Equation of state for dark energy in modified gravity theories

Main references:

[1] K. Bamba, C. Q. Geng and C. C. Lee,
JCAP 1011, 001 (2010) [arXiv:1007.0482 [astro-ph.CO]].
[2] K. Bamba, C. Q. Geng, C. C. Lee and L. W. Luo,
JCAP 1101, 021 (2011) [arXiv:1011.0508 [astro-ph.CO]].
[3] K. Bamba, S. Nojiri, S. D. Odintsov and M. Sasaki, arXiv:1104.2692 [hep-th].

KMI Inauguration Conference on ``Quest for the Origin of Particles and the Universe'' (KMIIN) on October 24, 2011
ES Hall, Engineering Science (ES) Building (KMI site),
Nagoya University, Nagoya

Presenter : **Kazuharu Bamba** (***KMI, Nagoya University***)

Collaborators : **Chao-Qiang Geng, Chung-Chi Lee, Ling-Wei Luo (*National Tsing Hua University*), Shin'ichi Nojiri (*KMI and Dep. of Physics, Nagoya University*), Sergei D. Odintsov (*ICREA and IEEC-CSIC*), Misao Sasaki (*YITP, Kyoto University and KIAS*)**

< Contents >

I. Introduction

- Current cosmic acceleration
- $f(R)$ gravity
- Crossing of the phantom divide

II. Future crossing of the phantom divide in $f(R)$ gravity

III. Equation of state for dark energy in $f(T)$ theory

IV. Effective equation of state for the universe and the finite-time future singularities in non-local gravity

V. Summary

∗ We use the ordinary metric formalism, in which the connection is written by the differentiation of the metric.

I. Introduction

2011 Nobel Prize in Physics

- Recent observations of Supernova (SN) Ia confirmed that the current expansion of the universe is accelerating.

[Perlmutter *et al*. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999)]

[Riess *et al*. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998)]

[Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

- There are two approaches to explain the current cosmic acceleration. **[Copeland, Sami and Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006)]**

[Tsujikawa, arXiv:1004.1493 [astro-ph.CO]]

< Gravitational field equation >

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

Gravity **Matter**

$G_{\mu\nu}$: Einstein tensor

$T_{\mu\nu}$: Energy-momentum tensor

$$\kappa^2 \equiv 8\pi / {M_{\mathrm{Pl}}}^2$$

M_{Pl} : Planck mass

(1) **General relativistic approach** ⟶ **Dark Energy**

(2) **Extension of gravitational theory**

(1) General relativistic approach

- Cosmological constant
- Scalar fields: X matter, Quintessence, Phantom, K-essence, Tachyon.
- Fluid: Chaplygin gas

$f(R)$: Arbitrary function of the Ricci scalar R

(2) Extension of gravitational theory

- $f(R)$ gravity [Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D 70, 043528 (2004)]

[Capozziello, Cardone, Carloni and Troisi, Int. J. Mod. Phys. D 12, 1969 (2003)]

- Scalar-tensor theories [Nojiri and Odintsov, Phys. Rev. D 68, 123512 (2003)]
- Ghost condensates [Arkani-Hamed, Cheng, Luty and Mukohyama, JHEP 0405, 074 (2004)]

$\mathcal{G}$: Gauss-Bonnet term

- Higher-order curvature term
- $f(\mathcal{G})$ gravity

T : torsion scalar

- DGP braneworld scenario [Dvali, Gabadadze and Porrati, Phys. Lett B 485, 208 (2000)]
- Non-local gravity [Deser and Woodard, Phys. Rev. Lett. 99, 111301 (2007)]
- $f(T)$ gravity [Bengochea and Ferraro, Phys. Rev. D 79, 124019 (2009)]

[Linder, Phys. Rev. D 81, 127301 (2010) [Erratum-ibid. D 82, 109902 (2010)]]

- Galileon gravity [Nicolis, Rattazzi and Trincherini, Phys. Rev. D 79, 064036 (2009)]

< Flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time >

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\boldsymbol{x}^2$$

$a(t)$: Scale factor

< Equation for $a(t)$ with a perfect fluid >

$$T_{\mu\nu} = \mathrm{diag}(\rho, P, P, P)$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\underline{(1+3w)\,\rho}$$

ρ : Energy density

P : Pressure

$\dot{} = \partial/\partial t$

$$\boxed{w \equiv \frac{P}{\rho}}$$

: Equation of state (EoS)

$\ddot{a} > 0$: **Accelerated expansion**

$$\Longrightarrow \boxed{w < -\frac{1}{3}}$$

: **Condition for accelerated expansion**

Cf. Cosmological constant $\Longrightarrow$ $w = -1$

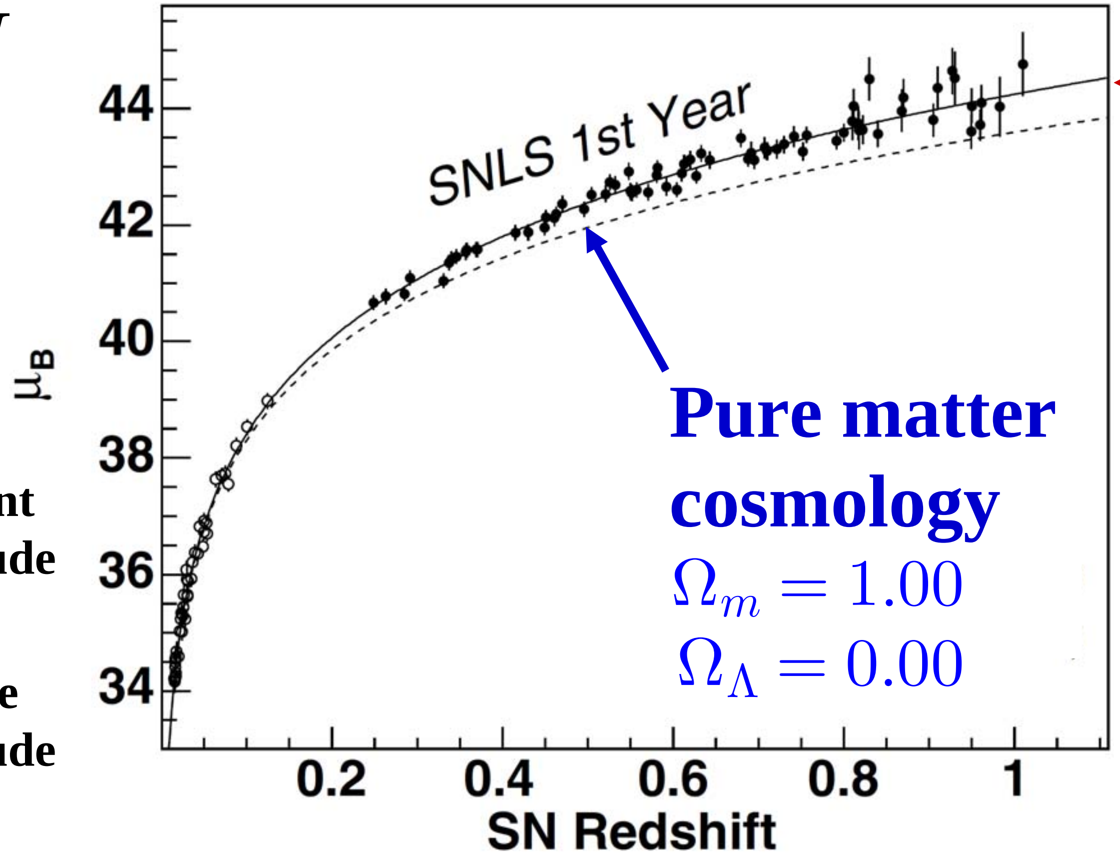
< SNLS data >

$m - M$

: **Distance estimator**

m : **Apparent magnitude**

M : **Absolute magnitude**



z

Flat Λ cosmology

$\Omega_m = 0.26$

$\Omega_\Lambda = 0.74$

Pure matter cosmology

$\Omega_m = 1.00$

$\Omega_\Lambda = 0.00$

From [Astier *et al.* [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

$$\frac{1}{H_0^2}\frac{\ddot{a}}{a} = -\frac{\Omega_m}{2}(1+z)^3 + \Omega_\Lambda$$

$$a_0 = 1$$

$$1 + z = \frac{a_0}{a},$$

z : Red shift

$$\Omega_m \equiv \frac{\kappa^2 \rho(t_0)}{3H_0^2}$$

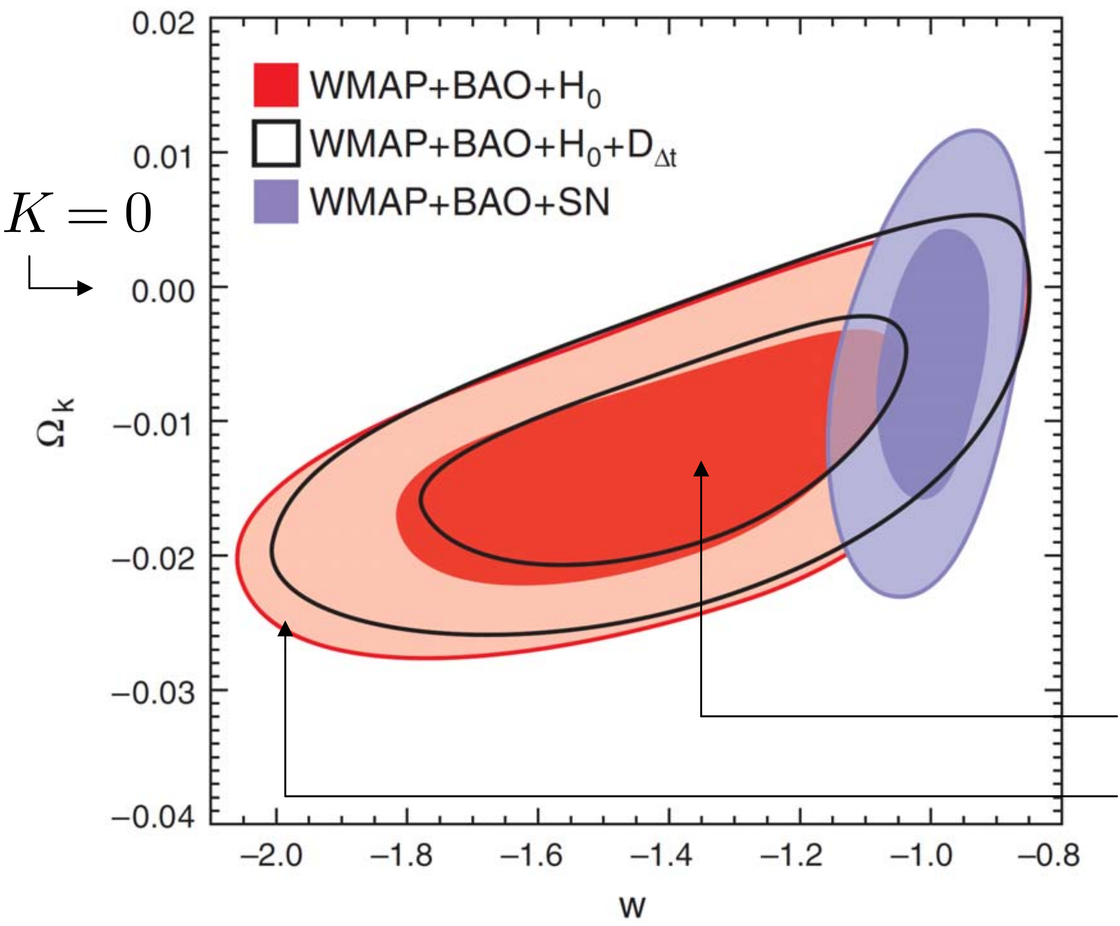
: Density parameter for matter

$$\Omega_\Lambda \equiv \frac{\Lambda}{3H_0^2}$$

: Density parameter for Λ

“0” denotes quantities at the present time t_0 .

< 7-year WMAP data on the current value of w >



$K = 0$

(68% CL)

(95% CL)

From [E. Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]]].

Hubble constant (H_0) measurement

Baryon acoustic oscillation (BAO)

: Special pattern in the large-scale correlation function of Sloan Digital Sky Survey (SDSS) luminous red galaxies

$D_{\Delta t}$: Time delay distance

$$\Omega_K \equiv \frac{K}{(a_0 H_0)^2}$$

: Density parameter for the curvature

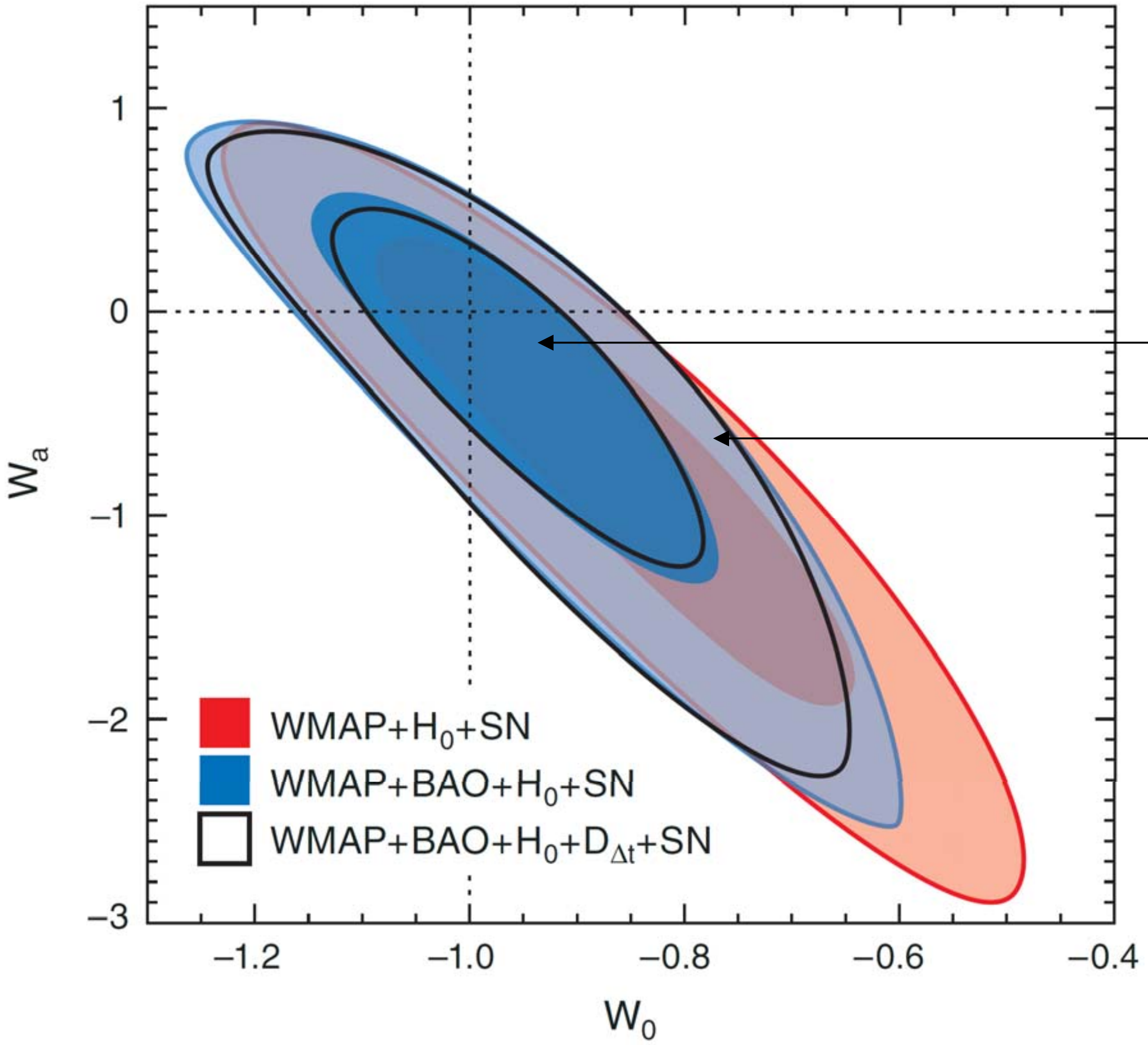
$K = 0$: Flat universe

- For the flat universe, constant w:

$$\boxed{w = -1.10 \pm 0.14 \ (68\% \ \mathrm{CL})}$$

(From $WMAP$ +BAO+H_0 .)

Cf. $\Omega_\Lambda = 0.725 \pm 0.016$ (68% CL)



From [E. Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]]].

(68% CL)

(95% CL)

Time-dependent w

$$w(a) = w_0 + w_a(1 - a)$$

$$a = \frac{1}{1+z}$$

w_0 : Current value of w

z : Redshift

(From $WMAP$+BAO+H_0+SN.)

- For the flat universe, a variable EoS :

$$\boxed{w_0 = -0.93 \pm 0.13\,,\ w_a = -0.41^{+0.72}_{-0.71} \quad (68\%\ \mathrm{CL})}$$

< Canonical scalar field >

$$S_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$g = \det(g_{\mu\nu})$

ϕ : Scalar field

$V(\phi)$: Potential of ϕ

- For a homogeneous scalar field $\phi = \phi(t)$:

$$\longrightarrow \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\Rightarrow \boxed{w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}}$$

If $\dot{\phi}^2 \ll V(\phi)$, $\underline{w_\phi \approx -1}$.

