# Relativistic matter in a magnetic field: New face of the chiral anomaly

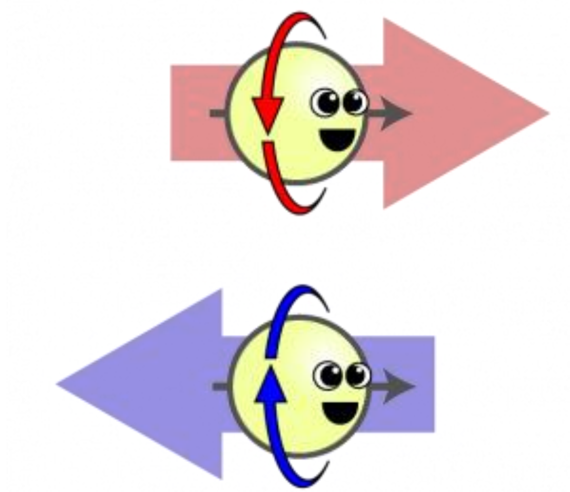
**Volodya Miransky***

**Western University, Canada**

*E. Gorbar, V. Miransky, I. Shovkovy, and Xinyang Wang,
Phys. Rev. B 88, 165105 (2013)

# Helicity/Chirality

- Helicities of (ultra-relativistic) massless particles are (approximately) conserved



**Right-handed**

**Left-handed**

- Conservation of chiral charge is a property of massless Dirac theory (classically)

- The symmetry is anomalous at quantum level

# Chiral magnetic effect

- Chiral charge is produced by topological QCD configurations

$$\frac{d(N_R - N_L)}{dt} = -\frac{g^2 N_f}{16\pi^2} \int d^3x \, F_a^{\mu\nu} \tilde{F}^a_{\mu\nu}$$

- Random fluctuations with nonzero chirality in each event

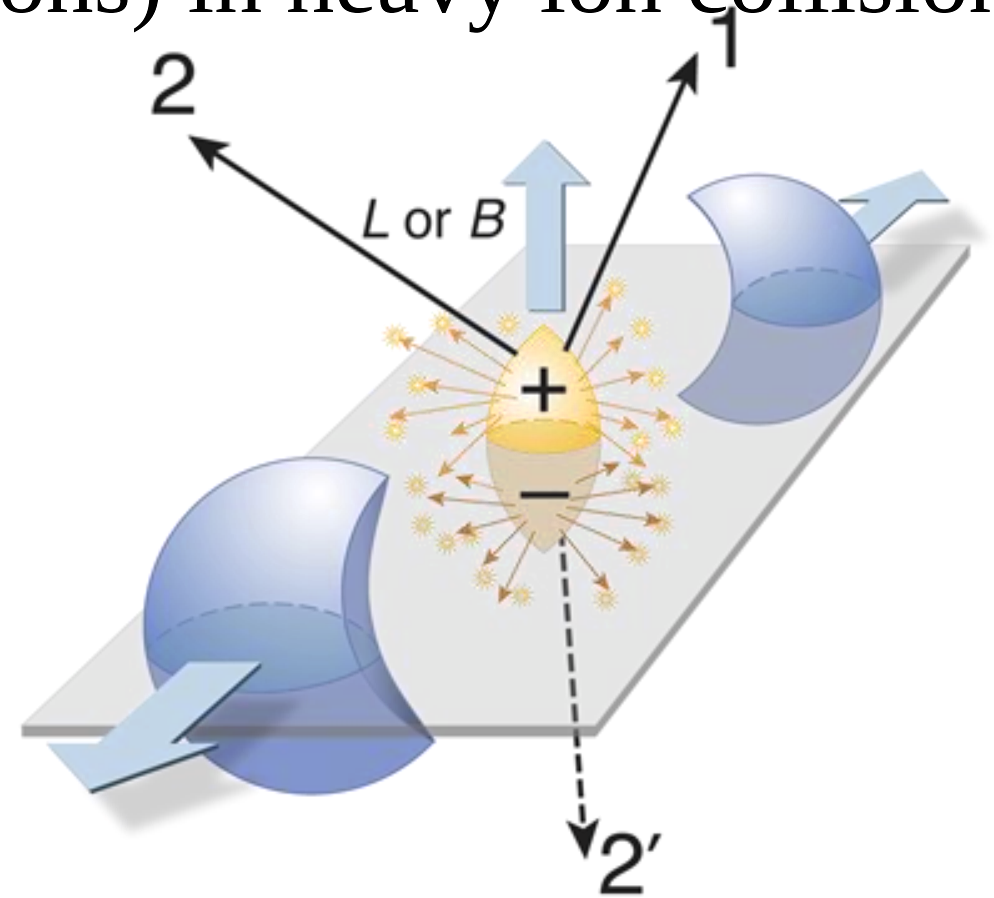
$$N_R - N_L \neq 0 \quad \Rightarrow \quad \mu_5 \neq 0$$

