, thus

$$a_0^2 + H_0^2 = \frac{H_0^2}{1 - H_0^2 r_0^2} .$$

and de Sitter local temperature felt by the Unruh detector is (Narnhofer-Thirring)

$$T_{dS}(r_0) = \frac{1}{2\pi}\sqrt{a_0^2 + H_0^2} .$$

de Sitter space in FRW coordinates

In this case the de Sitter metric is a SS dynamical BH

$$ds^2 = -dt^2 + e^{2H_0 t}(dr^2 + r^2 dS^2)\,, \quad a(t) = e^{H_0 t}.$$

with $H(t) = H_0$ constant. For Kodama observers $r = R_0 e^{-H_0 t}$ and denoting $V_0 = 1 - H_0^2 R_0^2$

$$\tau = \sqrt{V_0}\, t\,, \qquad a(\tau) = e^{\frac{H_0}{\sqrt{V_0}}\tau}\,,$$

and

$$\eta(\tau) = -\frac{1}{H_0} e^{-\frac{H_0}{\sqrt{V_0}}\tau}\,, \quad r(\tau) = R_0\, e^{-\frac{H_0}{\sqrt{V_0}}\tau}\,,$$

so, the geodesic distance is

$$\sigma^2(\tau, s) = -\frac{4\,V_0}{H_0^2} \sinh^2\left(\frac{H_0\, s}{2\sqrt{V_0}}\right)\,.$$

Again stationarity: $\sigma^2(\tau, s) = \sigma^2(s) = \sigma^2(-s)$, and

$$\dot{F}(E) = \frac{1}{2\pi} \frac{E}{e^{\frac{2\pi\sqrt{V_0}E}{H_0}} - 1},$$

In the FRW de Sitter space one detects a quantum system in thermal equilibrium at a temperature $T_0 = \frac{H_0}{2\pi\sqrt{V_0}}$.

Recall that 4-acceleration of a Kodama observer in a FRW space-time is

$$a^2 = a^\mu a_\mu = R_0^2 \left[\frac{\dot{H}(t) + (1 - H^2(t)R_0^2)H^2(t)}{(1 - H^2(t)R_0^2)^{\frac{3}{2}}} \right]^2$$

where $a^\mu := u^\nu \nabla_\nu u^\mu$, u^μ being the 4-velocity of the detector.

As a result, for dS space in a time dependent patch we have

$$a_0^2 = \frac{R_0^2 H_0^4}{1 - R_0^2 H_0^2},$$

showing that

$$\frac{H_0}{\sqrt{1 - H_0^2 R_0^2}} = \sqrt{H_0^2 + a_0^2}$$

in agreement with the dS static patch calculation.

This is an important check of the approach, since it shows the coordinate independence of the result for the important case of de Sitter space.

When $R_0 = 0$, one has $V_0 = 1$ and the classical Gibbons-Hawking result $T_{dS} = \frac{H_0}{2\pi}$ is recovered.

Wath about a generic FRW space-time?
In the stationary case, in the limit $\tau \to \infty$, one has

$$\dot{F}(E) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} ds\, e^{-iEs} \left(\frac{1}{\sigma^2(s)} + \frac{1}{s^2} \right) = \frac{1}{2\pi} \frac{E}{\exp\left(\frac{E}{T_0}\right) - 1}.$$

This is a sort of Fluctuation-Dissipation Theorem, since $\dot{F}(E)$ is a Fourier transform of a renormalized correlation function (Wightman function). Note that in this case Detailed Balance Principle holds

$$\frac{\dot{F}(E)}{\dot{F}(-E)} = e^{-\frac{E}{T_0}}, \quad \dot{F}(-E) - \dot{F}(E) = \frac{E}{2\pi}$$

Viceversa, if the above relations hold then $\dot{F}(E)$ is the Plank distribution. Thus we can define the local equilibrium temperature:

$$T_0 = \frac{E}{\ln \dot{F}(-E) - \ln \dot{F}(E)}$$

Alternatively

$$T_0 = \frac{E}{\ln\left(1 + \frac{E}{2\pi(\dot{F}(E))}\right)}$$

Thus, in general, we may define the local effective temperature by

$$\frac{\dot{F}(E,\tau)}{\dot{F}(-E,\tau)} = e^{-\frac{E}{T_0(\tau)}}.$$

Thus

$$T_0(\tau) = \frac{E}{\ln \dot{F}(-E,\tau) - \ln \dot{F}(E,\tau)}$$

or

$$T_0(\tau) = \frac{E}{\ln\left(1 + \frac{E}{2\pi(\dot{F}(E,\tau) - J(\tau))}\right)}$$