$\longrightarrow$ **Accelerated expansion can be realized.**

< *f*(*R*) gravity >

$$S = \int d^4x \sqrt{-g} \, \underline{\frac{f(R)}{2\kappa^2}}$$

$$\boxed{f(R) \text{ gravity}}$$

$f(R) = R$: General Relativity

[Nojiri and Odintsov, Phys. Rept. 505, 59 (2011) [arXiv:1011.0544 [gr-qc]]; Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007) [arXiv:hep-th/0601213]]

[Capozziello and Francaviglia, Gen. Rel. Grav. 40, 357 (2008)]

[Sotiriou and Faraoni, Rev. Mod. Phys. 82, 451 (2010)]

[De Felice and Tsujikawa, Living Rev. Rel. 13, 3 (2010)]

< Gravitational field equation >

$$f'(R) = df(R)/dR$$

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R) = 0$$

$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$: Covariant d'Alembertian

∇_μ : Covariant derivative operator

- In the flat FLRW background, gravitational field equations read

$$H^2 = \frac{\kappa^2}{3}\rho_{\rm eff}\,, \quad \dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\rm eff} + p_{\rm eff}\right)$$

$\rho_{\rm eff}$, $p_{\rm eff}$: Effective energy density and pressure from the term $f(R) - R$

$$\rho_{\rm eff} = \frac{1}{\kappa^2 f'(R)}\left[\frac{1}{2}\left(-f(R) + Rf'(R)\right) - 3H\dot{R}f''(R)\right]$$

$$p_{\rm eff} = \frac{1}{\kappa^2 f'(R)}\left[\frac{1}{2}\left(f(R) - Rf'(R)\right) + \left(2H\dot{R} + \ddot{R}\right)f''(R) + \dot{R}^2 f'''(R)\right]$$

$$\longrightarrow \boxed{w_{\rm eff} = \frac{p_{\rm eff}}{\rho_{\rm eff}} = \frac{\left(f(R) - Rf'(R)\right)/2 + \left(2H\dot{R} + \ddot{R}\right)f''(R) + \dot{R}^2 f'''(R)}{\left(-f(R) + Rf'(R)\right)/2 - 3H\dot{R}f''(R)}}$$

- Example: $f(R) = R - \underline{\frac{\mu^{2(n+1)}}{R^n}}$ **[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D <u>70</u>, 043528 (2004)]**

μ : Mass scale, n : Constant

Second term become important as R decreases.

⇨ $\underline{a \propto t^q}$, $\quad q = \frac{(2n+1)(n+1)}{n+2}$

$$w_{\rm eff} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}$$

(For $n = 1$, $\underline{q = 2 \text{ and } w_{\rm eff} = -2/3}$.)

If $q > 1$, accelerated expansion can be realized.

< Conditions for the viability of *f*(*R*) gravity >

(1) $f'(R) > 0 \longleftarrow$ **Positivity of the effective gravitational coupling**

$f'(R) \equiv df(R)/dR$ $G_{\mathrm{eff}} = G/f'(R) > 0$ G : Gravitational constant

(2) $f''(R) > 0 \longleftarrow$ **Stability condition:** $M^2 \approx 1/(3f''(R)) > 0$

$f''(R) \equiv d^2f(R)/dR^2$

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

M : Mass of a new scalar degree of freedom (“scalaron”) in the weak-field regime.

(3) $f(R) \to R - 2\Lambda$ for $R \gg R_0$. $\longleftarrow$ **Existence of a matter-dominated stage**

R_0 : Current curvature, Λ : Cosmological constant

(4) $0 < m \equiv Rf''(R)/f'(R) < 1 \longleftarrow$ **Stability of the late-time de Sitter point**

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

Cf. For general relativity, $m = 0.$

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]

(5) Constraints from the violation of the equivalence principle (Solar-system constraints)

[Chiba, Phys. Lett. B 575, 1 (2003)]

[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

$M = M(R) \longleftarrow$ Scale-dependence : **“Chameleon mechanism”**

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

< Models of *f*(*R*) gravity (examples) >

(i) Hu-Sawicki model

[Hu and Sawicki, Phys. Rev. D 76, 064004 (2007)]

Cf. [Nojiri and Odintsov, Phys. Lett. B 657, 238 (2007); Phys. Rev. D 77, 026007 (2008)]

$$f_{\mathrm{HS}} = R - \frac{c_1 R_{\mathrm{HS}} \left(R/R_{\mathrm{HS}}\right)^p}{c_2 \left(R/R_{\mathrm{HS}}\right)^p + 1}$$

$c_1,\ c_2,\ p(>0),\ R_{\mathrm{HS}}(>0)$: Constant parameters

(ii) Starobinsky's model

[Starobinsky, JETP Lett. 86, 157 (2007)]

$$f_{\mathrm{S}} = R + \lambda R_{\mathrm{S}} \left[\left(1 + \frac{R^2}{R_{\mathrm{S}}^2}\right)^{-n} - 1\right]$$

$\lambda(>0),\ n(>0),\ R_{\mathrm{S}}$: Constant parameters

(iii) Tsujikawa's model

[Tsujikawa, Phys. Rev. D 77, 023507 (2008)]

$$f_{\mathrm{T}} = R - \mu R_{\mathrm{T}} \tanh\left(\frac{R}{R_{\mathrm{T}}}\right)$$

$\mu(>0),\ R_{\mathrm{T}}(>0)$: Constant parameters

(iv) Exponential gravity model

[Cognola, Elizalde, Nojiri, Odintsov, Sebastiani and Zerbini, Phys. Rev. D 77, 046009 (2008)]

[Linder, Phys. Rev. D 80, 123528 (2009)]

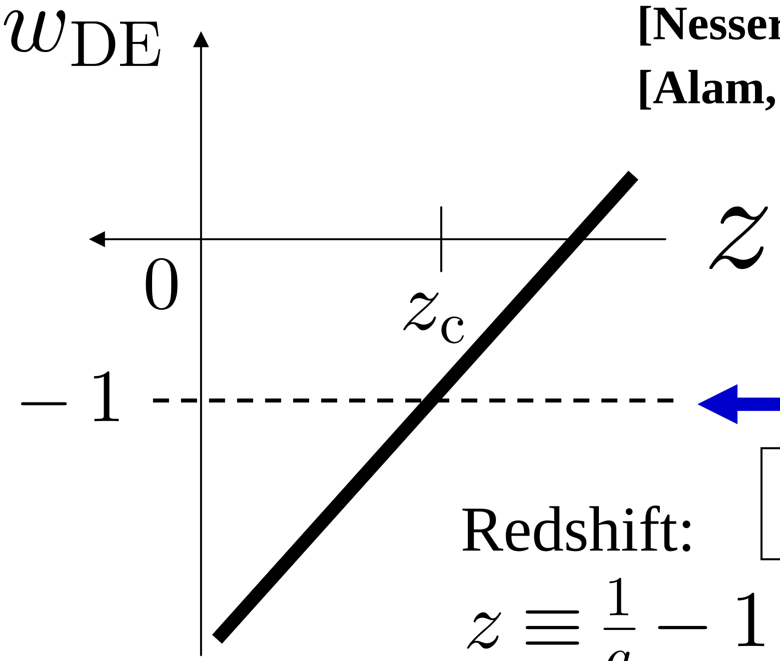
$$f_{\mathrm{E}} = R - \beta R_{\mathrm{E}} \left(1 - e^{-R/R_{\mathrm{E}}}\right)$$

$\beta,\ R_{\mathrm{E}}$: Constant parameters

< Crossing of the phantom divide >

- Various observational data (SN, Cosmic microwave background radiation (CMB), BAO) imply that the EoS of dark energy $w_{\rm DE}$ may evolve from larger than -1 (non-phantom phase) to less than -1 (phantom phase). Namely, it crosses -1 (the crossing of the phantom divide).

[Alam, Sahni and Starobinsky, JCAP 0406, 008 (2004)]
[Nesseris and Perivolaropoulos, JCAP 0701, 018 (2007)]
[Alam, Sahni and Starobinsky, JCAP 0702, 011 (2007)]



Redshift:
$z \equiv \frac{1}{a} - 1$

$z_{\rm c}$: Red shift at the crossing of the phantom divide

(a) $w_{\rm DE} > -1$

Non-phantom phase

(b) $w_{\rm DE} = -1$

Crossing of the phantom divide

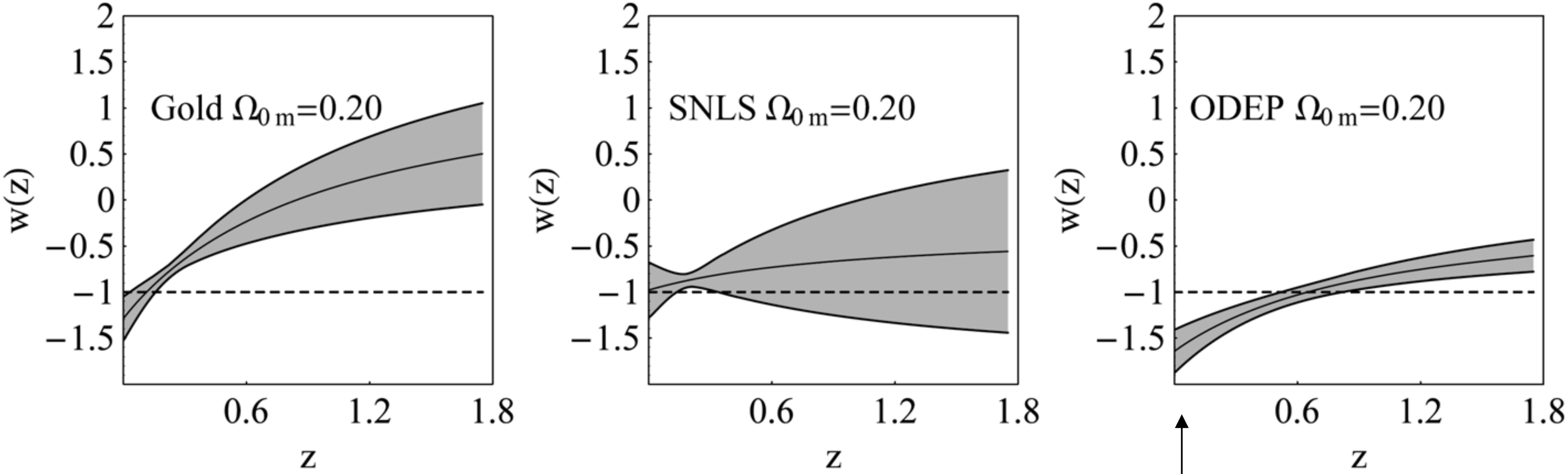
(c) $w_{\rm DE} < -1$

Phantom phase

< Data fitting of $w(z)$ >

$$\boxed{w(z) = w_0 + w_1 \frac{z}{1+z}}$$

From [Nesseris and L. Perivolaropoulos, JCAP 0701, 018 (2007)].



SN gold data set

[Riess *et al*. [Supernova Search Team Collaboration], Astrophys. J. 607, 665 (2004)]

SNLS data set

[Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

Shaded region shows 1σ error.

Cosmic microwave background radiation (CMB) data

[Spergel *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 170, 377 (2007)]

+ SDSS baryon acoustic peak (BAO) data

[Eisenstein *et al*. [SDSS Collaboration], Astrophys. J. 633, 560 (2005)]

- **It is known that in several viable *f*(*R*) gravity models, the crossing of the phantom divide can occur in the past.**

(i) Hu-Sawicki model

[Hu and Sawicki, Phys. Rev. D 76, 064004 (2007)]

[Martinelli, Melchiorri and Amendola, Phys. Rev. D 79, 123516 (2009)]

Cf. [Nozari and Azizi, Phys. Lett. B 680, 205 (2009)]

(ii) Starobinsky's model

[Motohashi, Starobinsky and Yokoyama, Prog. Theor. Phys. 123, 887 (2010); Prog. Theor. Phys. 124, 541 (2010)]

(iv) Exponential gravity model

[Linder, Phys. Rev. D 80, 123528 (2009)]

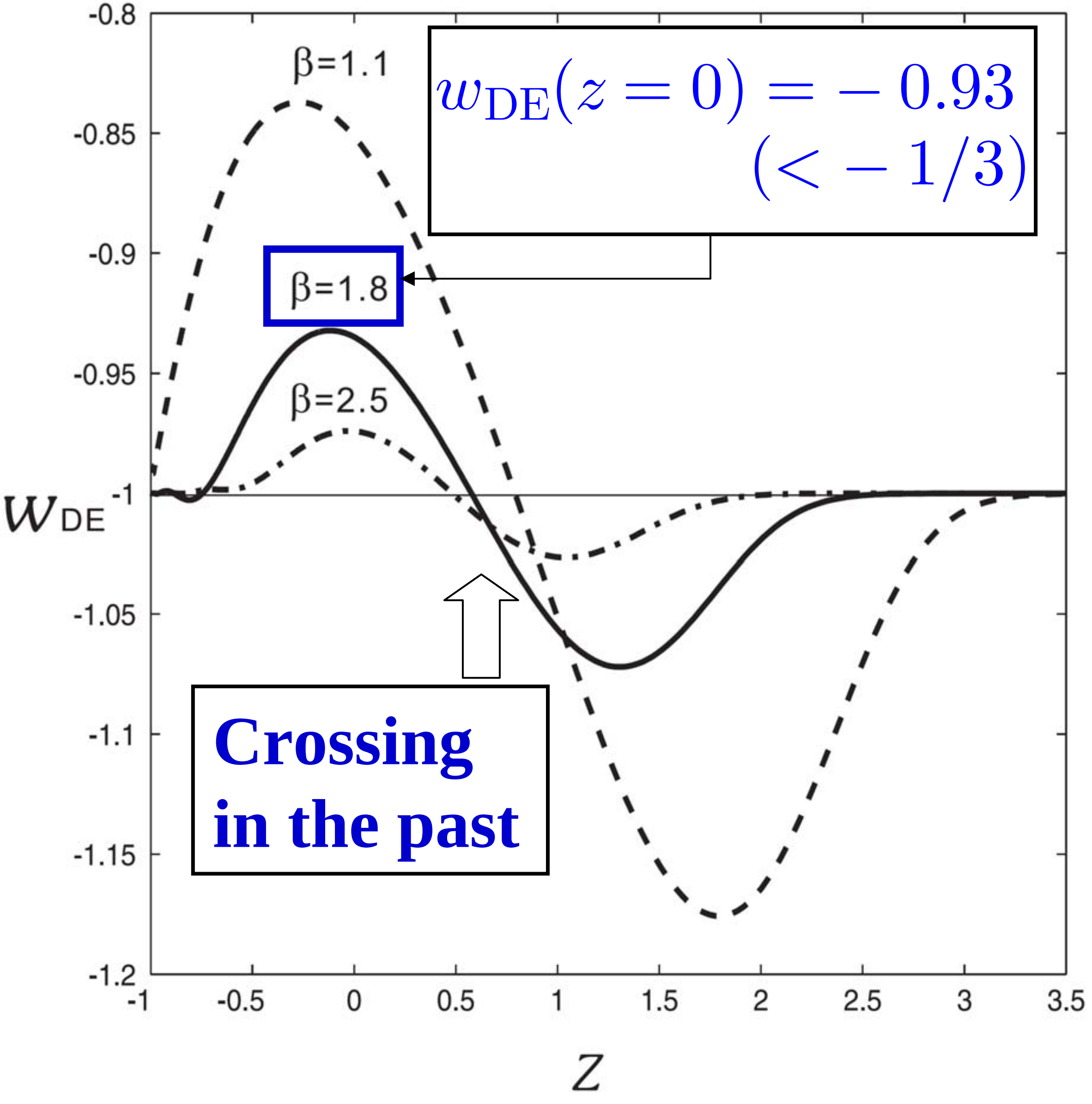
[KB, Geng and Lee, JCAP 1008, 021 (2010) arXiv:1005.4574 [astro-ph.CO]]

Cf. Appleby-Battye model

[Appleby, Battye and Starobinsky, JCAP 1006, 005 (2010)]

< Cosmological evolution of $w_{\rm DE}$ in the exponential gravity model >

From [KB, Geng and Lee, JCAP 1008, 021 (2010)].



$$w_{\rm DE}(z=0) = -\,0.93$$
$$(< -\,1/3)$$

Crossing in the past

$$f_{\rm E}(R) =$$
$$R - \beta R_{\rm E}\left(1 - e^{-R/R_{\rm E}}\right)$$

$$w_{\rm DE} = -\,1$$

Crossing of the phantom divide

$$\beta R_{\rm E} \simeq 18 H_0^2 \Omega_{\rm m}^{(0)}$$

We explicitly demonstrate that the future crossings of the phantom divide line $w_{\rm DE} = -1$ are the generic feature in the existing viable *f*(*R*) gravity models.

