Quantum Hall effect: what can be learned from curved space?

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Carlos Hoyos, DTS 2011

In memory of my father Dam Trung Bao (1929-2011)

Outline

- This talk is not be related to AdS/CFT, string theory
- but we will see how thinking about curve space helps us understand flat-space physics

Quantum Hall state

- simplest example: noninteracting electrons filling n Landau levels (interger QH effect)
- Fractional QH effect: much more complicated theory (Laughlin)
- gapped, no low-energy degree of freedom
- The effective action can be expanded in polynomials of external fields
- To lowest order: Chern-Simons action

$$S = \frac{\nu}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

encodes Hall conductivity $\quad \sigma_{xy} = \frac{\nu}{2\pi} \frac{e^2}{\hbar}$

What is missing

- CS action does not involve metric
- Stress-energy tensor = 0
- It is not how real quantum Hall system behaves

Hall viscosity

Avron et al 1995

- Turn on $h_{xy}(t)$ metric perturbations
- observe $T_{xx} = - T_{yy} \sim h'_{xy}(t)$
- there must be a term proportional first derivative of metric in the effective Lagrangian
- How? curvature ~ 2nd derivative

Wen-Zee term

- Hall viscosity: described by Wen-Zee term
  (W.Goldberger & N.Read unpublished; N.Read 2009 KITP talk)
- Introduce spatial vielbein (viel=2) $g_{ij}=e^a_i\, e^a_j$
- We can now define the spin connection

$$\omega_i = \frac{1}{2}\epsilon^{ab} e^{aj} \nabla_i e^{bj} \qquad \omega_0 = \frac{1}{2}\epsilon^{ab} e^{aj} \partial_0 e^{bj}$$

Vielbein defined up to a local O(2) rotation

$$e^a_i \to e^a_i + \lambda \epsilon^{ab} e^b_i \qquad \omega_\mu \to \omega_\mu - \partial_\mu \lambda$$

like an abelian gauge field

Vielbein and curvature

$$\partial_1 \omega_2 - \partial_2 \omega_1 = \frac{1}{2}\sqrt{g}\, R$$

Wen-Zee terms

in addition to the Chern-Simons term

$$\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}(\kappa\,\omega_\mu\partial_\nu A_\lambda + \kappa'\omega_\mu\partial_\nu\omega_\lambda)$$

will not be important for
futher discussions

The first term gives rise to

- Wen-Zee shift
- Hall viscosity

Wen-Zee shift

- Rewrite S_{WZ} as

$$\frac{\kappa}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu\omega_\lambda = \frac{\kappa}{4\pi}\sqrt{g}\,A_0 R + \cdots$$

Total particle number:

# of magnetic fluxes

Euler

$$Q = \int \mathrm{d}^2x\,\sqrt{g}\,j^0 = \int \mathrm{d}^2x\,\sqrt{g}\left(\frac{\nu}{2\pi}B + \frac{\kappa}{4\pi}R\right) = \nu N_\phi + \kappa\chi$$

On a sphere: $Q = \nu(N_\phi + \mathcal{S}),\quad \mathcal{S} = \frac{2\kappa}{\nu}$

'shift'

IQH states: ν=n, κ=n²/2

Laughlin's states: ν=1/n, κ=1/2

Hall viscosity from WZ term

$$S_{\rm WZ} = -\frac{\kappa B}{16\pi}\epsilon^{ij} h_{ik}\partial_t h_{jk} + \cdots$$

stress ~ time derivative of metric

$$\eta^a = \frac{\kappa B}{4\pi} = \frac{1}{4}\mathcal{S}n$$

derived by N.Read previously

Flat space physics

- But is this Wen-Zee term be important for physics in flat space?
- In this talk we will argue that it is
- Reason: nonrelativistic diffeomorphism
- For a nonrelativistic system of particles with the same charge/mass ratio, there is a nonrelativistic principle of equivalence
  - accelerated frame ~ electric field
  - rotating frame ~ magnetic field (Coriolis force ~ Lorentz force)
  - nonrelativistic diffeomorphism mixes metric and EM field