In the stationary case, $\dot{F}$ is time independent, and for large $\Delta\tau$

$$T_0 = \frac{E}{\ln\left(1 + \frac{E}{2\pi(\dot{F})}\right)}$$

dS asymptotically space-times

Important physical ex: flat FRW universe with $\Lambda > 0$ and matter $p = 0$. The solution of Friedman eq. (asymptotically dS)

$$a(t) = a_0 \sinh^{2/3}\left(\frac{3}{2}h\,t\right) \to e^{h\,t} \quad t \to \infty$$

with $h \equiv \sqrt{\Omega_\Lambda} H_0$, $\Omega_m + \Omega_\Lambda = 1$; $H_0 = \sqrt{8\pi\rho_{cr}/3}$.
Take for simplicity the comoving observer $R_0 = 0$. Expansion in $\Delta\tau$

$$\frac{1}{\sigma^2(\tau,s)} = \frac{1}{\sigma^2_{dS}(s)} - h^2 \sum_{n=1}^{\infty} \left(e^{-3hn\Delta\tau} \sum_{k=1}^{3n-1} g(n,k)\, e^{k\,hs} \right),$$

$g(n,k)$ computable numerical coefficients. Note pure de Sitter contribution,

$$\sigma^2_{dS}(s) = -4h^{-2} \sinh^2\left(\frac{hs}{2}\right)$$

with the effective Hubble constant $h = \sqrt{\Omega_\Lambda} H_0$.

Direct computation

$$\dot{F}_\tau(E) = \dot{F}_{dS}(E) + J_{dS,\tau}(E) \; - \; \frac{h^2}{2\pi^2} \sum_{n=1}^{\infty} \sum_{k=1}^{3n-1} g(n,k)\, e^{-3n\,h\Delta\tau} \times \frac{e^{hk\Delta\tau}(hk\cos(E\Delta\tau) + E\sin(E\Delta\tau)) - hk}{h^2k^2 + E^2} \tag{1}$$

$J_{dS,\tau}$ is the dS tail. The numerical coefficients $g(n,k)$ appear in another tail which decays exponentially in time with oscillating terms vanishing in the limit $\Delta\tau \to \infty$.

Thus the detector clicks close to a de Sitter response and reaches thermalization through decaying oscillations as $\Delta\tau$ is sufficiently large and the local effective temperature goes to dS temperature

$$T_{dS} = h = \sqrt{\Omega_\Lambda} H_0$$

Buchholz Quantum Thermometer

Another proposal to detect local temperature associated with stationary space-time admitting an event horizon has been put forward by Buchholz and collaborators. The idea may be substantiated by the following argument.

Let us start with a free massless quantum scalar field $\Phi(x)$ in thermal equilibrium at temperature T in flat space-time. It is well known that finite temperature effects may be investigated by considering the scalar field defined in the Euclidean manifold $S_1 \times R^3$,
with imaginary time $\tau = -it$, compactified in the circle S_1, with period $\beta = \frac{1}{T}$.

Let us consider the local quantity $<\Phi(x)^2>$. Formally, this is a divergent quantity: product of valued operator distribution in the same point x: regularisation and renormalisation are required.
Within zeta-function regularisation (Dowker, Hawking, and many others)

$$<\Phi(x)^2>=\zeta(1|L_\beta)(x)\,,$$

where $\zeta(z|L_\beta)(x)$ is the analytic continuation of the local zeta-function associated with the operator elliptic self-adjoint L_β

$$L_\beta=-\partial_\tau^2-\nabla^2\,,$$

defined on $S_1\times R^3$. The local zeta-function is defined with Re z sufficiently large by means

$$\zeta(z|L_\beta)(x)=\frac{1}{\Gamma(z)}\int_0^\infty dt t^{z-1}K_t(x,x)\,,$$

where the heat-kernel is (spectral theorem)

$$K_t(x,x)=<x|e^{-tL_\beta}|x>=\frac{1}{\beta(4\pi t)^{3/2}}\sum_n e^{-\frac{4\pi^2}{\beta^2}n^2}$$

Jacobi-Poisson formula leads to

$$K_t(x,x) = \frac{1}{(4\pi t)^2} \sum_n e^{-\frac{n^2\beta^2}{4t}} \,.$$

The term $n = 0$ leads to divergent integral $\int_0^\infty dt t^{z-3}$, but within Gelfand analytic continuation it vanishes and one has

$$\zeta(z|L_\beta)(x) = \frac{\Gamma(2-z)}{8\pi^2\Gamma(z)} \left(\frac{\beta^2}{4}\right)^{z-2} \zeta_R(4-2z)\,,$$

where $\zeta_R(z)$ is the Riemann zeta-function. This analytic continuation is regular at $z = 1$, recalling that $\zeta_R(2) = \frac{\pi^2}{6}$, one has

$$< \Phi(x)^2 >= \frac{1}{12\beta^2} = \frac{T^2}{12}\,.$$

The regularised vacuum expectation value of the observable Φ^2 gives the temperature of the quantum field in thermal equilibrium:
a quantum thermometer.