- Recent related study on the future crossings of the phantom divide:

[Motohashi, Starobinsky and Yokoyama, JCAP 1106, 006 (2011)]

II. Future crossing of the phantom divide in *f*(*R*) gravity

< Action >

$$g = \det(g_{\mu\nu})$$

$$I = \int d^4x \sqrt{-g} \frac{f(R)}{\underline{2\kappa^2}} + I_{\mathrm{matter}}(g_{\mu\nu}, \Upsilon_{\mathrm{matter}})$$

$g_{\mu\nu}$: Metric tensor

f : Arbitrary function of R

I_{matter} : Action of matter

$\Upsilon_{\mathrm{matter}}$: Matter fields

< Gravitational field equation >

$$FG_{\mu\nu} = \kappa^2 T^{(\mathrm{matter})}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (FR - f) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F$$

$$F(R) \equiv df(R)/dR$$

$T^{(\mathrm{matter})}_{\mu\nu}$: Energy-momentum tensor of all perfect fluids of matter

$$G_{\mu\nu} = R_{\mu\nu} - (1/2)\, g_{\mu\nu} R$$

$R_{\mu\nu}$: Ricci tensor

∇_μ : Covariant derivative operator

$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$: Covariant d'Alembertian

→ Gravitational field equations in the FLRW background:

$$3FH^2 = \kappa^2 \rho_{\mathrm{M}} + \frac{1}{2}(FR - f) - 3H\dot{F}$$

$$-2F\dot{H} = \kappa^2 (\rho_{\mathrm{M}} + P_{\mathrm{M}}) + \ddot{F} - H\dot{F}$$

$\dot{} = \partial/\partial t$

$H = \dot{a}/a$

: Hubble parameter

ρ_{M} and P_{M} : Energy density and pressure of all perfect fluids of matter, respectively.

⇨ $$\boxed{w_{\mathrm{DE}} \equiv \frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}}}$$

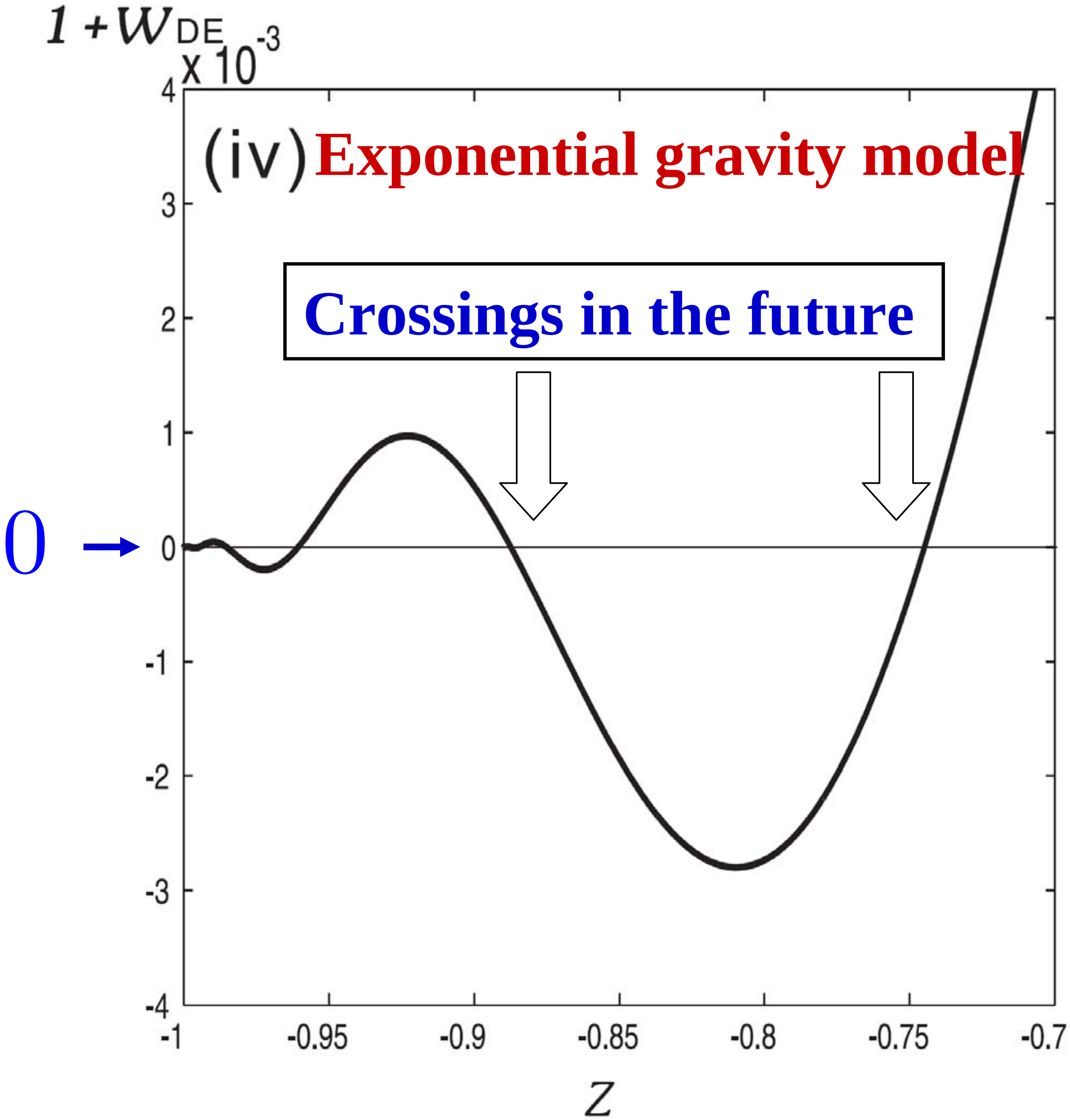
$$\rho_{\mathrm{DE}} = \frac{1}{\kappa^2}\left[\frac{1}{2}(FR - f) - 3H\dot{F} + 3(1 - F)H^2\right]$$

$$P_{\mathrm{DE}} = \frac{1}{\kappa^2}\left[-\frac{1}{2}(FR - f) + \ddot{F} + 2H\dot{F} - (1 - F)\left(2\dot{H} + 3H^2\right)\right]$$

- Analysis method: **[Hu and Sawicki, Phys. Rev. D <u>76</u>, 064004 (2007)]**

< Future evolutions of $1 + w_{\mathrm{DE}}$ as functions of z >

$$\boxed{1 + w_{\mathrm{DE}}}$$



(iv) Exponential gravity model

Crossings in the future

$1 + w_{\mathrm{DE}} = 0$

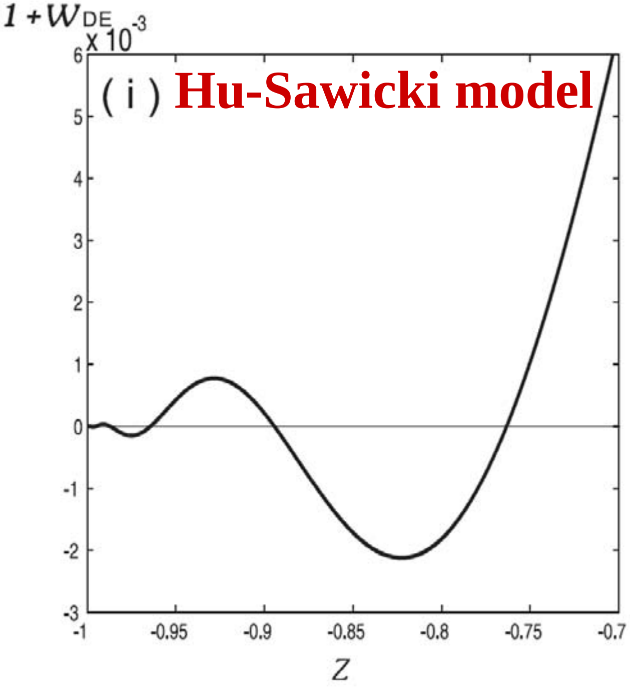
Crossing of the phantom divide

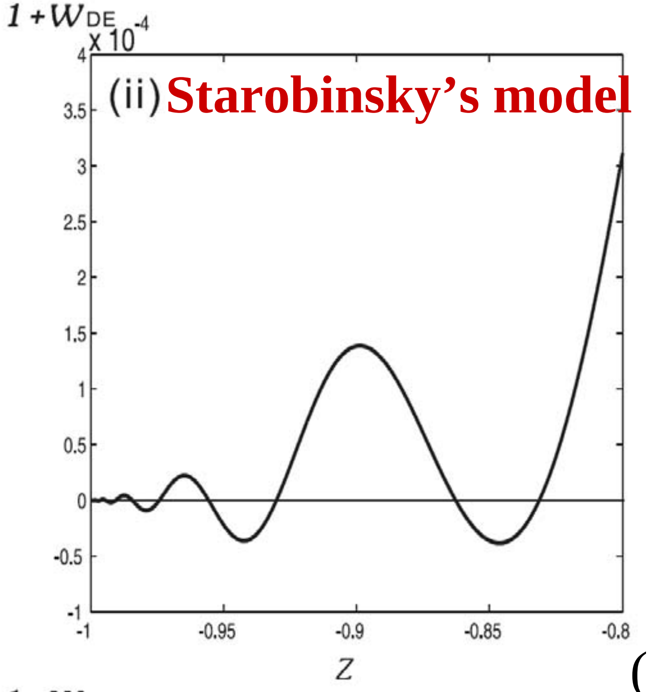
Redshift:

$$z \equiv \frac{1}{a} - 1$$

($z < 0$: Future)

$$\boxed{1 + w_{\rm DE}}$$

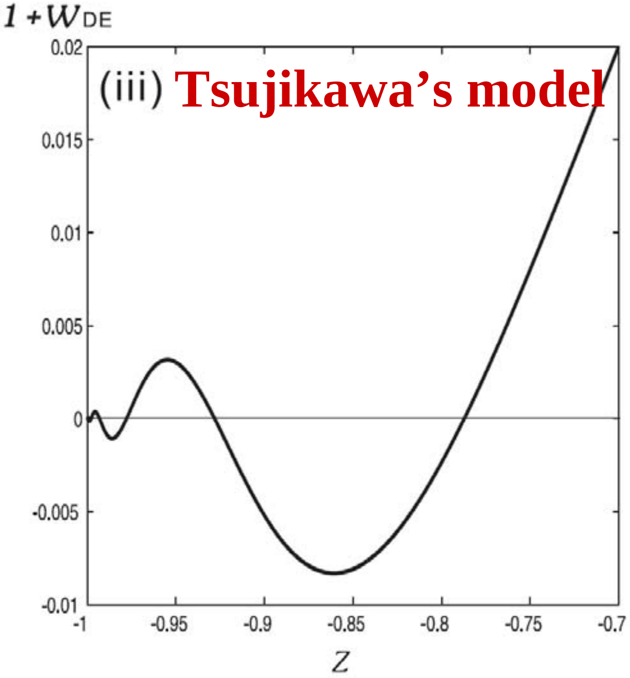

(i) **Hu-Sawicki model**


(ii) **Starobinsky's model**

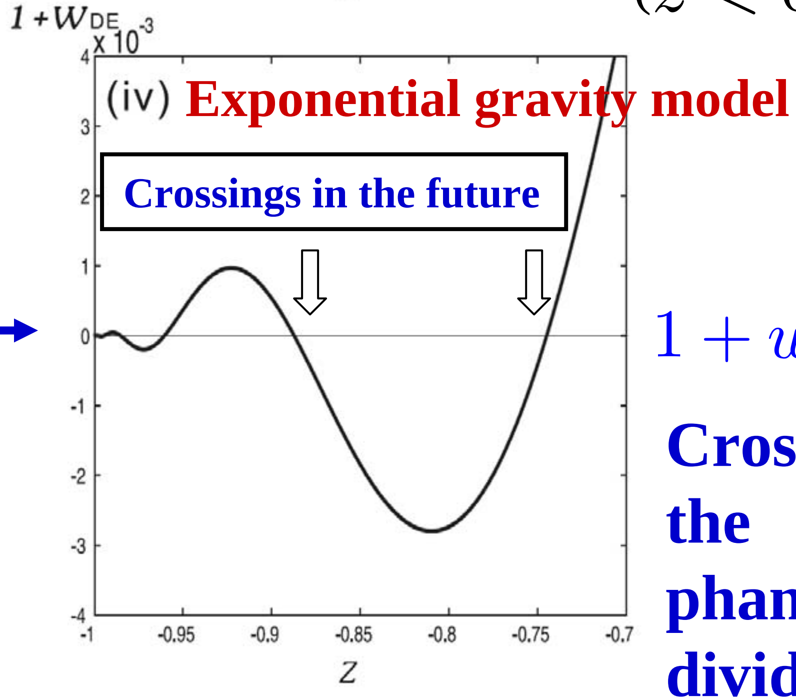
Redshift:

$$z \equiv \frac{1}{a} - 1$$

$(z < 0$: Future)


(iii) **Tsujikawa's model**

0 →


(iv) **Exponential gravity model**

Crossings in the future

$$1 + w_{\rm DE} = 0$$

Crossing of the phantom divide

III. Equation of state for dark energy in *f*(*T*) theory

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu$$

$e_A(x^\mu)$: Orthonormal tetrad components

An index A runs over 0, 1, 2, 3 for the tangent space at each point of x^μ the manifold.

μ and ν are coordinate indices on the manifold and also run over 0, 1, 2, 3, and $e_A(x^\mu)$ forms the tangent vector of the manifold.

$${T^\rho}_{\mu\nu} \equiv e^\rho_A \left(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu\right)$$: Torsion tensor

$${K^{\mu\nu}}_\rho \equiv -\frac{1}{2}\left({T^{\mu\nu}}_\rho - {T^{\nu\mu}}_\rho - {T_\rho}^{\mu\nu}\right)$$

: Contorsion tensor

$$T \equiv {S_\rho}^{\mu\nu} {T^\rho}_{\mu\nu}$$: Torsion scalar

$${S_\rho}^{\mu\nu} \equiv \frac{1}{2}\left({K^{\mu\nu}}_\rho + \delta^\mu_\rho \, {T^{\alpha\nu}}_\alpha - \delta^\nu_\rho \, {T^{\alpha\mu}}_\alpha\right)$$

→ **Instead of the Ricci scalar R for the Lagrangian density in general relativity, the teleparallel Lagrangian density is described by the torsion scalar T .**

< Modified teleparallel action for *f*(*T*) theory >

$$\boxed{I = \frac{1}{16\pi G}\int d^4x |e| \left(T + f(T)\right)}$$

$$|e| = \det\left(e^A_\mu\right) = \sqrt{-g}$$

$$F(T) \equiv T + f(T)$$

⇨ $$\frac{1}{e}\partial_\mu\left(eS_A{}^{\mu\nu}\right)F' - e_A^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}F' + S_A{}^{\mu\nu}\partial_\mu\left(T\right)F'' + \frac{1}{4}e_A^\nu F = 0$$

: Gravitational field equation

[Bengochea and Ferraro, Phys. Rev. D 79, 124019 (2009)]

∗ A prime denotes a derivative with respect to T .

- We assume the flat FLRW space-time with the metric.

⇨ $$\boxed{T = -6H^2}$$

→ Modified Friedmann equations in the flat FLRW background:

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\mathrm{M}} + \rho_{\mathrm{DE}}\right)$$

$$f_T \equiv df(T)/dT$$

$$\left(H^2\right)' = -8\pi G\left(\rho_{\mathrm{M}} + P_{\mathrm{M}} + \rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right)$$

$$f_{TT} \equiv d^2 f(T)/dT^2$$

$$\rho_{\mathrm{DE}} = \frac{1}{16\pi G}\left(-f + 2Tf_T\right), \qquad P_{\mathrm{DE}} = \frac{1}{16\pi G}\frac{f - Tf_T + 2T^2 f_{TT}}{1 + f_T + 2Tf_{TT}}$$

∗ A prime denotes a derivative with respect to $\ln a$.

⇨ $$\boxed{w_{\mathrm{DE}} \equiv \frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = -1 + \frac{T'}{3T}\frac{f_T + 2Tf_{TT}}{f/T - 2f_T} = -\frac{f/T - f_T + 2Tf_{TT}}{\left(1 + f_T + 2Tf_{TT}\right)\left(f/T - 2f_T\right)}}$$

- We consider only non-relativistic matter (cold dark matter and baryon).