- Driving electric current

$$\langle \vec{j} \rangle = -\frac{e^2 \vec{B}}{2\pi^2} \mu_5$$

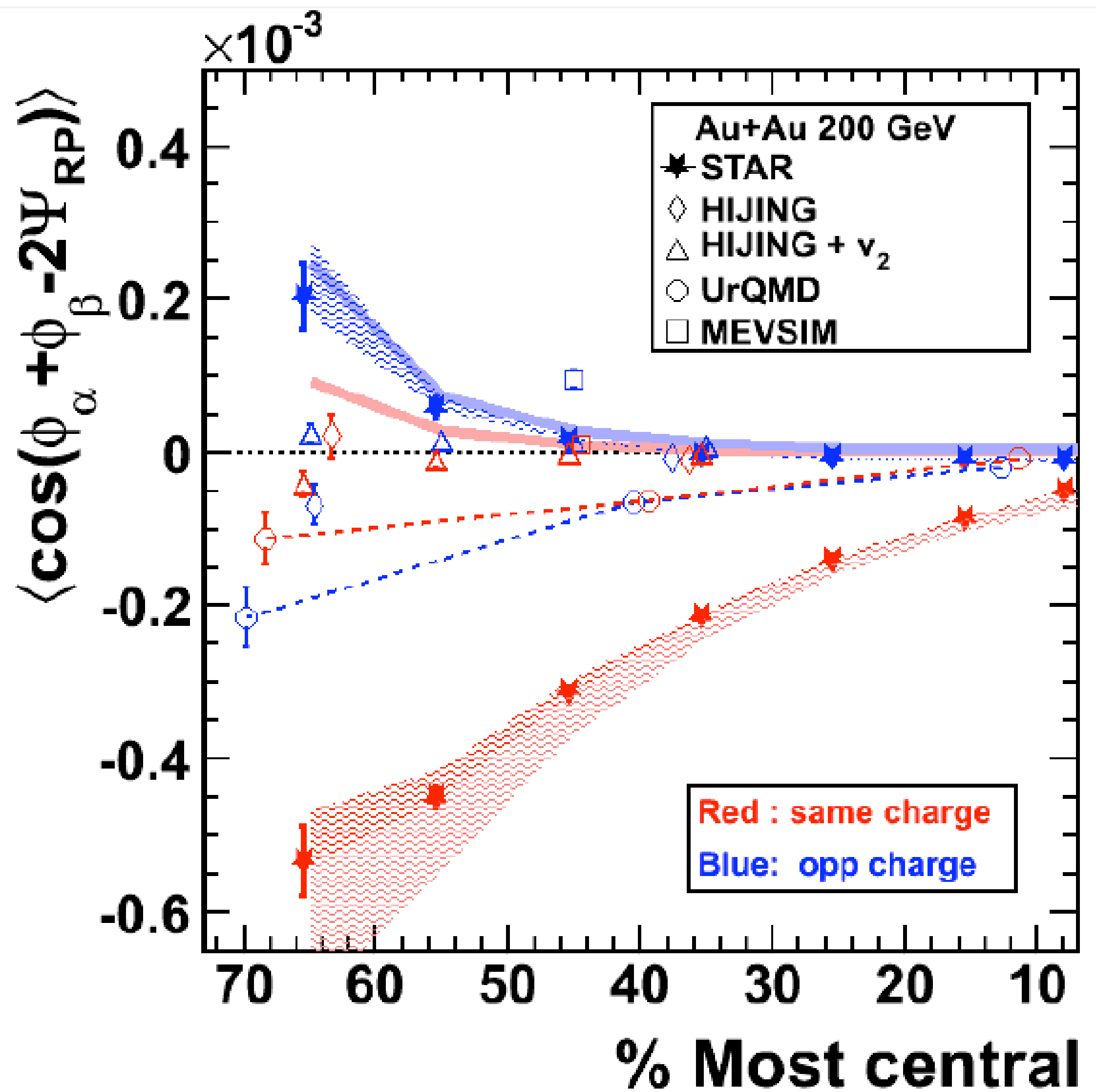
# Heavy ion collisions

- Dipole pattern of electric currents (charge correlations) in heavy ion collisions




[Kharzeev, Zhitnitsky, Nucl. Phys. A **797**, 67 (2007)]
[Kharzeev, McLerran, Warringa, Nucl. Phys. A **803**, 227 (2008)]
[Fukushima, Kharzeev, Warringa, Phys. Rev. D **78**, 074033 (2008)]

# Experimental evidence




[B. I. Abelev et al. [The STAR Collaboration], arXiv:0909.1739]
[B. I. Abelev et al. [STAR Collaboration], arXiv:0909.1717]

# Chiral separation effect

- Axial current induced by fermion chemical potential

$$\left\langle j_5^3 \right\rangle_{\text{free}} = -\frac{eB}{2\pi^2}\mu \qquad \text{(free theory!)}$$

[Vilenkin, Phys. Rev. D **22** (1980) 3067]
[Metlitski & Zhitnitsky, Phys. Rev. D **72**, 045011 (2005)]
[Newman & Son, Phys. Rev. D **73** (2006) 045006]

- Exact result (is it?), which follows from chiral anomaly relation

- No radiative correction expected…

# Possible implication

- Seed chemical potential ($\mu$) induces axial current

$$\left\langle j_5^3 \right\rangle_{\text{free}} = -\frac{eB}{2\pi^2}\mu$$

- Leading to separation of chiral charges:

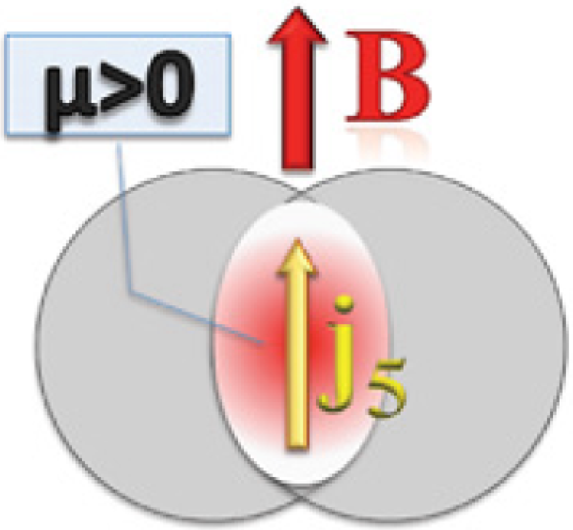
$\mu_5>0$ (one side) & $\mu_5<0$ (another side)

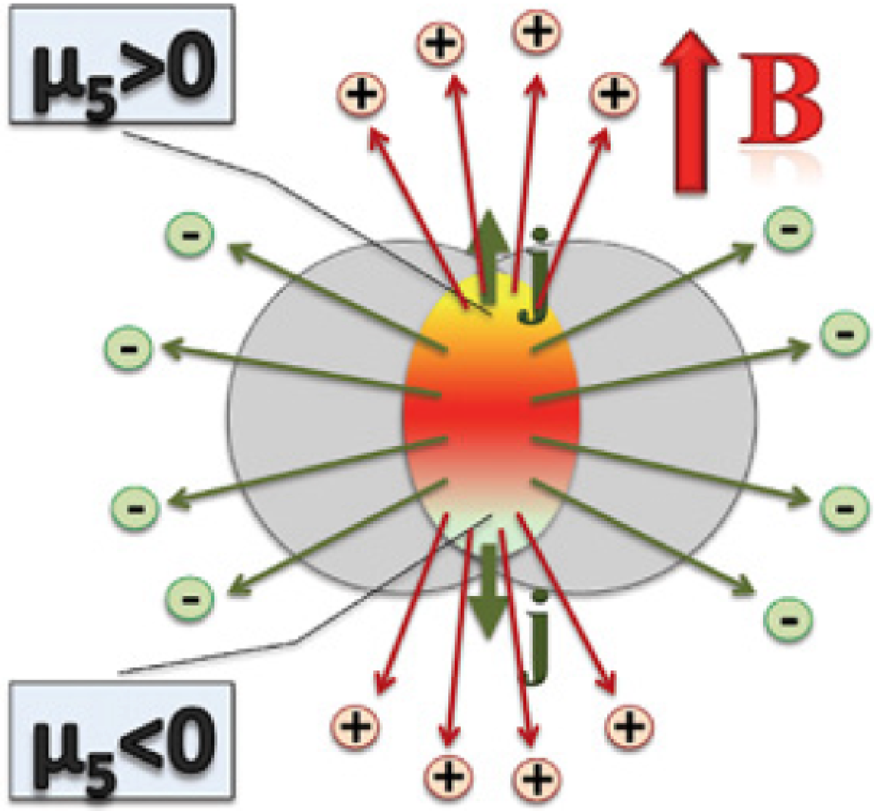
- In turn, chiral charges induce back-to-back electric currents through

$$\left\langle j^3 \right\rangle_{\text{free}} = -\frac{e^2 B}{2\pi^2}\mu_5$$

# Quadrupole CME

- Start from a small baryon density and B≠0




- Produce back-to-back electric currents

[Gorbar, V.M., Shovkovy, Phys. Rev. D **83**, 085003 (2011)]
[Burnier, Kharzeev, Liao, Yee, Phys. Rev. Lett. **107** (2011) 052303]