Symmetries of NR theory

DTS, M.Wingate 2006

Microscopic theory

$$S_0 = \int \mathrm{d}t\, \mathrm{d}^2x \,\sqrt{g}\Big[\frac{i}{2}\psi^\dagger \overleftrightarrow{D}_t\psi - \frac{g^{ij}}{2m}D_i\psi^\dagger D_j\psi\Big] \qquad D_\mu\psi \equiv (\partial_\mu - iA_\mu)\psi$$

Gauge invariance: $\psi \to e^{i\alpha}\psi \qquad A_\mu \to A_\mu + \partial_\mu\alpha$

General coordinate invariance:

$$\delta\psi = -\xi^k\partial_k\psi \equiv \mathcal{L}_\xi\psi$$

$$\delta A_0 = \xi^k\partial_k A_0 \equiv \mathcal{L}_\xi A_0$$

$$\delta A_i = -\xi^k\partial_k A_i - A_k\partial_i\xi^k \equiv \mathcal{L}_\xi A_i$$

$$\delta g_{ij} = -\xi^k\partial_k g_{ij} - g_{kj}\partial_i\xi^k - g_{ik}\partial_j\xi^k \equiv \mathcal{L}_\xi g_{ij}$$

Here ξ is time independent: $\xi=\xi(\mathbf{x})$

NR diffeomorphism

- These transformations can be generalized to be time-dependent: $\xi=\xi(t,\mathbf{x})$

$$\delta\psi = -\mathcal{L}_\xi \psi$$

$$\delta A_0 = -\mathcal{L}_\xi A_0 - A_k \dot{\xi}^k$$

$$\delta A_i = -\mathcal{L}_\xi A_i - m g_{ik} \dot{\xi}^k$$

$$\delta g_{ij} = -\mathcal{L}_\xi g_{ij}$$

Time dependent diffeomorphisms mix metric and gauge field

Galilean transformations: special case $\xi^i=v^i t$

Where does it come from

Start with complex scalar field

$$S = -\int \mathrm{d}x\,\sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + \phi^*\phi)$$

Take nonrelativistic limit:

$$g_{\mu\nu} = \begin{pmatrix} -1 + \dfrac{2A_0}{mc^2} & \dfrac{A_i}{mc} \\ \dfrac{A_i}{mc} & g_{ij} \end{pmatrix}$$

$$\phi = e^{-imcx^0}\frac{\psi}{\sqrt{2mc}}$$

$$S = \int \mathrm{d}t\,\mathrm{d}\mathbf{x}\,\sqrt{g}\left[\frac{i}{2}\psi^\dagger\overleftrightarrow{\partial}_t\psi + A_0\psi^\dagger\psi - \frac{g^{ij}}{2m}(\partial_i\psi^\dagger + iA_i\psi^\dagger)(\partial_j\psi - iA_j\psi)\right].$$

Relativistic diffeomorphism

$$x^\mu \to x^\mu + \xi^\mu$$

μ =0: gauge transform

$$\phi = e^{-imcx^0} \frac{\psi}{\sqrt{2mc}}$$

μ =i: general coordinate transformations

Interactions

- Interactions can be introduced that preserve nonrelativistic diffeomorphism
  - interactions mediated by fields
- For example, Coulomb interactions: mediated by photon propagating in 3+1 dimensions

$$S = S_0 + \int \mathrm{d}t\, \mathrm{d}^2x\, \sqrt{g}\, a_0(\psi^\dagger\psi - n_0) + \frac{2\pi\varepsilon}{e^2}\int \mathrm{d}t\, \mathrm{d}^2x\, \mathrm{d}z\, \sqrt{g}\, \left[g^{ij}\partial_i a_0 \partial_j a_0 + (\partial_z a_0)^2\right]$$

$$\delta a_0 = -\xi^k \partial_k a_0$$

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But this cannot be achieved by local terms

Resolution

- Higher order terms contain inverse powers of B

$$\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{m}{B}g^{ij}E_iE_j + \cdots$$

- Quantum Hall state with diff. invariance does not exist at zero magnetic field!