< Combined *f*(*T*) theory >

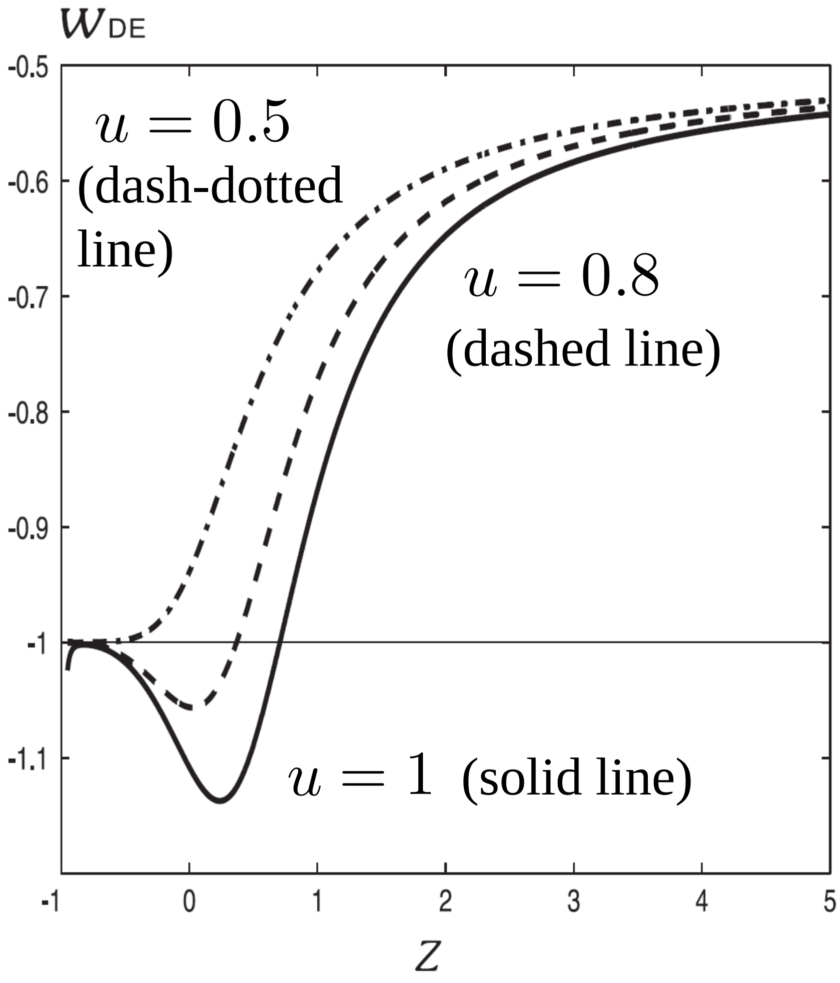
$$f(T) = \gamma \left[\underline{T_0 \left(\frac{u T_0}{T} \right)^{-1/2} \ln \left(\frac{u T_0}{T} \right)} \underline{- T \left(1 - e^{u T_0 / T} \right)} \right]$$

$u (> 0)$: Positive constant

$T_0 = T(z = 0)$

Logarithmic term

Exponential term



$$\gamma \equiv \frac{1 - \Omega_{\mathrm{m}}^{(0)}}{2u^{-1/2} + \left[1 - (1 - 2u)\, e^{u} \right]}$$

← $w_{\mathrm{DE}} = -1$

Crossing of the phantom divide

- The model contains only one parameter u if one has the value of $\Omega_{\mathrm{m}}^{(0)}$.

IV. Effective equation of state for the universe and the finite-time future singularities in non-local gravity

Non-local gravity ⇐ **produced by quantum effects**

[Deser and Woodard, Phys. Rev. Lett. 99, 111301 (2007)]

- It is known that so-called matter instability occurs in $F(R)$ gravity.

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

→ This implies that the curvature inside matter sphere becomes very large and hence the curvature singularity could appear.

[Arbuzova and Dolgov, Phys. Lett. B 700, 289 (2011)]

⇒ It is important to examine whether there exists the curvature singularity, i.e., “**the finite-time future singularities**” **in non-local gravity**.

A. Non-local gravity

< Action >

$g = \det(g_{\mu\nu})$ $g_{\mu\nu}$: Metric tensor

f : Some function Λ : Cosmological constant

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R \left(1 + \underbrace{f(\Box^{-1} R)}_{\text{Non-local gravity}} \right) - 2\Lambda \right] + \mathcal{L}_{\text{matter}} (Q; g) \right\}$$

By introducing two scalar fields η and ξ , we find

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R \left(1 + f(\eta) \right) + \xi \underline{\left(\Box\eta - R \right)} - 2\Lambda \right] + \mathcal{L}_{\text{matter}} \right\}$$

$$= \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \left[R \left(1 + f(\eta) \right) - \partial_\mu \xi \partial^\mu \eta - \xi R - 2\Lambda \right] + \mathcal{L}_{\text{matter}} \right\}$$

- By the variation of the action in the first expression over ξ , we obtain

$\underline{\Box\eta = R}$ (or $\eta = \Box^{-1} R$)

→ Substituting this equation into the action in the first expression, one re-obtains the starting action.

∇_μ : Covariant derivative operator

$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$

: Covariant d'Alembertian

$\mathcal{L}_{\text{matter}} (Q; g)$

: Matter Lagrangian

Q : Matter fields

< Gravitational field equation >

$$0 = \frac{1}{2} g_{\mu\nu} \left[R \left(1 + f(\eta) - \xi \right) - \partial_\rho \xi \partial^\rho \eta - 2\Lambda \right] - R_{\mu\nu} \left(1 + f(\eta) - \xi \right)$$

$$+ \frac{1}{2} \left(\partial_\mu \xi \partial_\nu \eta + \partial_\mu \eta \partial_\nu \xi \right) - \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) \left(f(\eta) - \xi \right) + \kappa^2 T_{\mathrm{matter}\,\mu\nu}$$

$T_{\mathrm{matter}\,\mu\nu} \equiv -(2/\sqrt{-g}) \left(\delta \sqrt{-g} \mathcal{L}_{\mathrm{matter}} / \delta g^{\mu\nu} \right)$: Energy-momentum tensor of matter

- The variation of the action with respect to η gives

$$0 = \Box \xi + f'(\eta) R$$

$*\ '$ (prime) : Derivative with respect to η

- We assume the flat FLRW space-time with the metric and consider the case in which the scalar fields η and ξ only depend on time.

⇨ Gravitational field equations in the FLRW background:

$$0 = -3H^2 \left(1 + f(\eta) - \xi \right) + \frac{1}{2} \dot{\xi} \dot{\eta} - 3H \left(f'(\eta) \dot{\eta} - \dot{\xi} \right) + \Lambda + \kappa^2 \rho_{\mathrm{m}}$$

$$0 = \left(2\dot{H} + 3H^2 \right) \left(1 + f(\eta) - \xi \right) + \frac{1}{2} \dot{\xi} \dot{\eta} + \left(\frac{d^2}{dt^2} + 2H \frac{d}{dt} \right) \left(f(\eta) - \xi \right) - \Lambda + \kappa^2 P_{\mathrm{m}}$$

ρ_{m} and P_{m} : Energy density and pressure of matter.

< Equations of motion for η and ξ >

$$0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2, \quad 0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2 \right) f'(\eta)$$

→ In the flat FLRW space-time, we analyze an asymptotic solution of the gravitational field equations in the limit of the time t_{s} when the finite-time future singularities appear.

- We consider the case in which the Hubble parameter is expressed as

$$\boxed{H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}}$$

h_{s} : Positive constant, $0 < t < t_{\mathrm{s}}$

q : Non-zero constant larger than -1 $(q > -1, q \neq 0)$

- When $t \longrightarrow t_{\mathrm{s}}$, $R = 6\dot{H} + 12H^2 \longrightarrow \infty$.

→ **Scale factor** :

$$\boxed{a \sim a_{\mathrm{s}} \exp\left[\frac{h_{\mathrm{s}}}{q-1}(t_{\mathrm{s}} - t)^{-(q-1)}\right]}$$

a_{s} : Constant

- We have

$$\boxed{\eta = -\int^t \frac{1}{a^3}\left(\int^{\bar{t}} Ra^3 d\bar{t}\right) dt} \; .$$

η_{C}, ξ_{C} : Integration constants

- We take a form of $f(\eta)$ as

$$\boxed{f(\eta) = f_{\mathrm{s}} \eta^{\sigma}} \; .$$

$f_{\mathrm{s}}(\neq 0)$, $\sigma(\neq 0)$: Non-zero constants

→

$$\boxed{\xi = \int^t \frac{1}{a^3}\left(\int^{\bar{t}} \frac{df(\eta)}{d\eta} Ra^3 d\bar{t}\right) dt}$$

- It is known that the finite-time future singularities can be classified in the following manner:

[Nojiri, Odintsov and Tsujikawa, Phys. Rev. D 71, 063004 (2005)]

	In the limit $t \to t_{\mathrm{s}}$,
Type I ("Big Rip"):	$\underline{a \longrightarrow \infty\,, \quad \rho_{\mathrm{eff}} \to \infty\,, \quad \vert P_{\mathrm{eff}}\vert \to \infty}$ * The case in which ρ_{eff} and P_{eff} becomes finite values at $t = t_{\mathrm{s}}$ is also included.
Type II ("sudden"):	$a \longrightarrow a_{\mathrm{s}}\,, \quad \rho_{\mathrm{eff}} \to \rho_{\mathrm{s}}\,, \quad \underline{\vert P_{\mathrm{eff}}\vert \to \infty}$
Type III:	$a \longrightarrow a_{\mathrm{s}}\,, \quad \underline{\rho_{\mathrm{eff}} \to \infty\,, \quad \vert P_{\mathrm{eff}}\vert \to \infty}$
Type IV:	$a \longrightarrow a_{\mathrm{s}}\,, \quad \rho_{\mathrm{eff}} \to 0\,, \quad \vert P_{\mathrm{eff}}\vert \to 0$ * Higher derivatives of H diverge. * The case in which ρ_{eff} and/or $\vert P_{\mathrm{eff}}\vert$ asymptotically approach finite values is also included.

- The finite-time future singularities described by the expression of H in non-local gravity have the following properties:

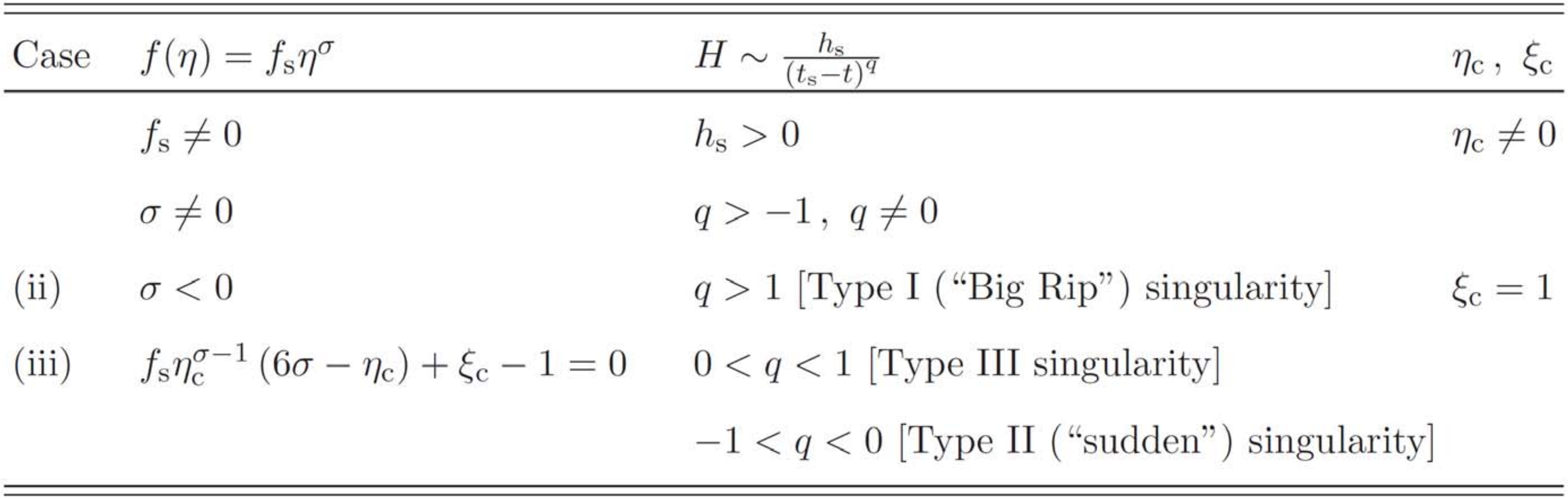
$$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}$$

For $q > 0$, Type I ("Big Rip")

For $-1 < q < 0$, Type II ("sudden")

For $q > -1$, Type III

* Range and conditions for the value of parameters of $f(\eta)$, H, and η_{c} and ξ_{c} in order that the finite-time future singularities can exist.



Case	$f(\eta) = f_{\mathrm{s}}\eta^{\sigma}$	$H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}}-t)^q}$	η_{c}, ξ_{c}
	$f_{\mathrm{s}} \neq 0$	$h_{\mathrm{s}} > 0$	$\eta_{\mathrm{c}} \neq 0$
	$\sigma \neq 0$	$q > -1$, $q \neq 0$	
(ii)	$\sigma < 0$	$q > 1$ [Type I ("Big Rip") singularity]	$\xi_{\mathrm{c}} = 1$
(iii)	$f_{\mathrm{s}}\eta_{\mathrm{c}}^{\sigma-1}(6\sigma - \eta_{\mathrm{c}}) + \xi_{\mathrm{c}} - 1 = 0$	$0 < q < 1$ [Type III singularity]	
		$-1 < q < 0$ [Type II ("sudden") singularity]	

→ We examine the asymptotic behavior of w_{eff} in the limit $t \to t_{\mathrm{s}}$ by taking the leading term in terms of $(t_{\mathrm{s}} - t)$.

- For $q > 1$ [Type I ("Big Rip") singularity], w_{eff} evolves from the non-phantom phase or the phantom one and asymptotically approaches $w_{\mathrm{eff}} = -1$.
- For $0 < q < 1$ [Type III singularity], w_{eff} evolves from the non-phantom phase to the phantom one with realizing a crossing of the phantom divide or evolves in the phantom phase.

  ⇨ The final stage is the eternal phantom phase.
- For $-1 < q < 0$ [Type II ("sudden") singularity], $w_{\mathrm{eff}} > 0$ at the final stage.

→ We estimate the present value of $w_{\rm eff}$.

- For case (ii) $[q > 1, \ \sigma < 0]$,

$\sigma = -1$

$q = 2$

$h_{\rm s} = 1 \, [{\rm GeV}]^{-1}$

$t_{\rm s} = 2t_{\rm p}$

$H_{\rm p} = 2.1h \times 10^{-42}{\rm GeV}$

: Current value of H , $h = 0.7$ **[Freedman *et al*. [HST Collaboration], Astrophys. J. 553, 47 (2001)]**

* We regard $w_{\rm eff} \approx w_{\rm DE}$ at the present time because the energy density of dark energy is dominant over that of non-relativistic matter at the present time.

$$\boxed{\begin{array}{l} f_{\rm s} = -3.0 \times 10^{-43} \\ \underline{w_{\rm eff} = -1.10} \end{array}}$$

$$\boxed{\begin{array}{l} f_{\rm s} = -2.1 \times 10^{-43} \\ \underline{w_{\rm eff} = -0.93} \end{array}}$$

$t_{\rm p}$: The present time

$h_{\rm s}$ has the dimension of $[{\rm Mass}]^{q-1}$.

- For $0 < q < 1$,

$q = 1/2$

$h_{\rm s} = 1 \, [{\rm GeV}]^{1/2}$

$\eta_{\rm c} = 1$

$t_{\rm s} = 2t_{\rm p}$

$$\boxed{\begin{array}{l} f_{\rm s} = 7.9 \times 10^{-2} \\ \underline{w_{\rm eff} = -1.10} \end{array}}$$

$$\boxed{\begin{array}{l} f_{\rm s} = 6.6 \times 10^{-2} \\ \underline{w_{\rm eff} = -0.93} \end{array}}$$

- For $-1 < q < 0$, $w_{\rm eff} > 0$.