# Motivation

- Any additional consequences of the CSE relation?

$$\left\langle j_5^3 \right\rangle_{\text{free}} = -\frac{eB}{2\pi^2}\mu \qquad \text{(free theory!)}$$

[Metlitski & Zhitnitsky, Phys. Rev. D **72**, 045011 (2005)]

- Any dynamical parameter Δ ("chiral shift") associated with this condensate?

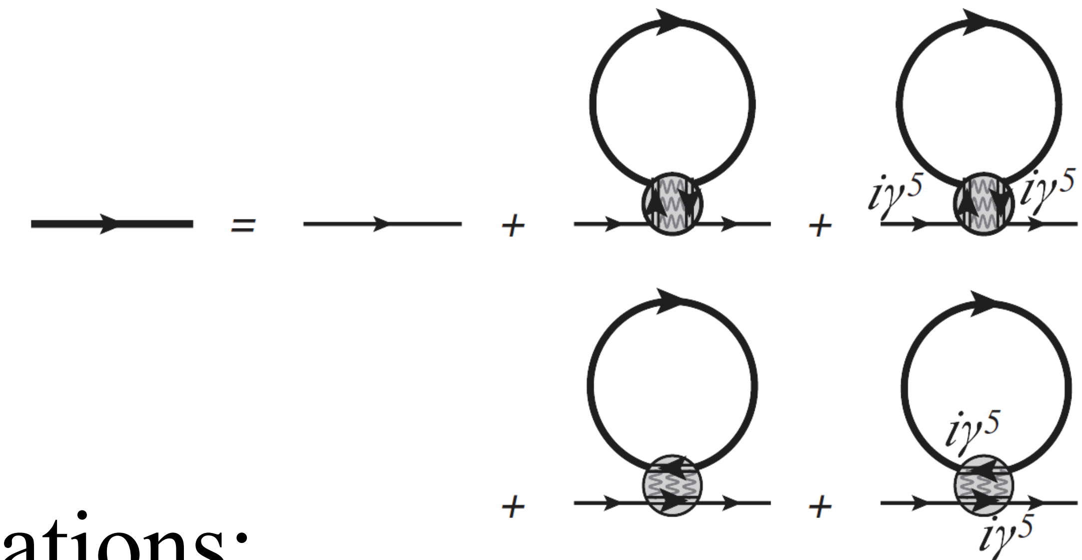
$$\mathcal{L} = \mathcal{L}_0 + \Delta\overline{\psi}\gamma^3\gamma^5\psi$$

- Note: Δ=0 is not protected by any symmetry

# Chiral shift in NJL model

[Gorbar, V.M., Shovkovy, Phys. Rev. C **80**, 032801(R) (2009)]

- NJL model (local interaction)




- "Gap" equations:

$$\mu = \mu_0 - \frac{1}{2} G_{\text{int}} \langle j^0 \rangle$$ ("effective" chemical potential)

$$m = m_0 - G_{\text{int}} \langle \bar{\psi} \psi \rangle$$ (dynamical mass)

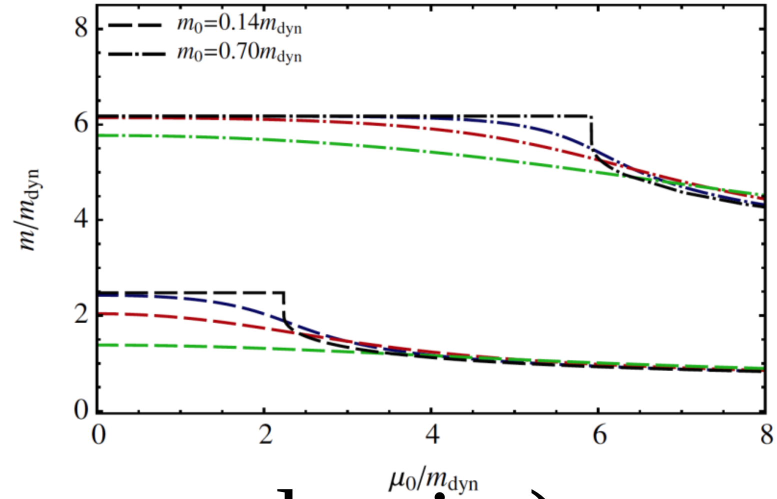
$$\Delta = -\frac{1}{2} G_{\text{int}} \langle j_5^3 \rangle$$ (chiral shift parameter)

# Solutions

- Magnetic catalysis solution (vacuum state):

$$m_{\mathrm{dyn}}^2 \simeq \frac{|eB|}{\pi} \exp\left(-\frac{4\pi^2}{G_{\mathrm{int}}|eB|}\right)$$

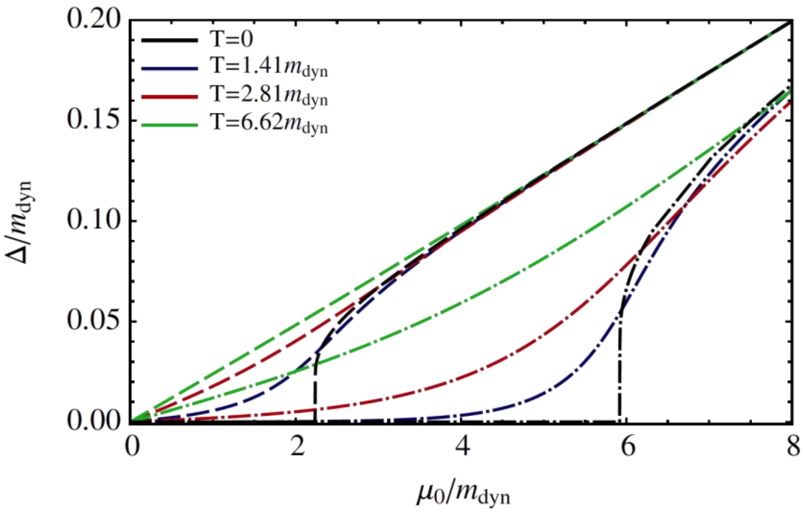
$$\Delta = 0 \quad \& \quad \mu = \mu_0$$