Another derivation (more traditional)

Quantum thermal Euclidean Wigthman function periodic of period β in the imaginary time is

$$W_\beta(x,x') =< \Phi(x)\Phi(x') >= \frac{1}{4\pi^2} \sum_n \frac{1}{|\vec{x} - \vec{x'}|^2 + (\tau - \tau' + n\beta)^2}$$

The term $n = 0$ is singular when $x \to x'$.

Subtracting this term (renormalization), one has in the coincidence limit

$$W_\beta(x,x) =< \Phi(x)^2 >= \frac{1}{2\pi^2\beta^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Again

$$< \Phi(x)^2 >= \frac{1}{12\beta^2} = \frac{T^2}{12}$$

Motivated by this argument, consider again the quantum probe in a FRW conformally flat space-time.
Off-diagonal Wigthman function is

$$W(x,x') =< \Phi(x)\Phi(x') >= \frac{1}{4\pi^2}\frac{1}{\sigma^2(x,x')}\,,$$

where

$$\sigma^2(x,x') = a(x)a(x')(x-x')^2\,,$$

recall $a(x)$ being the conformal factor. In the limit $x \to x'$, formally one has

$$< \Phi(x)^2 >= W(x,x)\,,$$

but $W(x,x)$ is ill defined:
again regularisation and renormalisation

Use point splitting regularization: take $W(x,x')$ and evaluate the limit $x' \to x$. Then subtract the leading divercence and define the local temperature as

$$\frac{T^2}{12} = < \Phi(x)^2 >_R = F.P. \lim_{x' \to x} W(x,x')\,,$$

where $F.P.$ stands for finite part prescription, the two points are joined by a Kodama trajectory parametrized by the proper time.

In the stationary space-times we have considered, the relevant Wightman functions of the probe all have the specific form

$$W(s,s+\varepsilon) = -\frac{1}{4\pi^2}\frac{\alpha_0^2}{4\sinh^2(\varepsilon\,\frac{\alpha_0}{2})}\,.$$

In fact,

for a generic static black hole

$$\alpha_0 = \frac{\kappa_H}{\sqrt{V_0}}\,, \qquad \kappa_H = \frac{V'_H}{2}\,,$$

and for the de Sitter space-time in flat FRW form

$$\alpha_0 = \frac{H_0}{\sqrt{1 - R_0^2 H_0^2}}\,.$$

In invariant form

$$\alpha_0 = \frac{\kappa_H}{\sqrt{\chi_0}}$$

Within point splitting regularization

$$W(s, s+\varepsilon) = -\frac{1}{4\pi^2}\frac{\alpha_0^2}{4\,\sinh^2(\varepsilon\,\frac{\alpha_0)}{2}} = -\frac{1}{4\pi^2\varepsilon^2} + \frac{1}{12}\left(\frac{\alpha_0}{2\pi}\right)^2 + O(\varepsilon)\,.$$

Naive Renormalization (F.P. prescription) gives

$$T_0 = \frac{\alpha_0}{2\pi} = \frac{T_H}{\sqrt{\chi_0}}$$

in agreement with H-J tunneling and Unruh-DeWitt detector technique.

Again, the local dependence comes from the localisation of the Kodama observer at $R = R_0$, R being the areal radius of BH.

Summary and Concluding remarks

Aim of the talk: understand the temperature-versus-surface gravity paradigm.

The tunnelling in its H-J covariant version leads to a very simple result for a spherically symmetric dynamical black hole

$$\text{Rate}\Gamma \simeq e^{-\frac{\omega_H}{T_H}} \simeq e^{-\frac{E}{T_0}}, \quad T_0 = \frac{T_H}{\sqrt{\chi_0}}$$

where k_H is the Hayward dynamical surface gravity. To what extend one may talk of dynamical temperature?

Answer has been tested with QFT techniques as the Unruh-DeWitt detector and the Buchholz proposal.

In the stationary cases (static black holes, dS-FRW) complete agreement with QFT methods.

In the dynamical case, positive answer for asymptotically dS space-times. In general, still an unsolved issue. Work in progress.

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