⇒ **In our models, $w_{\rm eff}$ can have the present observed value of $w_{\rm DE}$.**

V. Summary

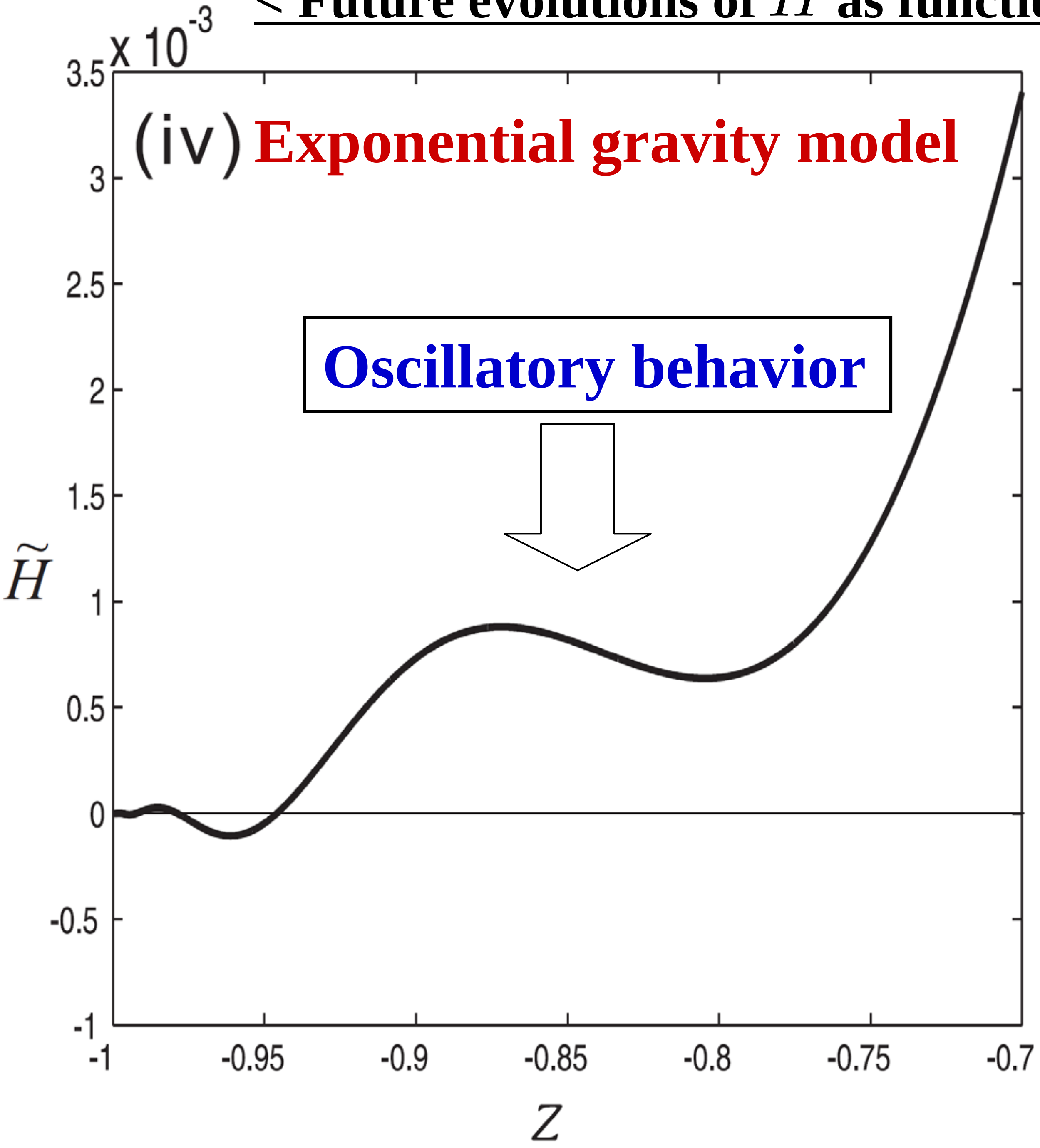
- **We have discussed modified gravitational theories in order to explain the current accelerated expansion of the universe, so-called dark energy problem.**
- We have investigated the equation of state for dark energy w_{DE} in *f*(*R*) gravity as well as *f*(*T*) theory.

⇨ **The future crossings of the phantom divide line $w_{\mathrm{DE}} = -1$ are the generic feature in the existing viable *f*(*R*) gravity models.**

⇨ **The crossing of the phantom divide line can be realized in the combined *f*(*T*) theory.**

- **We have studied the effective equation of state for the universe when the finite-time future singularities occur in non-local gravity.**

< Future evolutions of H as functions of z >



$$\boxed{\tilde{H} \equiv \bar{H} - \bar{H}_{\mathrm{f}}}$$

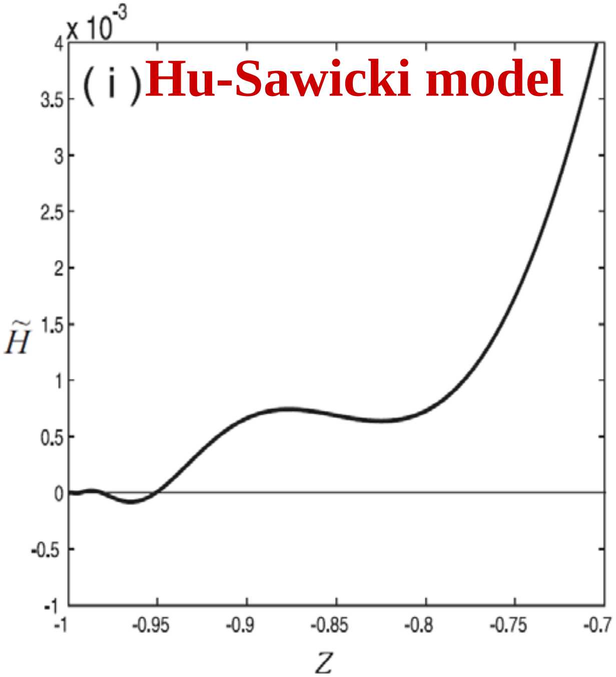
$$\bar{H} \equiv H / H_0$$

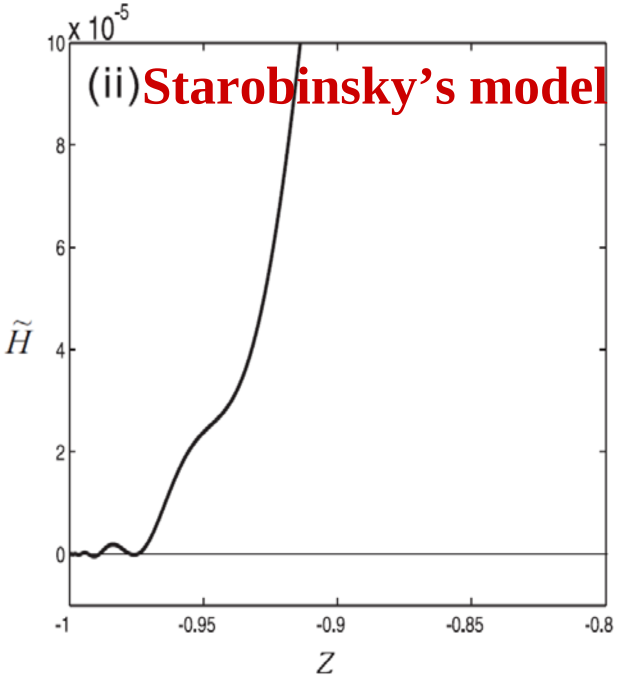
$$\bar{H}_{\mathrm{f}} \equiv \frac{H(z=-1)}{H_0}$$

: ‘f’ denotes the value at the final stage $z = -1.$

$$H_0 = 71 \mathrm{km/s/Mpc}$$

: Present value of the Hubble parameter

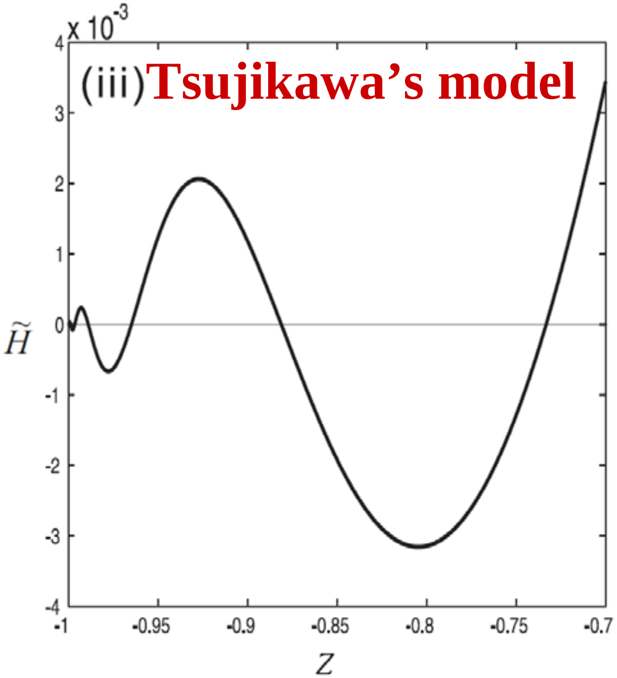

(i) **Hu-Sawicki model**

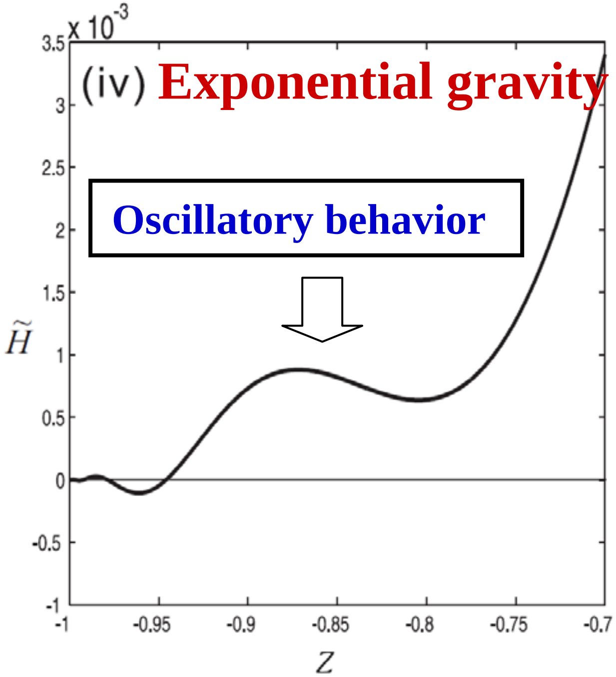

(ii) **Starobinsky's model**

$$\boxed{\tilde{H} \equiv \bar{H} - \bar{H}_{\mathrm{f}}}$$

$$\bar{H} \equiv H/H_0$$

$$\bar{H}_{\mathrm{f}} \equiv \frac{H(z=-1)}{H_0}$$


(iii) **Tsujikawa's model**


(iv) **Exponential gravity model**

Oscillatory behavior

$H_0 = 71\mathrm{km/s/Mpc}$

: Present value of the Hubble parameter

- In the future ($-1 \leq z \lesssim -0.74$), the crossings of the phantom divide are the generic feature for all the existing viable *f*(*R*) models.
- As z decreases ($-1 \leq z \lesssim -0.90$), dark energy becomes much more dominant over non-relativistic matter ($\underline{\Xi = \Omega_{\mathrm{m}}/\Omega_{\mathrm{DE}} \lesssim 10^{-5}}$).

< Effective equation of state for the universe >

$$\boxed{w_{\mathrm{eff}} \equiv -1 - \frac{2}{3}\frac{\dot{H}}{H^2} = \frac{P_{\mathrm{tot}}}{\rho_{\mathrm{tot}}}}$$

$$\rho_{\mathrm{tot}} \equiv \rho_{\mathrm{DE}} + \rho_{\mathrm{m}} + \rho_{\mathrm{r}}$$

: Total energy density of the universe

$$P_{\mathrm{tot}} \equiv P_{\mathrm{DE}} + P_{\mathrm{m}} + P_{\mathrm{r}}$$

: Total pressure of the universe

$$\Rightarrow \boxed{w_{\mathrm{DE}} \approx w_{\mathrm{eff}}}$$

P_{DE} : Pressure of dark energy

P_{m} : Pressure of non-relativistic matter (cold dark matter and baryon)

P_{r} : Pressure of radiation

$$\boxed{w_{\mathrm{eff}} = -1 - \frac{2}{3}\frac{\dot{H}}{H^2}}$$

(a) $\dot{H} < 0 \Longrightarrow w_{\mathrm{eff}} > -1$ **Non-phantom phase**

(b) $\dot{H} = 0 \Longrightarrow w_{\mathrm{eff}} = -1$ **Crossing of the phantom divide**

(c) $\dot{H} > 0 \Longrightarrow w_{\mathrm{eff}} < -1$ **Phantom phase**

- The physical reason why the crossing of the phantom divide appears in the farther future $\underline{(-1 \leq z \lesssim -0.90)}$ is that **the sign of $\dot{H}$ changes from negative to positive due to the dominance of dark energy over non-relativistic matter.**
- As $w_{\mathrm{DE}} \approx w_{\mathrm{eff}}$ in the farther future, w_{DE} oscillates around the phantom divide line $w_{\mathrm{DE}} = -1$ because **the sign of $\dot{H}$ changes and consequently multiple crossings can be realized.**

< Conditions for the viability of *f*(*R*) gravity >

(1) $f'(R) > 0$ ← **Positivity of the effective gravitational coupling**

$f'(R) \equiv df(R)/dR$ $G_{\mathrm{eff}} = G/f'(R) > 0$ G : Gravitational constant

(2) $f''(R) > 0$ ⟵ **Stability condition:** $M^2 \approx 1/(3f''(R)) > 0$

$f''(R) \equiv d^2 f(R)/dR^2$

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

M : Mass of a new scalar degree of freedom (“scalaron”) in the weak-field regime.

(3) $f(R) \to R - 2\Lambda$ for $R \gg R_0$. ← **Existence of a matter-dominated stage**

R_0 : Current curvature, Λ : Cosmological constant

(4) $0 < m \equiv Rf''(R)/f'(R) < 1$ ⟵ **Stability of the late-time de Sitter point**

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

Cf. For general relativity, $m = 0$.

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]

(5) **Constraints from the violation of the equivalence principle**

$M = M(R)$ ⟵ Scale-dependence : **“Chameleon mechanism”**

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

(6) **Solar-system constraints**

[Chiba, Phys. Lett. B 575, 1 (2003)]

[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

< Flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time >

$$ds^2 = -dt^2 + a^2(t)d\boldsymbol{x}^2$$

$a(t)$: Scale factor

→ Gravitational field equations in the FLRW background:

$$3FH^2 = \kappa^2\rho_{\mathrm{M}} + \frac{1}{2}(FR - f) - 3H\dot{F}$$

$$-2F\dot{H} = \kappa^2(\rho_{\mathrm{M}} + P_{\mathrm{M}}) + \ddot{F} - H\dot{F}$$

$\dot{} = \partial/\partial t$

$H = \dot{a}/a$: Hubble parameter

ρ_{M} and P_{M} : Energy density and pressure of all perfect fluids of matter, respectively.

< Analysis method >

[Hu and Sawicki, Phys. Rev. D 76, 064004 (2007)]

$$H^2 - (F-1)\left(H\frac{dH}{d\ln a} + H^2\right) + \frac{1}{6}(f - R) + H^2F'\frac{dR}{d\ln a} = \frac{\kappa^2\rho_{\mathrm{M}}}{3} \quad (1)$$

- Ricci scalar:

$$R = 12H^2 + 6H\frac{dH}{d\ln a} \quad (2)$$

$'$ (prime): Derivative with respect to R

→ We solve Equations (1) and (2) by introducing the following variables:

$$\boxed{y_H \equiv \frac{\rho_{\mathrm{DE}}}{\rho_{\mathrm{m}}^{(0)}} = \frac{H^2}{\bar{m}^2} - a^{-3} - \chi a^{-4}}$$

$$\bar{m}^2 \equiv \frac{\kappa^2 \rho_{\mathrm{m}}^{(0)}}{3}$$

$$\chi \equiv \frac{\rho_{\mathrm{r}}^{(0)}}{\rho_{\mathrm{m}}^{(0)}} \simeq 3.1 \times 10^{-4}$$

$$\boxed{y_R = \frac{R}{\bar{m}^2} - 3a^{-3}}$$

- '(0)'denotes the present values.

ρ_{DE} : Energy density of dark energy

ρ_{m} : Energy density of non-relativistic matter (cold dark matter and baryon)

ρ_{r} : Energy density of radiation

⇨

$$\frac{dy_H}{d\ln a} = \frac{y_R}{3} - 4y_H \tag{3}$$

$$\frac{dy_R}{d\ln a} = 9a^{-3} - \frac{1}{y_H + a^{-3} + \chi a^{-4}} \frac{1}{\bar{m}^2 F'} \times \left[y_H - (F-1)\left(\frac{1}{6}y_R - y_H - \frac{1}{2}a^{-3} - \chi a^{-4}\right) + \frac{1}{6}\frac{f-R}{\bar{m}^2}\right] \tag{4}$$

⟶ Combining Equations (3) and (4), we obtain

$$\boxed{\frac{d^2 y_H}{d\left(\ln a\right)^2} + J_1 \frac{d y_H}{d \ln a} + J_2 y_H + J_3 = 0}$$

: Equation for y_H

$$J_1 = 4 + \frac{1}{y_H + a^{-3} + \chi a^{-4}} \frac{1-F}{6\bar{m}^2 F'}$$

$$J_2 = \frac{1}{y_H + a^{-3} + \chi a^{-4}} \frac{2-F}{3\bar{m}^2 F'}$$

$$J_3 = -3a^{-3} - \frac{\left(1-F\right)\left(a^{-3} + 2\chi a^{-4}\right) + \left(R-f\right) / \left(3\bar{m}^2\right)}{y_H + a^{-3} + \chi a^{-4}} \frac{1}{6\bar{m}^2 F'}$$

< Equation of state for (the component corresponding to) dark energy >

$$\boxed{w_{\mathrm{DE}} \equiv \frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}}}$$

$$\rho_{\mathrm{DE}} = \frac{1}{\kappa^2}\left[\frac{1}{2}\left(FR - f\right) - 3H\dot{F} + 3\left(1 - F\right)H^2\right]$$

$$P_{\mathrm{DE}} = \frac{1}{\kappa^2}\left[-\frac{1}{2}\left(FR - f\right) + \ddot{F} + 2H\dot{F} - \left(1 - F\right)\left(2\dot{H} + 3H^2\right)\right]$$

< Continuity equation for dark energy >

$$\dot{\rho}_{\mathrm{DE}} + 3H\left(1 + w_{\mathrm{DE}}\right)\rho_{\mathrm{DE}} = 0$$

$$\Rightarrow \quad \boxed{w_{\mathrm{DE}} = -1 - \frac{1}{3}\frac{1}{y_H}\frac{dy_H}{d\ln a}}$$

⇨ $$\frac{1}{e}\partial_\mu\left(eS_A{}^{\mu\nu}\right)F' - e_A^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}F' + S_A{}^{\mu\nu}\partial_\mu\left(T\right)F'' + \frac{1}{4}e_A^\nu F = 0$$

: Gravitational field equation

∗ A prime denotes a derivative with respect to T .