- State with a chiral shift (nonzero density):

$$m_{\mathrm{dyn}} = 0 \quad \& \quad \mu \simeq \frac{\mu_0}{1 + g/(\Lambda l)^2}$$

$$\Delta = \frac{g s_\perp \mu}{(\Lambda l)^2 + \frac{1}{2} g (\Lambda l)^2}$$



# Chiral shift @ Fermi surface

- Chirality is $\approx$ well defined at Fermi surface ($|\boldsymbol{k}^3| \gg \boldsymbol{m}$)
- L-handed Fermi surface:

$$n=0: \quad k^3 = +\sqrt{(\mu - s_\perp \Delta)^2 - m^2}$$

$$n>0: \quad k^3 = +\sqrt{(\sqrt{\mu^2 - 2n|eB|} - s_\perp \Delta)^2 - m^2}$$

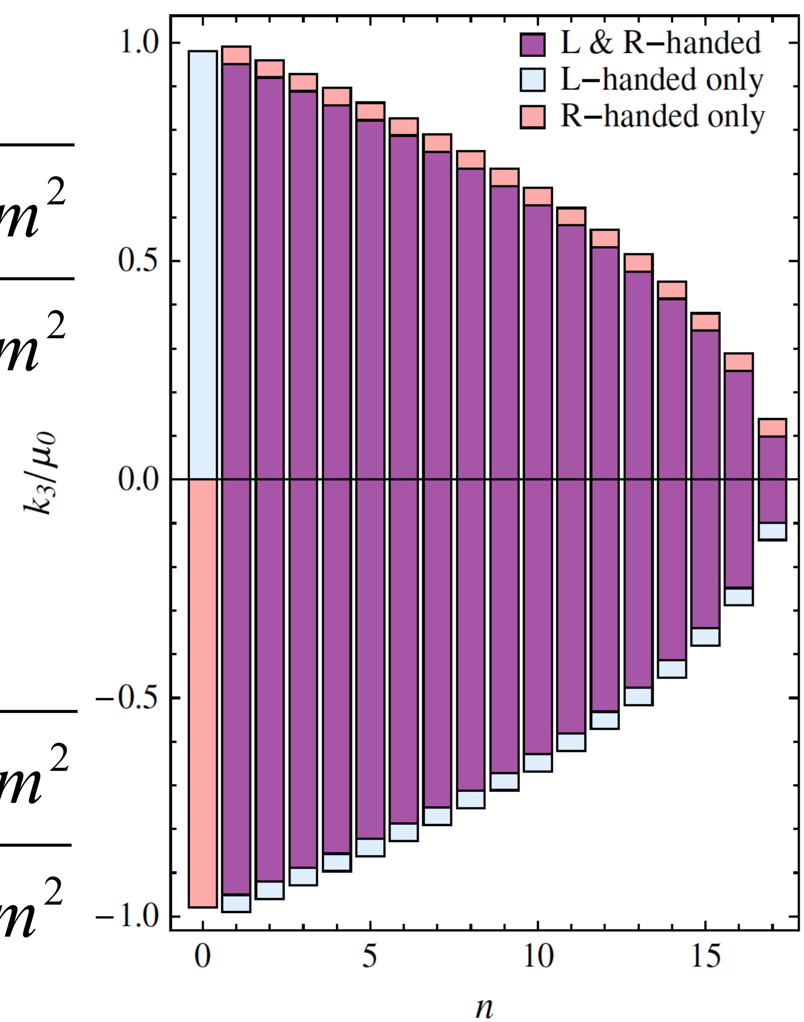
$$k^3 = -\sqrt{(\sqrt{\mu^2 - 2n|eB|} + s_\perp \Delta)^2 - m^2}$$

- R-handed Fermi surface:

$$n=0: \quad k^3 = -\sqrt{(\mu - s_\perp \Delta)^2 - m^2}$$

$$n>0: \quad k^3 = -\sqrt{(\sqrt{\mu^2 - 2n|eB|} - s_\perp \Delta)^2 - m^2}$$

$$k^3 = +\sqrt{(\sqrt{\mu^2 - 2n|eB|} + s_\perp \Delta)^2 - m^2}$$



# Chiral shift vs. axial anomaly

- Does the chiral shift modify the axial anomaly relation?

- Using point splitting method, one derives

$$\langle \partial_\mu j_5^\mu(u) \rangle = -\frac{e^2 \epsilon^{\beta\mu\lambda\sigma} F_{\alpha\mu} F_{\lambda\sigma} \epsilon^\alpha \epsilon_\beta}{8\pi^2 \epsilon^2} \left( e^{-is_\perp \Delta \epsilon^3} + e^{is_\perp \Delta \epsilon^3} \right)$$

$$\to -\frac{e^2}{16\pi^2} \epsilon^{\beta\mu\lambda\sigma} F_{\beta\mu} F_{\lambda\sigma} \qquad \text{for} \quad \epsilon \to 0$$

[Gorbar, V.M., Shovkovy, Phys. Lett. B **695** (2011) 354]

- Therefore, the chiral shift does **not** affect the conventional axial anomaly relation

# Axial current

- Does the chiral shift give any contribution to the axial current?

- In the point splitting method, one has

$$\left\langle j_5^\mu \right\rangle_{\text{singular}} = -\frac{\Delta}{2\pi^2\varepsilon^2}\delta_\mu^3 \cong \frac{\Lambda^2\Delta}{2\pi^2}\delta_\mu^3$$

[Gorbar, V.M., Shovkovy, Phys. Lett. B **695** (2011) 354]

- This is consistent with the NJL calculations

- Since $\Delta \sim g\mu\, eB/\Lambda^2$, the correction to the axial current should be finite

# Axial current in QED

[Gorbar, V.M., Shovkovy, Wang, Phys. Rev. D **88**, 025025 (2013); *ibid*. D **88,** 025043 (2013)]

- Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \overline{\psi}\left(i\gamma^{\mu} D_{\mu} + \mu\gamma^{0} - m\right)\psi + (\text{counterterms})$$

- Axial current

$$\left\langle j_3^5 \right\rangle = -Z_2 \operatorname{tr}\left[\gamma^3\gamma^5 G(x,x)\right]$$