Diff invariant terms

$$\mathcal{L}_1 = \frac{\nu}{4\pi}\Big(\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{m}{B}g^{ij}E_iE_j\Big)$$

Kohn's theorem ~ 1960

$$\mathcal{L}_2 = \frac{\kappa}{2\pi}\Big(\varepsilon^{\mu\nu\lambda}\omega_\mu\partial_\nu A_\lambda + \frac{1}{2B}\,g^{ij}\partial_i B\,E_j\Big)$$

$$\mathcal{L}_3 = -\epsilon(B) - \frac{m}{B}\epsilon''(B)g^{ij}\partial_i B\,E_j$$

$\sigma_{xy}(q)$

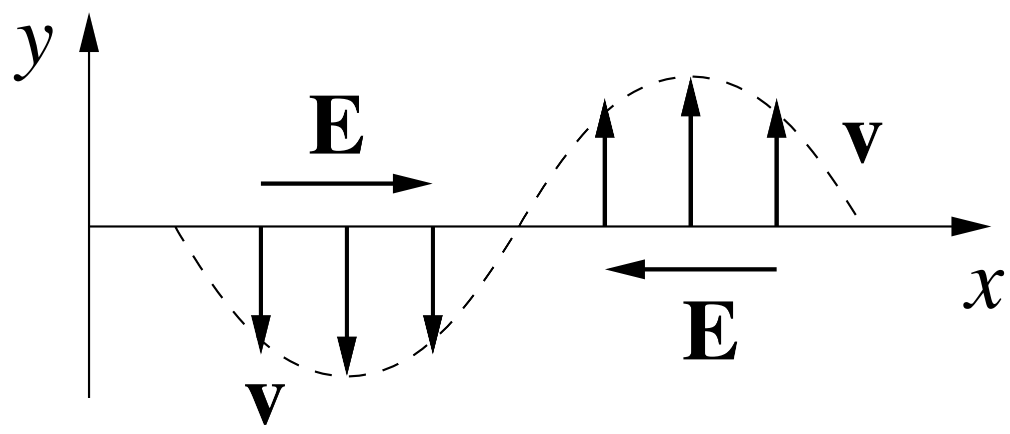
ground state
energy density

black: leading   blue: subleading

Kohn's theorem

- Response of the system on uniform electric field does not depend on interactions
- Effective action captures first order in omega corrections to conductivities at q=0

$\sigma_{xy}(q)$: new prediction



$$E_x = E\,\mathrm{e}^{iqx}$$

$$j_y = \sigma_{xy}(q) E_x$$

From effective field theory

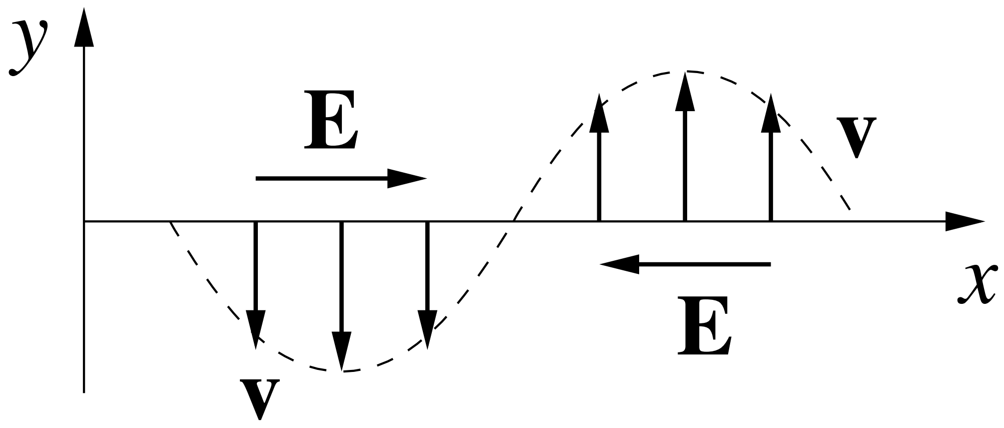
$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 + C_2 (q\ell)^2 + \mathcal{O}(q^4 \ell^4)$$

$$C_2 = \frac{\eta^a}{\hbar n} - \frac{2\pi}{\nu} \frac{\ell^2}{\hbar \omega_c} B^2 \epsilon''(B)$$

$$\frac{\eta^a}{\hbar n} = \mathcal{S}/4$$

Physical interpretation

- First term: Hall viscosity



$$\partial_x v_y + \cancel{\partial_y v_x} \neq 0$$

$$T_{xx} = T_{xx}(x) \neq 0$$

additional force $F_x \sim \partial_x T_{xx}$
Hall effect: additional contribution to v_y