[Bengochea and Ferraro, Phys. Rev. D <u>79</u>, 124019 (2009)]

- We assume the flat FLRW space-time with the metric,

$$ds^2 = dt^2 - a^2(t)d\boldsymbol{x}^2 \quad \Rightarrow \quad \boxed{T = -6H^2}$$

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2),$$
$$e_\mu^A = (1, a, a, a)$$

→ Modified Friedmann equations in the flat FLRW background:

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\mathrm{M}} + \rho_{\mathrm{DE}}\right)$$

$$\left(H^2\right)' = -8\pi G\left(\rho_{\mathrm{M}} + P_{\mathrm{M}} + \rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right)$$

$$\rho_{\mathrm{DE}} = \frac{1}{16\pi G}\left(-f + 2Tf_T\right)$$

$$P_{\mathrm{DE}} = \frac{1}{16\pi G}\frac{f - Tf_T + 2T^2 f_{TT}}{1 + f_T + 2Tf_{TT}}$$

$$f_T \equiv df(T)/dT$$

$$f_{TT} \equiv d^2f(T)/dT^2$$

∗ A prime denotes a derivative with respect to $\ln a$.

⇨ $$\boxed{w_{\rm DE} \equiv \frac{P_{\rm DE}}{\rho_{\rm DE}} = -1 + \frac{T'}{3T}\frac{f_T + 2Tf_{TT}}{f/T - 2f_T} = -\frac{f/T - f_T + 2Tf_{TT}}{\left(1 + f_T + 2Tf_{TT}\right)\left(f/T - 2f_T\right)}}$$

We consider only non-relativistic matter (cold dark matter and baryon) with $\rho_{\rm M} = \rho_{\rm m}$ and $P_{\rm M} = P_{\rm m} = 0$.

→ Continuity equation: $$\frac{d\rho_{\rm DE}}{dN} \equiv \rho'_{\rm DE} = -3\left(1 + w_{\rm DE}\right)\rho_{\rm DE}$$

$$N \equiv \ln a$$

- We define a dimensionless variable : $$\boxed{y_H \equiv \frac{H^2}{\bar{m}^2} - a^{-3} = \frac{\rho_{\rm DE}}{\rho_{\rm m}^{(0)}}}$$

$$\bar{m}^2 \equiv \frac{8\pi G\rho_{\rm m}^{(0)}}{3}$$

$$\boxed{y'_H = -3\left(1 + w_{\rm DE}\right)y_H}$$: Evolution equation of the universe

< Gravitational field equation >

$$0 = \frac{1}{2} g_{\mu\nu} \left[R\left(1 + f(\eta) - \xi\right) - \partial_\rho \xi \partial^\rho \eta - 2\Lambda \right] - R_{\mu\nu}\left(1 + f(\eta) - \xi\right)$$
$$+ \frac{1}{2}\left(\partial_\mu \xi \partial_\nu \eta + \partial_\mu \eta \partial_\nu \xi\right) - \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right)\left(f(\eta) - \xi\right) + \kappa^2 T_{\mathrm{matter}\,\mu\nu}$$

$$T_{\mathrm{matter}\,\mu\nu} \equiv -\left(2/\sqrt{-g}\right)\left(\delta\sqrt{-g}\mathcal{L}_{\mathrm{matter}}/\delta g^{\mu\nu}\right)$$

: Energy-momentum tensor of matter

- The variation of the action with respect to η gives

$$0 = \Box\xi + f'(\eta) R$$

$'$ (prime) : Derivative with respect to η

< Flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric >

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} \left(dx^i\right)^2$$

$a(t)$: Scale factor

- We consider the case in which the scalar fields η and ξ only depend on time.

→ Gravitational field equations in the FLRW background:

$$0 = -3H^2\left(1 + f(\eta) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\eta} - 3H\left(f'(\eta)\dot{\eta} - \dot{\xi}\right) + \Lambda + \kappa^2\rho_{\mathrm{m}}$$

$$0 = \left(2\dot{H} + 3H^2\right)\left(1 + f(\eta) - \xi\right) + \frac{1}{2}\dot{\xi}\dot{\eta} + \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)\left(f(\eta) - \xi\right) - \Lambda + \kappa^2 P_{\mathrm{m}}$$

$\dot{} = \partial/\partial t$ $\quad$ $H = \dot{a}/a$: Hubble parameter

ρ_{m} and P_{m} : Energy density and pressure of matter, respectively.

→ For a perfect fluid of matter: $T_{\mathrm{matter}\,00} = \rho_{\mathrm{m}}$

$T_{\mathrm{matter}\,ij} = P_{\mathrm{m}}\delta_{ij}$

< Equations of motion for η and ξ >

$$0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2$$

$$0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right) f'(\eta)$$

$$R = 6\dot{H} + 12H^2$$

A. Finite-time future singularities

→ In the flat FLRW space-time, we analyze an asymptotic solution of the gravitational field equations in the limit of the time t_{s} when the finite-time future singularities appear.

- We consider the case in which the Hubble parameter is expressed as

$$\boxed{H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}}$$

h_{s} : Positive constant

q : Non-zero constant larger than -1 $(q > -1, q \neq 0)$

We only consider the period $0 < t < t_{\mathrm{s}}$.

- When $t \longrightarrow t_{\mathrm{s}}$, $R = 6\dot{H} + 12H^2 \longrightarrow \infty$

Scale factor

$$\boxed{a \sim a_{\mathrm{s}} \exp\left[\frac{h_{\mathrm{s}}}{q-1}(t_{\mathrm{s}} - t)^{-(q-1)}\right]}$$

a_{s} : Constant

- By using $\ddot{\eta} + 3H\dot{\eta} = a^{-3}d\left(a^3\dot{\eta}\right)/dt$ and $0 = \ddot{\eta} + 3H\dot{\eta} + 6\dot{H} + 12H^2$,

$$\boxed{\eta = -\int^t \frac{1}{a^3}\left(\int^{\bar{t}} Ra^3 d\bar{t}\right) dt}$$

η_c : Integration constant

- We take a form of $f(\eta)$ as $\boxed{f(\eta) = f_\mathrm{s}\eta^\sigma}$.

$f_\mathrm{s}(\neq 0)$, $\sigma(\neq 0)$

: Non-zero constants

- By using $\ddot{\xi} + 3H\dot{\xi} = a^{-3}d\left(a^3\dot{\xi}\right)/dt$ and $0 = \ddot{\xi} + 3H\dot{\xi} - \left(6\dot{H} + 12H^2\right)f'(\eta)$,

$$\boxed{\xi = \int^t \frac{1}{a^3}\left(\int^{\bar{t}} \frac{df(\eta)}{d\eta} Ra^3 d\bar{t}\right) dt}$$

ξ_c : Integration constant

⇨ There are three cases.

(i) $[q > 1,\ \sigma > 0]$: $\eta \propto (t_\mathrm{s} - t)^{-(q-1)}$, $\xi \propto (t_\mathrm{s} - t)^{-(q-1)\sigma}$

(ii) $[q > 1,\ \sigma < 0]$: $\eta \propto (t_\mathrm{s} - t)^{-(q-1)}$, $\xi \sim \xi_\mathrm{c}$

(iii) $[-1 < q < 0\,,\ 0 < q < 1]$: $\eta \sim \eta_\mathrm{c}$, $\xi \sim \xi_\mathrm{c}$

< Other model >

- **Appleby-Battye model** **[Appleby and Battye, Phys. Lett. B 654, 7 (2007)]**

$$f_{\mathrm{AB}}(R) = \frac{R}{2} + \frac{1}{2b_1} \log\left[\cosh(b_1 R) - \tanh(b_2)\sinh(b_1 R)\right]$$

$$b_1(> 0), \quad b_2$$

: Constant parameters

Future crossing of the phantom divide

(i) Hu-Sawicki model

[Hu and Sawicki, Phys. Rev. D 76, 064004 (2007)]

$$f_{\mathrm{HS}} = R - \frac{c_1 R_{\mathrm{HS}} \left(R/R_{\mathrm{HS}}\right)^p}{c_2 \left(R/R_{\mathrm{HS}}\right)^p + 1}$$

$$\boxed{\begin{array}{l} p = 1 \\ c_1 = 1\,,\ c_2 = 1 \end{array}}$$

(ii) Starobinsky's model

[Starobinsky, JETP Lett. 86, 157 (2007)]

$$f_{\mathrm{S}} = R + \lambda R_{\mathrm{S}} \left[\left(1 + \frac{R^2}{R_{\mathrm{S}}^2} \right)^{-n} - 1 \right]$$

$$\boxed{\begin{array}{l} n = 2 \\ \lambda = 1.5 \end{array}}$$

(iii) Tsujikawa's model

[Tsujikawa, Phys. Rev. D 77, 023507 (2008)]

$$f_{\mathrm{T}} = R - \mu R_{\mathrm{T}} \tanh \left(\frac{R}{R_{\mathrm{T}}} \right)$$

$$\boxed{\mu = 1}$$

(iv) Exponential gravity model

[Cognola, Elizalde, Nojiri, Odintsov, Sebastiani and Zerbini, Phys. Rev. D 77, 046009 (2008)]
[Linder, Phys. Rev. D 80, 123528 (2009)]

$$f_{\mathrm{E}} = R - \beta R_{\mathrm{E}} \left(1 - e^{-R/R_{\mathrm{E}}} \right)$$

$$\boxed{\beta = 1.8}$$

→ We examine the behavior of each term of the gravitational field equations in the limit $t \to t_\mathrm{s}$, in particular that of the leading terms, and study the condition that an asymptotic solution can be obtained.

- For case (ii) $[q > 1,\ \sigma < 0]$, $\boxed{\xi_\mathrm{c} = 1}$
- For case (iii) $[-1 < q < 0\,,\ 0 < q < 1]$,

$$\boxed{f_\mathrm{s}\eta_\mathrm{c}^{\sigma-1}\left(6\sigma-\eta_\mathrm{c}\right)+\xi_\mathrm{c}-1=0}$$

⇒ the leading term vanishes in both gravitational field equations.

$$\boxed{H \sim \frac{h_\mathrm{s}}{\left(t_\mathrm{s}-t\right)^{q}}}$$

⇒ Thus, the expression of the Hubble parameter can be a leading-order solution in terms of $\left(t_\mathrm{s}-t\right)$ for the gravitational field equations in the flat FLRW space-time.

⇓

This implies that there can exist the finite-time future singularities in non-local gravity.

B. Relations between the model parameters and the property of the finite-time future singularities

- $\boxed{f(\eta) = f_{\mathrm{s}} \eta^{\sigma}}$ ⟶ f_{s} and σ characterize the theory of non-local gravity.
- $\boxed{H \sim \frac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^{q}}}$ ⟶ h_{s} , t_{s} and q specify the property of the finite-time future singularity.
- η_{c} and ξ_{c} determine a leading-order solution in terms of $(t_{\mathrm{s}} - t)$ for the gravitational field equations in the flat FLRW space-time.
- When $t \longrightarrow t_{\mathrm{s}}$,

for $q > 1$, $\quad a \longrightarrow \infty$

for $-1 < q < 0$ and $0 < q < 1$, $\quad a \longrightarrow a_{\mathrm{s}}$

for $q > 0$, $\quad H \longrightarrow \infty$, $\rho_{\mathrm{eff}} = 3H^2/\kappa^2 \longrightarrow \infty$

for $-1 < q < 0$, H asymptotically becomes finite and also ρ_{eff} asymptotically approaches a finite constant value ρ_{s} .

for $q > -1$, $\quad \dot{H} \sim q h_{\mathrm{s}} (t_{\mathrm{s}} - t)^{-(q+1)} \rightarrow \infty$, $P_{\mathrm{eff}} = -\left(2\dot{H} + 3H^2\right)/\kappa^2 \rightarrow \infty$

B. Estimation of the current value of the effective equation of state parameter for non-local gravity

[Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011)]

- The limit on a constant equation of state for dark energy in a flat universe has been estimated as

$$\boxed{w_{\mathrm{DE}} = -1.10 \pm 0.14 \, (68\% \, \mathrm{CL})}$$

by combining the data of Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations with the latest distance measurements from the baryon acoustic oscillations (BAO) in the distribution of galaxies and the Hubble constant measurement.

- For a time-dependent equation of state for dark energy, by using a linear form $w_{\mathrm{DE}}(a) = w_{\mathrm{DE}\,0} + w_{\mathrm{DE}\,a}\,(1-a)\,,$ constraints on $w_{\mathrm{DE}\,0}$ and $w_{\mathrm{DE}\,a}$ have been found as

$w_{\mathrm{DE}\,0}$: Current value of w_{DE}

$w_{\mathrm{DE}\,a}$: Derivative of w_{DE}

$$\boxed{\begin{aligned} w_{\mathrm{DE}\,0} &= -0.93 \pm 0.13\,, \\ w_{\mathrm{DE}\,a} &= -0.41^{+0.72}_{-0.71} \, (68\% \, \mathrm{CL}) \end{aligned}}$$

from the combination of the WMAP data with the BAO data, the Hubble constant measurement and the high-redshift SNe Ia data.

IV. Effective equation of state for the universe and phantom-divide crossing

A. Cosmological evolution of the effective equation of state for the universe

- The effective equation of state for the universe :

$$\boxed{w_{\mathrm{eff}} \equiv \frac{P_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = -1 - \frac{2\dot{H}}{3H^2}}$$

$$\rho_{\mathrm{eff}} = \frac{3H^2}{\kappa^2} \,, \quad P_{\mathrm{eff}} = -\frac{2\dot{H} + 3H^2}{\kappa^2}$$

$\dot{H} < 0$: **The non-phantom (quintessence) phase**

$$\longrightarrow \quad w_{\mathrm{eff}} > -1$$

$$\dot{H} = 0 \quad \longrightarrow \quad w_{\mathrm{eff}} = -1$$

Phantom crossing

$\dot{H} > 0$: **The phantom phase**

$$\longrightarrow \quad w_{\mathrm{eff}} < -1$$

(1) General relativistic approach

- **Cosmological constant**

Canonical field

- **Scalar field : X matter, Quintessence** ←

[Chiba, Sugiyama and Nakamura, Mon. Not. Roy. Astron. Soc. 289, L5 (1997)]

[Caldwell, Dave and Steinhardt, Phys. Rev. Lett. 80, 1582 (1998)]

Cf. Pioneering work: [Fujii, Phys. Rev. D 26, 2580 (1982)]

Phantom ← Wrong sign kinetic term

[Caldwell, Phys. Lett. B 545, 23 (2002)]

K-essence ← Non canonical kinetic term

[Chiba, Okabe and Yamaguchi, Phys. Rev. D 62, 023511 (2000)]

[Armendariz-Picon, Mukhanov and Steinhardt, Phys. Rev. Lett. 85, 4438 (2000)]

Tachyon ← String theories

[Padmanabhan, Phys. Rev. D 66, 021301 (2002)]

$A > 0$: Constant

ρ : Energy density

p : Pressure

- **Chaplygin gas** ← $p = -A/\rho$

[Kamenshchik, Moschella and Pasquier, Phys. Lett. B 511, 265 (2001)]

(2) Extension of gravitational theory

Cf. Application to inflation:
[Starobinsky, Phys. Lett. B 91, 99 (1980)]

- ***f*(*R*) gravity**

$f(R)$: Arbitrary function of the Ricci scalar R

[Capozziello, Cardone, Carloni and Troisi, Int. J. Mod. Phys. D 12, 1969 (2003)]

[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D 70, 043528 (2004)]

[Nojiri and Odintsov, Phys. Rev. D 68, 123512 (2003)]

$f_i(\phi)$: Arbitrary function of a scalar field ϕ
$(i = 1, 2)$

- **Scalar-tensor theories** $\longleftarrow f_1(\phi)R$

[Boisseau, Esposito-Farese, Polarski and Starobinsky, Phys. Rev. Lett. 85, 2236 (2000)]

[Gannouji, Polarski, Ranquet and Starobinsky, JCAP 0609, 016 (2006)]

- **Ghost condensates**

[Arkani-Hamed, Cheng, Luty and Mukohyama, JHEP 0405, 074 (2004)]

- **Higher-order curvature term**

Gauss-Bonnet term with a coupling to a scalar field: $f_2(\phi)\mathcal{G}$

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$R_{\mu\nu}$: Ricci curvature tensor

[Nojiri, Odintsov and Sasaki, Phys. Rev. D 71, 123509 (2005)]

$R_{\mu\nu\rho\sigma}$: Riemann tensor

- $f(\mathcal{G})$ **gravity** $\longleftarrow \frac{R}{2\kappa^2} + f(\mathcal{G})$

$\kappa^2 \equiv 8\pi G$

[Nojiri and Odintsov, Phys. Lett. B 631, 1 (2005)]

G : Gravitational constant

- **DGP braneworld scenario**

  [Dvali, Gabadadze and Porrati, Phys. Lett B 485, 208 (2000)]

  [Deffayet, Dvali and Gabadadze, Phys. Rev. D 65, 044023 (2002)]

- ***f(T)* gravity** : Extended teleparallel Lagrangian density described by the torsion scalar T.