- To leading order in coupling $\alpha = e^2/(4\pi)$

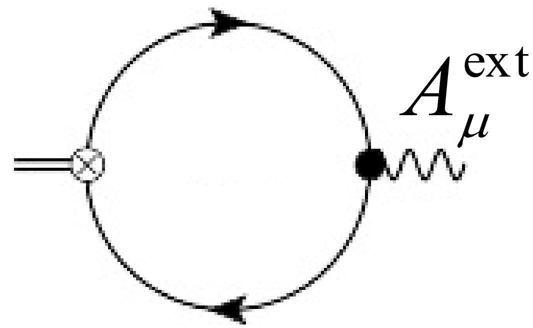
$$G(x,y) = S(x,y) + i\int d^4u\, d^4v\, S(x,u)\Sigma(u,v)S(v,y)$$

# Expansion in external field

- Use expansion of $S(x,y)$ in powers of $A_\mu^{\text{ext}}$

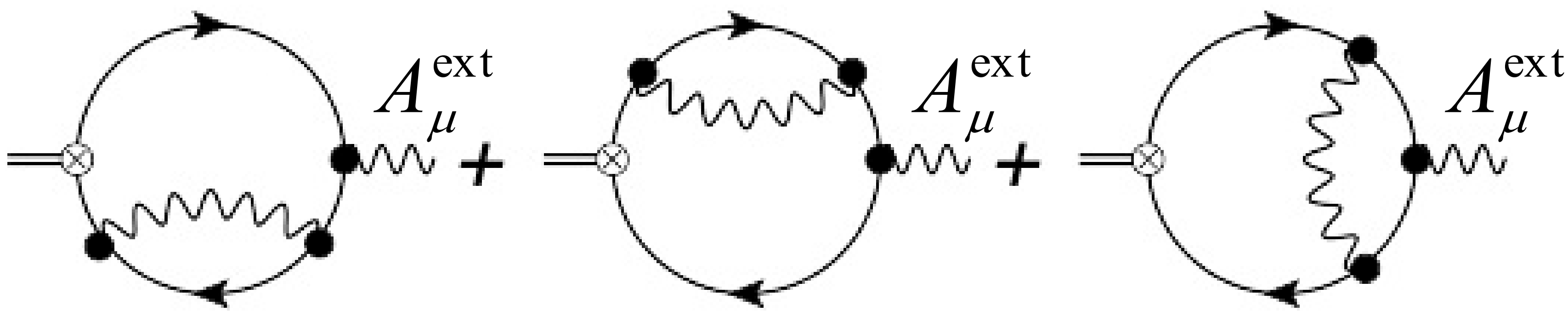
- To leading order in coupling,

$$\left\langle j_5^3 \right\rangle_0 =$$




- The radiative correction is

$$\left\langle j_5^3 \right\rangle_\alpha =$$



# Alternative form of expansion

- Expand $S(x,y) = e^{i\Phi(x,y)}\overline{S}(x-y)$ in field

$$S(x,y) = \underbrace{\overline{S}^{(0)}(x-y) + \overline{S}^{(1)}(x-y)}_{\text{Translation invariant part}} + i\underbrace{\Phi(x,y)}_{\text{Schwinger phase}} S^{(0)}(x-y)$$

- The Schwinger phase (in Landau gauge)

$$\Phi(x,y) = -\frac{eB}{2}(x_1 + y_1)(x_2 - y_2)$$

- Note: the phase is not translation invariant

# Translation invariant parts

- Fourier transforms

$$\bar{S}^{(0)}(k) = i\frac{(k_0+\mu)\gamma^0 - \mathbf{k}\cdot\gamma + m}{\left(k_0+\mu+i\varepsilon\,\mathrm{sign}(k_0)\right)^2 - \mathbf{k^2} - m^2}$$

$$\bar{S}^{(1)}(k) = -\gamma^1\gamma^2 eB\frac{(k_0+\mu)\gamma^0 - k_3\gamma^3 + m}{\left[\left(k_0+\mu+i\varepsilon\,\mathrm{sign}(k_0)\right)^2 - \mathbf{k^2} - m^2\right]^2}$$

- Note the singularity near the Fermi surface…

# Fermi surface singularity

- “Vacuum” + “matter” parts

$$\frac{1}{\left[\left(k_0+\mu+i\varepsilon\,\mathrm{sign}(k_0)\right)^2-\mathbf{k}^2-m^2\right]^n}=\text{"Vac."}+\text{"Mat."}$$

where

$$\text{"Vac."}=\frac{1}{\left[\left(k_0+\mu\right)^2-\mathbf{k}^2-m^2+i\varepsilon\right]^n}$$

$$\text{"Mat."}=\frac{2\pi i(-1)^{n-1}}{(n-1)!}\theta\left(|\mu|-|k_0|\right)\theta\left(-k_0\mu\right)\delta^{(n-1)}\left[\left(k_0+\mu\right)^2-\mathbf{k}^2-m^2\right]$$

# Axial current (0th order)

- From definition

$$\left\langle j_5^3 \right\rangle_0 = -\int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\left[\gamma^3\gamma^5 \bar{S}^{(1)}(k)\right]$$

- After integrating over energy

$$\left\langle j_5^3 \right\rangle_0 = -\frac{eB\,\mathrm{sign}(\mu)}{4\pi^3} \int d^3\mathbf{k}\, \underbrace{\delta\left(\mu^2 - \mathbf{k}^2 - m^2\right)}_{\text{Matter part}}$$

and finally

$$\left\langle j_5^3 \right\rangle_0 = -\frac{eB\,\mathrm{sign}(\mu)}{2\pi^2} \sqrt{\mu^2 - m^2}$$

- Note the role of the Fermi surface (!)

# Conventional wisdom

- Only the lowest (n=0) Landau level contributes

$$\left\langle j_5^3 \right\rangle_0 = \frac{eB}{4\pi^2} \int dk_3 \left[ \theta\left(-\mu - \sqrt{k_3^2 + m^2}\right) - \theta\left(\mu - \sqrt{k_3^2 + m^2}\right) \right]$$

giving same answer

$$\left\langle j_5^3 \right\rangle_0 = -\frac{eB\,\mathrm{sign}(\mu)}{2\pi^2} \sqrt{\mu^2 - m^2}$$