Physical interpretation (II)

- 2nd term: more complicated interpretation

Fluid has nonzero angular velocity

$$\Omega(x) = \frac{1}{2}\partial_x v_y = -\frac{cE_x'(x)}{2B}$$

$$\delta B = 2mc\Omega/e$$

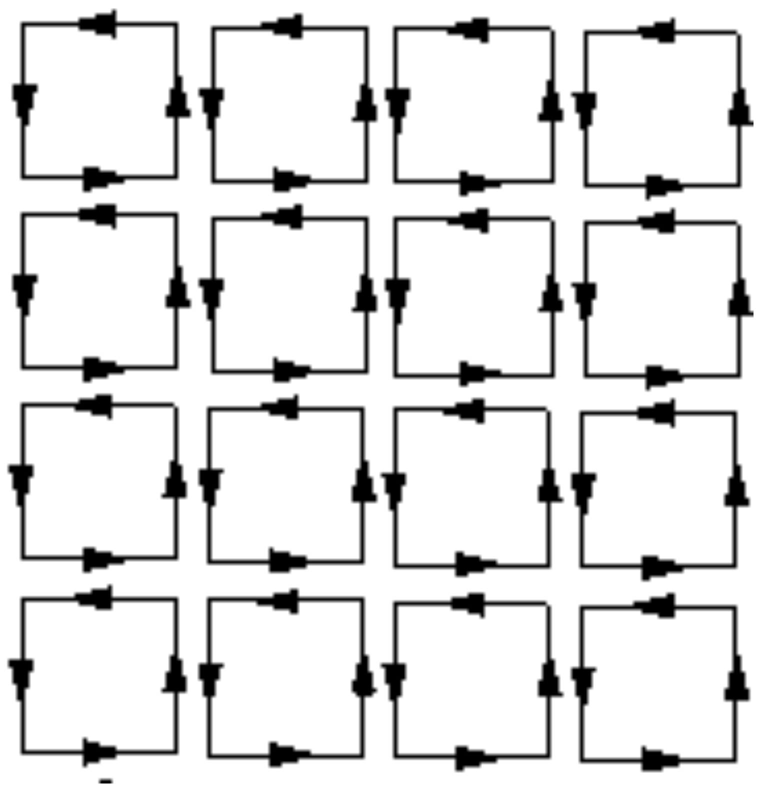
Coriolis=Lorentz

Hall fluid is diamagnetic: $d\epsilon = -MdB$

M is spatially dependent M=M(x)

Extra contribution to current $\mathbf{j} = c\,\hat{\mathbf{z}} \times \nabla M$

Current ~ gradient of magnetization



$$\mathbf{j} = c\,\hat{\mathbf{z}} \times \nabla M$$

High B limit

- In the limit of high magnetic field: ϵ(B) known: free fermions
  - n Landau levels for IQH states
  - first Landau level for FQH states with ν<1
- Wen-Zee shift is known

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 - \frac{3n}{4}(q\ell)^2 + \mathcal{O}(q^4\ell^4)$$ ν=n

$$\frac{\sigma_{xy}(q)}{\sigma_{xy}(0)} = 1 + \frac{2n-3}{4}(q\ell)^2 + \mathcal{O}(q^4\ell^4), \quad \nu = \frac{1}{2n+1}$$

exact nonperturbative results!

Conclusions

- Thinking about the curved space is productive in nonrelativistic physics
- Reason: NR principle of equivalence
- NR diffeomorphism mixes metric and EM field
- Nontrivial consequences in quantum Hall physics
- Wen-Zee term in the action leads to one contribution to the Hall conductivity at finite q