  [Bengochea and Ferraro, Phys. Rev. D 79, 124019 (2009)]

  [Linder, Phys. Rev. D 81, 127301 (2010) [Erratum-ibid. D 82, 109902 (2010)]]

  - "Teleparallelism" : One could use the Weitzenböck connection, which has no curvature but torsion, rather than the curvature defined by the Levi-Civita connection.

  [Hayashi and Shirafuji, Phys. Rev. D 19, 3524 (1979) [Addendum-ibid. D 24, 3312 (1982)]]

- **Galileon gravity** **[Nicolis, Rattazzi and Trincherini, Phys. Rev. D 79, 064036 (2009)]** **Review: [Tsujikawa, Lect. Notes Phys. 800, 99 (2010)]**

  $\Box\phi(\partial^{\mu}\phi\partial_{\mu}\phi) \longleftarrow$ Longitudinal graviton (i.e. a branebending mode ϕ)

  - The equations of motion are invariant under the Galilean shift: $\partial_{\mu}\phi \rightarrow \partial_{\mu}\phi + b_{\mu}$

    ⇨ One can keep the equations of motion up to the second-order.

    ⟶ This property is welcome to avoid the appearance of an extra degree of freedom associated with ghosts.

  $\Box$: Covariant d'Alembertian

- **Non-local gravity** $\longleftarrow$ Quantum effects

  [Deser and Woodard, Phys. Rev. Lett. 99, 111301 (2007)]

< Conditions for the viability of *f*(*R*) gravity >

(1) $\boxed{f'(R) > 0}$

- **Positivity of the effective gravitational coupling**

  $G_{\mathrm{eff}} = G_0 / f'(R) > 0$ G_0 : Gravitational constant

  (The graviton is not a ghost.)

(2) $\boxed{f''(R) > 0}$ **[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]**

- **Stability condition:** $M^2 \approx 1/(3f''(R)) > 0$

  M : **Mass of a new scalar degree of freedom (called the "scalaron") in the weak-field regime.**

  (The scalaron is not a tachyon.)

(3) $\boxed{f(R) \to R - 2\Lambda \quad \text{for} \quad R \gg R_0}$

R_0 : Current curvature

Λ : Cosmological constant

- **Realization of the ΛCDM-like behavior in the large curvature regime**

  ↑ Standard cosmology [Λ + Cold dark matter (CDM)]

(4) Solar system constraints

$f(R)$ gravity ⟺ **Brans-Dicke theory with $\omega_{\mathrm{BD}} = 0$**

Equivalent

ω_{BD} : Brans-Dicke parameter

[Bertotti, Iess and Tortora, Nature 425, 374 (2003).] ⟶ Observational constraint: $|\omega_{\mathrm{BD}}| > 40000$

[Chiba, Phys. Lett. B 575, 1 (2003)]

[Erickcek, Smith and Kamionkowski, Phys. Rev. D 74, 121501 (2006)]

[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

- However, if the mass of the scalar degree of freedom M is large, namely, the scalar becomes short-ranged, it has no effect at Solar System scales.
- $\boxed{M = M(R)}$ ← Scale-dependence: "**Chameleon mechanism**"

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

⇨ The scalar degree of freedom may acquire a large effective mass at terrestrial and Solar System scales, shielding it from experiments performed there.

(5) Existence of a matter-dominated stage and that of a late-time cosmic acceleration

- **Combing local gravity constraints, it is shown that $\boxed{m \equiv Rf''(R)/f'(R)}$ has to be several orders of magnitude smaller than unity.**

- For general relativity, $m = 0$.

m quantifies the deviation from the Λ CDM model.

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

(6) Stability of the de Sitter space

$$\frac{(f_{\mathrm{d}}')^2 - 2f_{\mathrm{d}}f_{\mathrm{d}}''}{f_{\mathrm{d}}'f_{\mathrm{d}}''} > 0$$

$$f_{\mathrm{d}} = f(R_{\mathrm{d}})$$

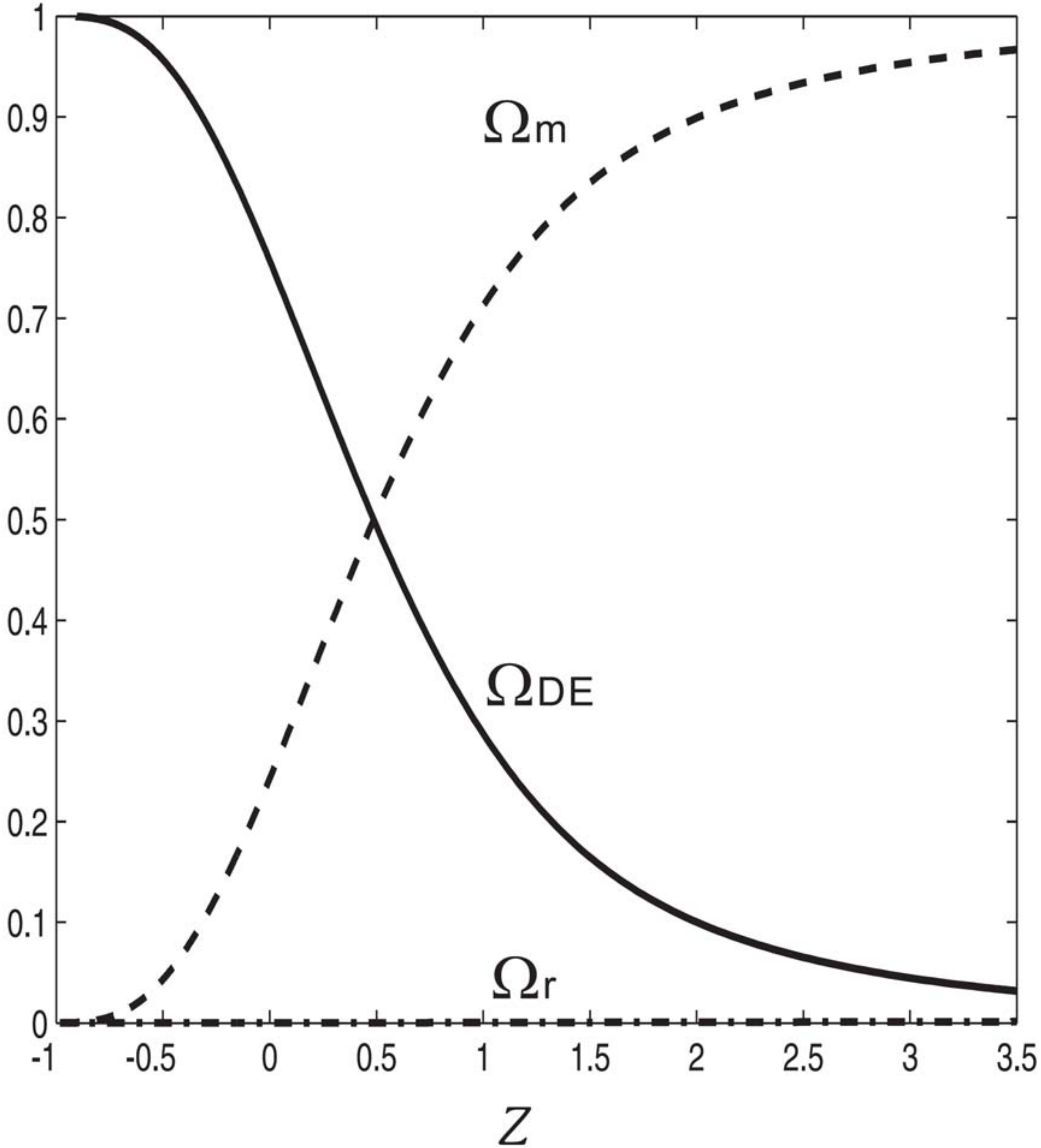
R_{d} : Constant curvature in the de Sitter space

- **Linear stability of the inhomogeneous perturbations in the de Sitter space** **[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]**

Cf. $R_{\mathrm{d}} = 2f_{\mathrm{d}}/f_{\mathrm{d}}'$ $\Longrightarrow$ $\boxed{m < 1}$

< Cosmological evolutions of $\Omega_{\rm DE}$, $\Omega_{\rm m}$ and $\Omega_{\rm r}$ in the exponential gravity model >

From [KB, Geng and Lee, JCAP 1008, 021 (2010)].



$$f_{\rm E}(R) =$$

$$R - \beta R_{\rm E}\left(1 - e^{-R/R_{\rm E}}\right)$$

$$\beta = 1.8$$

$$\beta R_{\rm E} \simeq 18 H_0^2 \Omega_{\rm m}^{(0)}$$

$$\Omega_{\rm DE} \equiv \rho_{\rm DE}/\rho_{\rm crit}^{(0)}$$

$$\Omega_{\rm m} \equiv \rho_{\rm m}/\rho_{\rm crit}^{(0)}$$

$$\Omega_{\rm r} \equiv \rho_{\rm r}/\rho_{\rm crit}^{(0)}$$

$$\rho_{\rm crit}^{(0)} = 3H_0^2/\kappa^2$$

< Conclusions of Sec. II >

- We have explicitly shown that the future crossings of the phantom divide are the generic feature in the existing viable *f*(*R*) gravity models.
- We have also illustrated that the cosmological horizon entropy oscillates with time due to the oscillatory behavior of the Hubble parameter.
- The new cosmological ingredient obtained in this study is that in the future the sign of $\dot{H}$ changes from negative to positive due to the dominance of dark energy over non-relativistic matter.

⇨ This is a common physical phenomena to the existing viable *f*(*R*) models and thus it is one of the peculiar properties of *f*(*R*) gravity models characterizing the deviation from the Λ CDM model.

< Conclusions of Sec. III >

- We have investigated the cosmological evolution in the exponential *f*(*T*) theory.

⇨ **The phase of the universe depends on the sign of the parameter p , i.e., for $p < 0 (> 0)$ the universe is always in the non-phantom (phantom) phase without the crossing of the phantom divide.**

- We have presented the logarithmic type *f*(*T*) model.

⇨ **It does not allow the crossing of the phantom divide.**

- **To realize the crossing of the phantom divide, we have constructed an *f*(*T*) theory by combining the logarithmic and exponential terms.**

⇨ **The crossing in the combined *f*(*T*) theory is from $w_{\mathrm{DE}} > -1$ to $w_{\mathrm{DE}} < -1$, which is opposite to the typical manner in *f*(*R*) gravity models.**

⇨ **This combined theory is consistent with the recent observational data of SNe Ia, BAO and CMB.**

< Conclusions of Sec. IV >

- We have explicitly shown that three types of the finite-time future singularities (Type I, II and III) can occur in non-local gravity and examined their properties.

- We have investigated the behavior of the effective equation of state for the universe when the finite-time future singularities occur.

→ Continuity equation: $\frac{d\rho_{\mathrm{DE}}}{dN} \equiv \rho'_{\mathrm{DE}} = -3\left(1+w_{\mathrm{DE}}\right)\rho_{\mathrm{DE}}$

$N \equiv \ln a$

- We define a dimensionless variable : $\boxed{y_H \equiv \frac{H^2}{\bar{m}^2} - a^{-3} = \frac{\rho_{\mathrm{DE}}}{\rho_{\mathrm{m}}^{(0)}}}$

$\bar{m}^2 \equiv \frac{8\pi G \rho_{\mathrm{m}}^{(0)}}{3}$

$\boxed{y'_H = -3\left(1+w_{\mathrm{DE}}\right) y_H}$: Evolution equation of the universe

< (a). Exponential *f*(*T*) theory >

$$\boxed{f(T) = \alpha T\left(1 - e^{pT_0/T}\right)}$$

p : Constant

$T_0 = T(z=0)$

$\Omega_{\mathrm{m}}^{(0)} \equiv \rho_{\mathrm{m}}^{(0)}/\rho_{\mathrm{crit}}^{(0)}$

$\rho_{\mathrm{crit}}^{(0)} = 3H_0^2/\left(8\pi G\right)$

$$\alpha = -\frac{1-\Omega_{\mathrm{m}}^{(0)}}{1-\left(1-2p\right)e^p}$$

- The case in which $p = 0$ corresponds to the Λ CDM model.
- This theory contains only one parameter p if the value of $\Omega_{\mathrm{m}}^{(0)}$ is given.

⇨ $\frac{1}{e}\partial_\mu \left(e S_A{}^{\mu\nu}\right) F' - e_A^\lambda T^\rho{}_{\mu\lambda} S_\rho{}^{\nu\mu} F' + S_A{}^{\mu\nu} \partial_\mu (T) F'' + \frac{1}{4} e_A^\nu F = 0$ A prime denotes a derivative with respect to T .

: Gravitational field equation **[Bengochea and Ferraro, Phys. Rev. D <u>79</u>, 124019 (2009)]**

- We assume the flat FLRW space-time with the metric,

$$ds^2 = dt^2 - a^2(t) d\boldsymbol{x}^2 \;⇨\; \boxed{T = -6H^2}$$

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2),$$
$$e_\mu^A = (1, a, a, a)$$

→ Modified Friedmann equations in the flat FLRW background:

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\mathrm{M}} + \rho_{\mathrm{DE}}\right)$$

$$\left(H^2\right)' = -8\pi G \left(\rho_{\mathrm{M}} + P_{\mathrm{M}} + \rho_{\mathrm{DE}} + P_{\mathrm{DE}}\right)$$

$$\rho_{\mathrm{DE}} = \frac{1}{16\pi G}\left(-f + 2T f_T\right)$$

$$P_{\mathrm{DE}} = \frac{1}{16\pi G} \frac{f - T f_T + 2T^2 f_{TT}}{1 + f_T + 2T f_{TT}}$$

$$f_T \equiv df(T)/dT$$

$$f_{TT} \equiv d^2 f(T)/dT^2$$

A prime denotes a derivative with respect to $\ln a$.

⇨ $$\boxed{w_{\mathrm{DE}} \equiv \frac{P_{\mathrm{DE}}}{\rho_{\mathrm{DE}}} = -1 + \frac{T'}{3T} \frac{f_T + 2T f_{TT}}{f/T - 2f_T} = -\frac{f/T - f_T + 2T f_{TT}}{\left(1 + f_T + 2T f_{TT}\right)\left(f/T - 2f_T\right)}}$$

We consider only non-relativistic matter (cold dark matter and baryon) with $\rho_{\mathrm{M}} = \rho_{\mathrm{m}}$ and $P_{\mathrm{M}} = P_{\mathrm{m}} = 0$.