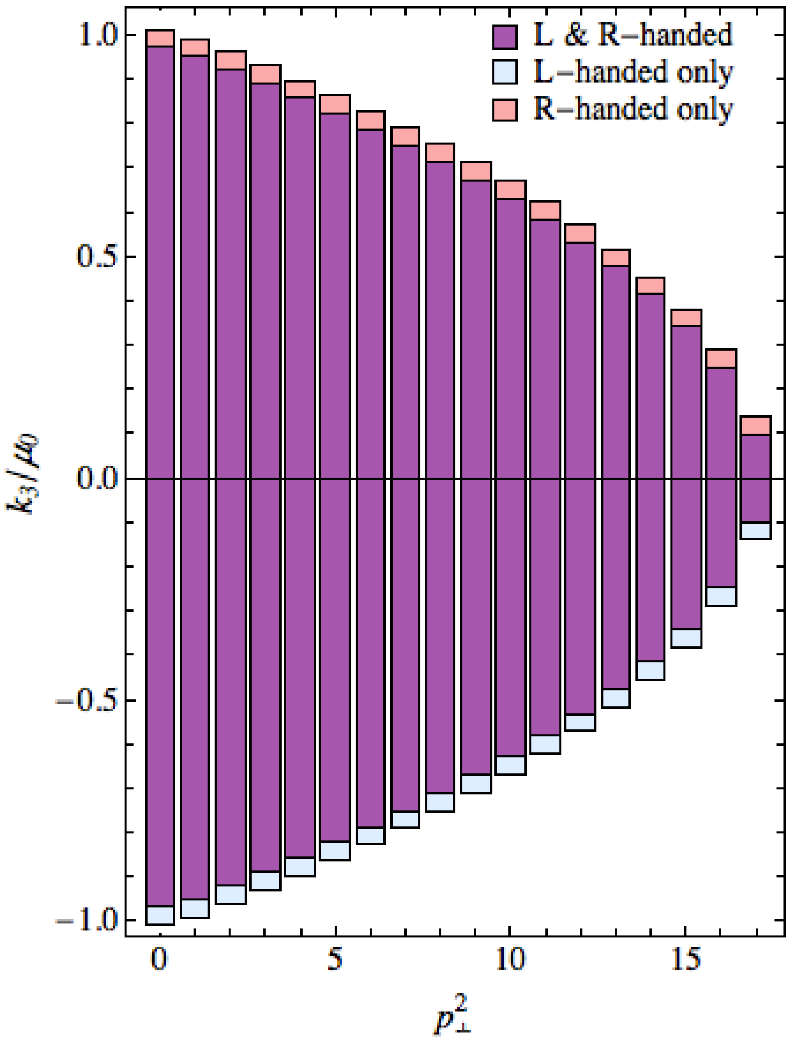
- There are no contributions from higher Landau levels (n≥1)

- There is a connection with the index theorem

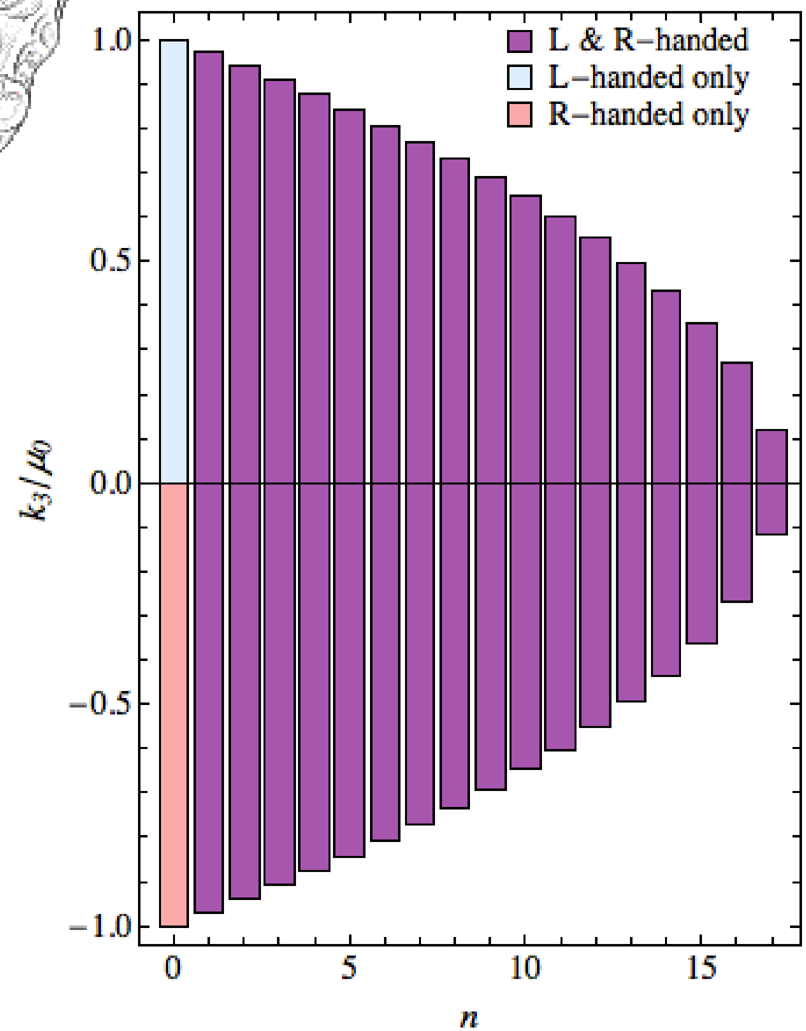
# Two facets

- Two ways to look at the same result

$B \to 0$




$B \neq 0$



# Radiative correction

- Original two-loop expression

$$\langle j_5^3\rangle_\alpha = 32\pi\alpha eB \int \frac{d^4p\, d^4k}{(2\pi)^8} \frac{1}{(P-K)_\Lambda^2} \left[ \frac{(k_0+\mu)[3(p_0+\mu)^2+\mathbf{p}^2+m^2] - 4(p_0+\mu)(\mathbf{p}\cdot\mathbf{k}+2m^2)}{(P^2-m^2)^3(K^2-m^2)} - \frac{(k_0+\mu)[3(p_0+\mu)^2-\mathbf{p}^2+3m^2] - 2(p_0+\mu)(\mathbf{p}\cdot\mathbf{k})}{3(P^2-m^2)^2(K^2-m^2)^2} \right] + \langle j_5^3\rangle_{\rm ct}.$$

- After integration by parts

$$\langle j_5^3\rangle_\alpha = 64i\pi^2\alpha eB \int \frac{d^4pd^4k}{(2\pi)^8} \left[ \frac{(k_0+\mu)(p_0+\mu) - \mathbf{p}\cdot\mathbf{k} - 2m^2}{(P-K)_\Lambda^2(K^2-m^2)} \delta'\left[\mu^2-m^2-\mathbf{p}^2\right]\delta(p_0) + \frac{3(p_0+\mu)^2 - 3(k_0+\mu)(p_0+\mu) + \mathbf{p}^2 - \mathbf{p}\cdot\mathbf{k} + 3m^2}{3(P-K)_\Lambda^2(P^2-m^2)^2} \delta\left(\mu^2-m^2-\mathbf{k}^2\right)\delta(k_0) \right] + \langle j_5^3\rangle_{\rm ct}$$

# Result (m<<μ)

- Loop contribution

$$f_1 + f_2 + f_3 = \frac{\alpha\, eB\, \mu}{2\pi^3}\left( \ln\frac{\Lambda}{2\mu} + \frac{11}{12} \right) + \frac{\alpha\, eB\, m^2}{2\pi^3 \mu}\left( \ln\frac{\Lambda}{2^{3/2}\mu} + \frac{1}{6} \right)$$

- Counterterm

$$\left\langle j_5^3 \right\rangle_{\rm ct} = -\frac{\alpha\, eB\, \mu}{2\pi^3}\left( \ln\frac{\Lambda}{m} + \ln\frac{m_\gamma^2}{m^2} + \frac{9}{4} \right) - \frac{\alpha\, eB\, m^2}{2\pi^3 \mu}\left( \ln\frac{\Lambda}{m_\gamma} - \frac{3}{4} \right)$$

- Final result

$$\left\langle j_5^3 \right\rangle_{\alpha} = -\frac{\alpha\, eB\, \mu}{2\pi^3}\left( \ln\frac{2\mu}{m} + \ln\frac{m_\gamma^2}{m^2} + \frac{4}{3} \right) - \frac{\alpha\, eB\, m^2}{2\pi^3 \mu}\left( \ln\frac{2^{3/2}\mu}{m_\gamma} - \frac{11}{12} \right)$$

# Sign of nonperturbative physics

- Unphysical dependence on photon mass

$$\left\langle j_5^3 \right\rangle_\alpha = -\frac{\alpha eB\mu}{2\pi^3}\left(\ln\frac{2\mu}{m} + \ln\frac{m_\gamma^2}{m^2} + \frac{4}{3}\right) - \frac{\alpha eB m^2}{2\pi^3\mu}\left(\ln\frac{2^{3/2}\mu}{m_\gamma} - \frac{11}{12}\right)$$

- Infrared physics with

$$m_\gamma \le |k_0|, |k_3| \le \sqrt{|eB|}$$

not captured properly

- Note: similar problem exists in calculation of Lamb shift

# Nonperturbative effects (?)

- Perpendicular momenta cannot be defined with accuracy better than

$$\left|\Delta \mathbf{k}_\perp\right|_{\min} \sim \sqrt{|eB|}$$

(In contrast to the tacit assumption in using expansion in powers of *B*-field)

- Screening effects provide a natural infrared regulator

$$m_\gamma \Rightarrow \sqrt{\alpha}\,\mu$$

(Formally, this goes beyond the leading order in coupling)

# Nonperturbative result (?)

- Conjectured nonpertubative modification

(1) If non-conservation of momentum dominates

$$\left\langle j_5^3 \right\rangle_\alpha = -\frac{\alpha\, eB\, \mu}{2\pi^3}\left(\ln\frac{\mu|eB|}{m^3} + O(1)\right) - \frac{\alpha\, eB\, m^2}{2\pi^3 \mu}\left(\ln\frac{\mu}{\sqrt{|eB|}} + O(1)\right)$$

(2) If photon screening is more important

$$\left\langle j_5^3 \right\rangle_\alpha = -\frac{\alpha\, eB\, \mu}{2\pi^3}\left(\ln\frac{\alpha\, \mu^3}{m^3} + O(1)\right) - \frac{\alpha\, eB\, m^2}{2\pi^3 \mu}\left(\ln\frac{1}{\sqrt{\alpha}} + O(1)\right)$$

# Summary (1)

- Weak *B*-field limit: **new interpretation** of the topological contribution to CSE relation
- Radiative corrections are **nonzero**
- Radiative corrections vanish without “matter” part with **singularity on Fermi surface**
- **Nonperturbative** physics complicates the infrared contribution
- With **logarithmic accuracy**, the result can be conjectured

# Self-energy at $B \neq 0$

- Self-energy

$$\Sigma(x,y) = -4i\pi\gamma^\mu S(x,y)\gamma^\nu D_{\mu\nu}(x-y)$$

- General structure

$$\Sigma(x,y) = \exp\left(i\Phi(x,y)\right)\overline{\Sigma}(x-y)$$

- Translation invariant part:

$$\overline{\Sigma}(p) = -4i\pi\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\,\overline{S}(k)\gamma^\nu D_{\mu\nu}(k-p)$$

# Contribution linear in $B$

$$\overline{\Sigma}^{(1)}(p) = -4i\pi \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \overline{S}^{(1)}(k) \gamma^\nu D_{\mu\nu}(k-p)$$

- The result has the form

$$\overline{\Sigma}^{(1)}(p) = \gamma^3\gamma^5\Delta + \gamma^0\gamma^5\mu_5(p)$$

where

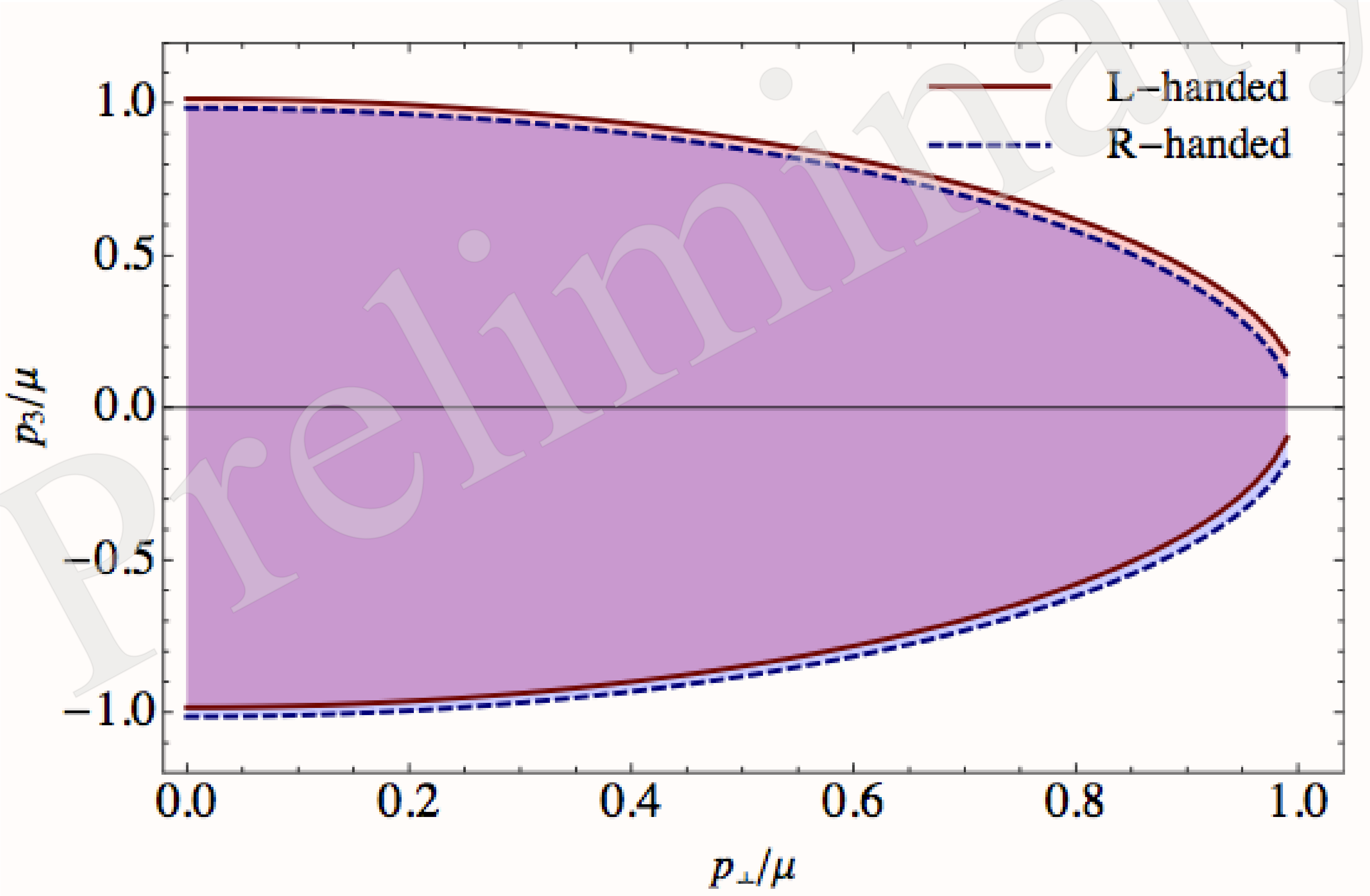
$$\Delta \approx \frac{\alpha eB\mu}{\pi m^2}\left(\ln\frac{m^2}{2\mu(|\mathbf{p}| - p_F)} - 1\right)$$

$$\mu_5(p) \approx -\frac{\alpha eB\mu}{\pi m^2}\frac{p_3}{p_F}\left(\ln\frac{m^2}{2\mu(|\mathbf{p}| - p_F)} - 1\right)$$

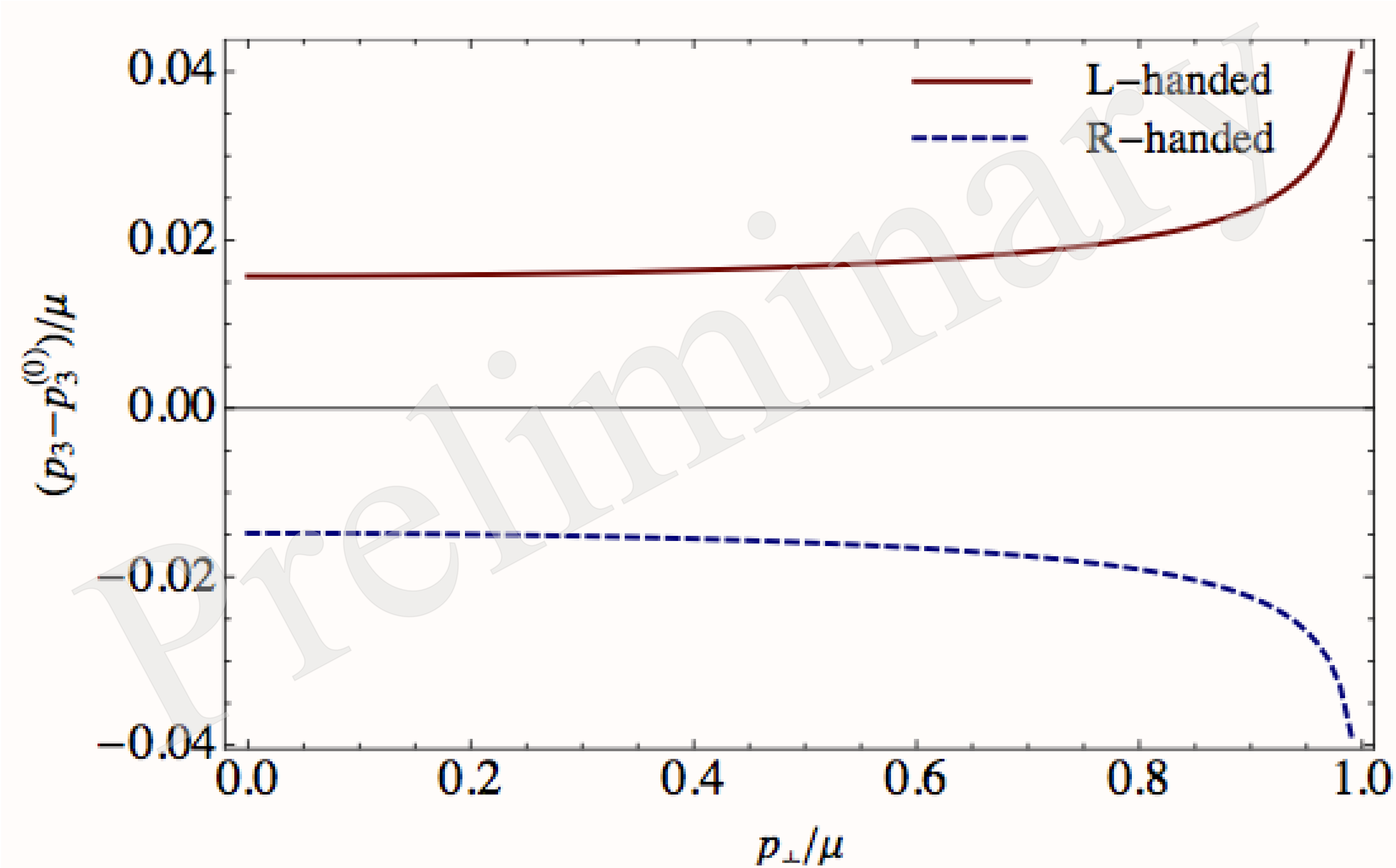
# Dispersion relations

- Let us use the condition

$$\mathrm{Det}\left[i\bar{S}^{-1}(p)+\bar{\Sigma}^{(1)}(p)\right]=0$$



# L/R-Fermi surface shift



# Summary (2)

- Chiral shift is generated in magnetized matter (evidence from renormalizable model now)
- The magnitude of chiral shift scales as

$$\Delta \propto \frac{\alpha eB \mu}{m^2} \ln \alpha$$

- Chiral shift induces a chiral asymmetry at the Fermi surface
- Chiral shift contributes to the axial current