→ Continuity equation: $\frac{d\rho_{\mathrm{DE}}}{dN} \equiv \rho'_{\mathrm{DE}} = -3\left(1 + w_{\mathrm{DE}}\right)\rho_{\mathrm{DE}}$

$N \equiv \ln a$

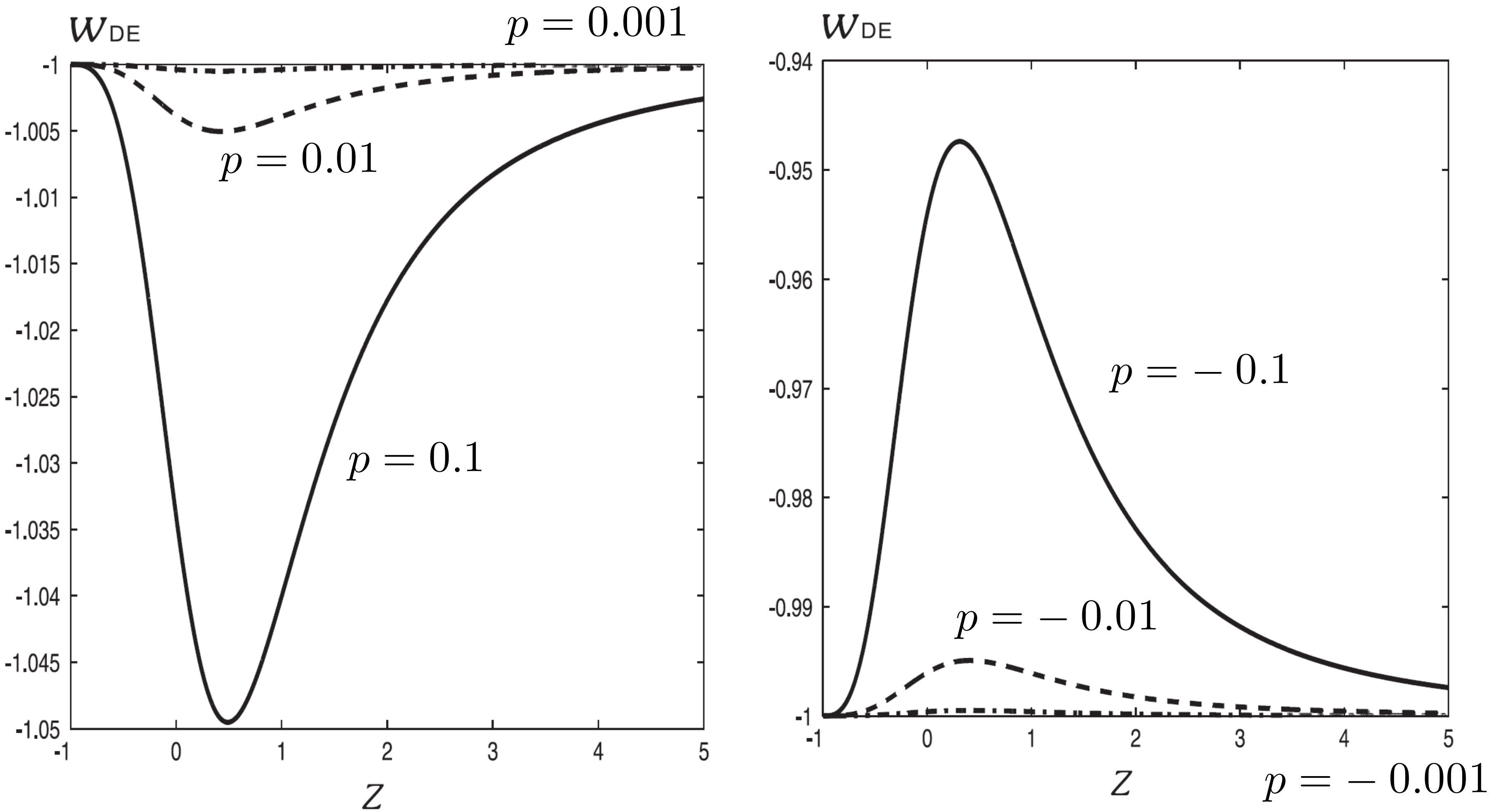
- We define a dimensionless variable : $\boxed{y_H \equiv \frac{H^2}{\bar{m}^2} - a^{-3} = \frac{\rho_{\mathrm{DE}}}{\rho_{\mathrm{m}}^{(0)}}}$

$\bar{m}^2 \equiv \frac{8\pi G \rho_{\mathrm{m}}^{(0)}}{3}$

$\boxed{y'_H = -3\left(1 + w_{\mathrm{DE}}\right) y_H}$: Evolution equation of the universe

<(a). Exponential *f*(*T*) theory >

$$f(T) = \alpha T \left(1 - e^{pT_0/T}\right)$$



$p > 0$ $p < 0$

- $|p| = 0.1$ (solid line), 0.01 (dashed line), 0.001 (dash-dotted line)
- $\Omega_{\rm m}^{(0)} = 0.26$

w_{DE} does not cross the phantom divide line $w_{\mathrm{DE}} = -1$ in the exponential *f*(*T*) theory.

- For $p > 0$, the universe always stays in the non-phantom (quintessence) phase ($w_{\mathrm{DE}} > -1$), whereas for $p < 0$ it in the phantom phase ($w_{\mathrm{DE}} < -1$).
- The larger $|p|$ is, the larger the deviation of the exponential *f*(*T*) theory from the Λ CDM model is.
- We have taken the initial conditions at $z = 0$ as $y_H(z = 0) = 2.8$.

< (b). Logarithmic *f*(*T*) theory >

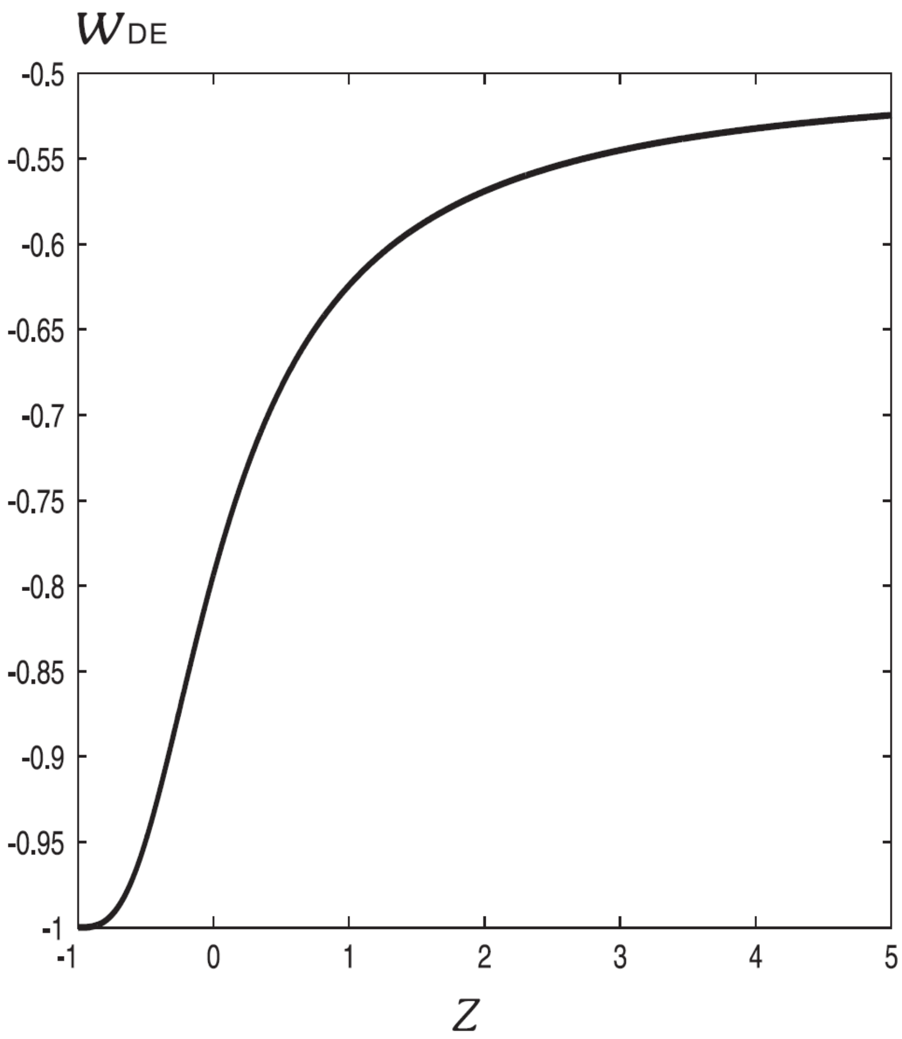
$$\boxed{f(T) = \beta T_0 \left(\frac{qT_0}{T}\right)^{-1/2} \ln \left(\frac{qT_0}{T}\right)}$$

$$\beta \equiv \frac{1-\Omega_{\mathrm{m}}^{(0)}}{2q^{-1/2}}$$

$q(>0)$

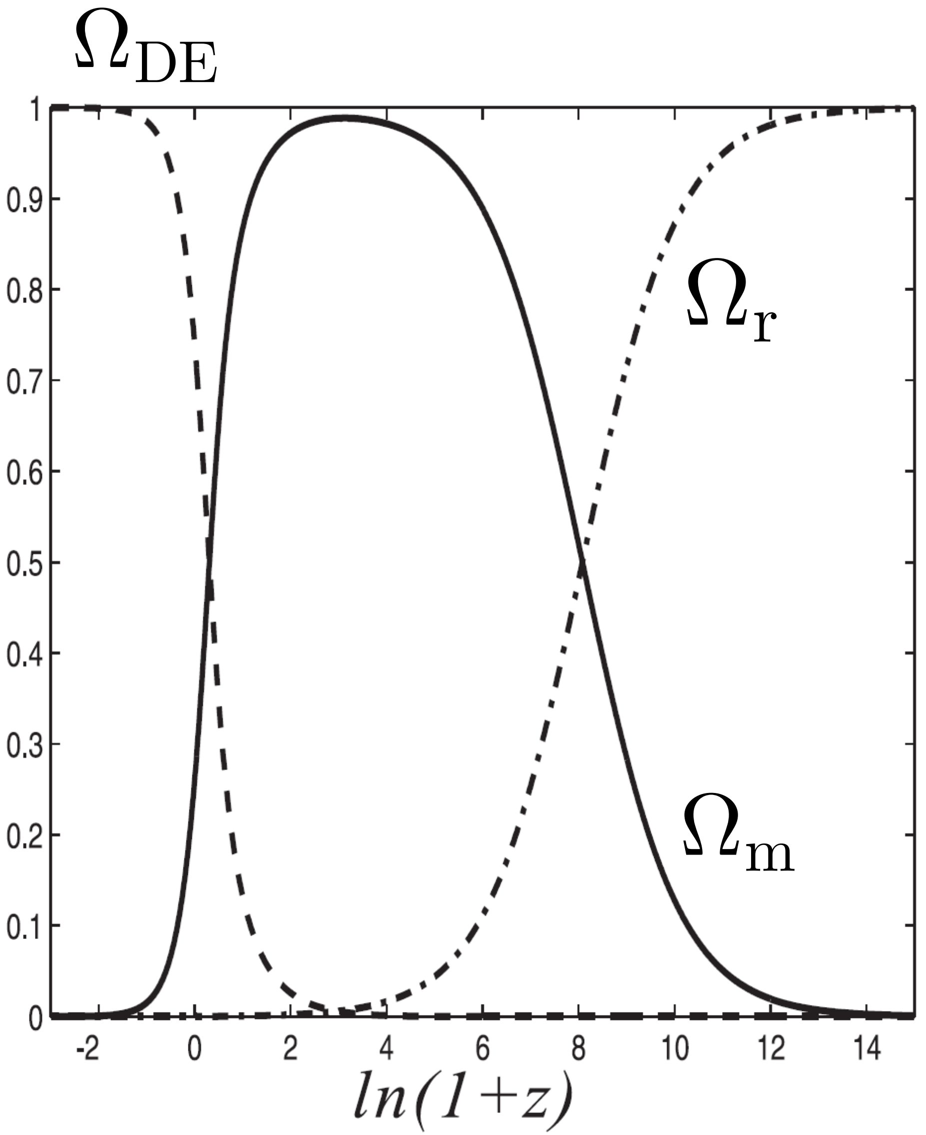
: Positive constant

- This theory contains only one parameter q if the value of $\Omega_{\mathrm{m}}^{(0)}$ is obtained.



w_{DE} does not cross the phantom divide line $w_{\mathrm{DE}} = -1$.

< Cosmological evolutions of $\Omega_{\rm DE}$, $\Omega_{\rm m}$ and $\Omega_{\rm r}$ >



$u = 1$

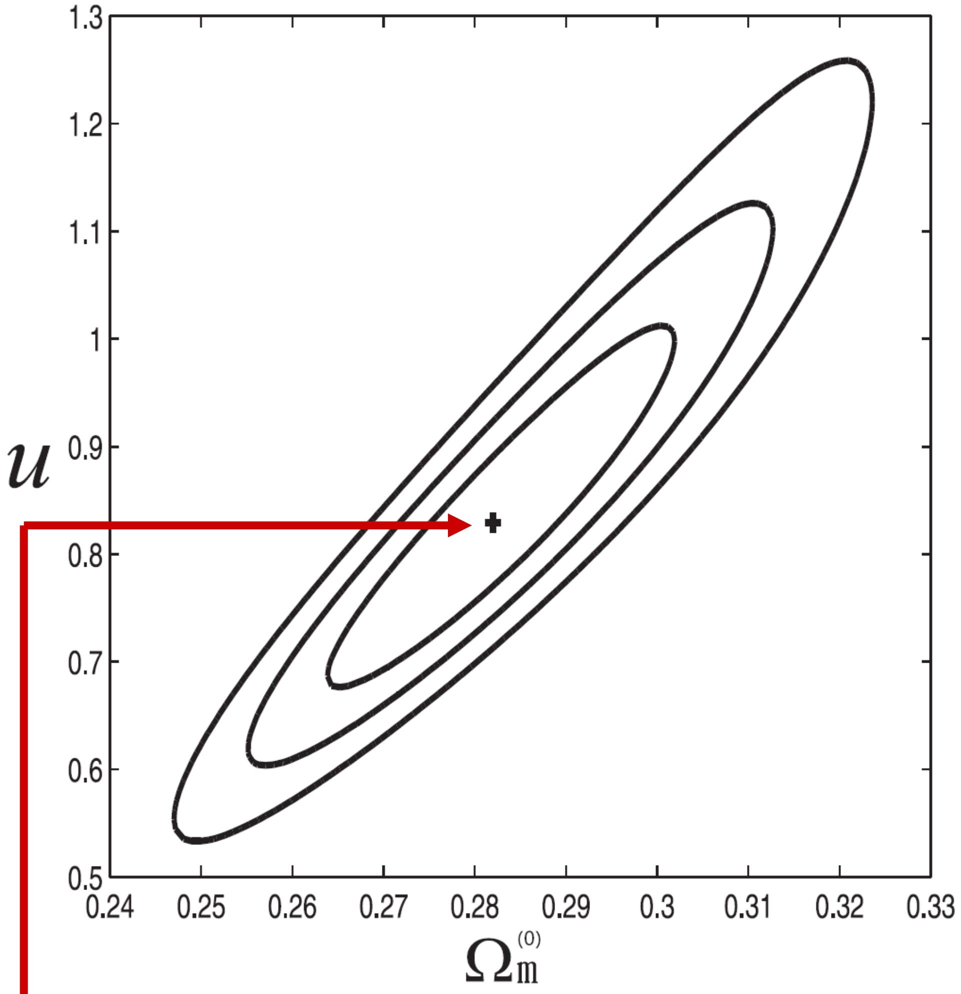
Radiation-dominated stage [$z \gtrsim 3225$]

$(\Omega_{\rm r} \gg \Omega_{\rm DE},\ \Omega_{\rm r} > \Omega_{\rm m})$

Matter-dominated stage

$(\Omega_{\rm m} > \Omega_{\rm DE},\ \Omega_{\rm m} \gg \Omega_{\rm r})$

Dark energy becomes dominant over matter ($z < 0.36$).



< The best-fit values >

Model	u	$\Omega_m^{(0)}$	h	$\chi^2_{\min}$
$f(T)$	0.829	0.282	0.691	544.56
ΛCDM		0.275	0.707	545.23

The minimum χ^2 ($\chi^2_{\min}$) of the combined *f*(*T*) theory is slightly smaller than that of the Λ CDM model.

The combined *f*(*T*) theory can fit the observational data well.

Contours of $68.27\%\ (1\sigma)$, $95.45\%\ (2\sigma)$ and $99.73\%\ (3\sigma)$ confidence levels in the $(\Omega_m^{(0)}, u)$ plane from SNe Ia, BAO and CMB data.

The plus sign depicts the best-fit point.

Backup Slides A

(4) Solar system constraints

$f(R)$ gravity ⇔ **Brans-Dicke theory with $\omega_{\mathrm{BD}} = 0$**

Equivalent

ω_{BD} : Brans-Dicke parameter

[Bertotti, Iess and Tortora, Nature 425, 374 (2003).] $\longrightarrow$ Observational constraint: $|\omega_{\mathrm{BD}}| > 40000$

[Chiba, Phys. Lett. B 575, 1 (2003)]

[Erickcek, Smith and Kamionkowski, Phys. Rev. D 74, 121501 (2006)]

[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

- However, if the mass of the scalar degree of freedom M is large, namely, the scalar becomes short-ranged, it has no effect at Solar System scales.
- $\boxed{M = M(R)}$ ← Scale-dependence: "**Chameleon mechanism**"

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

⇨ The scalar degree of freedom may acquire a large effective mass at terrestrial and Solar System scales, shielding it from experiments performed there.

(5) Existence of a matter-dominated stage and that of a late-time cosmic acceleration

- **Combing local gravity constraints, it is shown that $\boxed{m \equiv Rf''(R)/f'(R)}$ has to be several orders of magnitude smaller than unity.**

m quantifies the deviation from the Λ CDM model.

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

(6) Stability of the de Sitter space

$$\frac{(f'_{\mathrm{d}})^2 - 2f_{\mathrm{d}}f''_{\mathrm{d}}}{f'_{\mathrm{d}}f''_{\mathrm{d}}} > 0$$

$f_{\mathrm{d}} = f(R_{\mathrm{d}})$

R_{d} : Constant curvature in the de Sitter space

- **Linear stability of the inhomogeneous perturbations in the de Sitter space** **[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]**

Cf. $R_{\mathrm{d}} = 2f_{\mathrm{d}}/f'_{\mathrm{d}}$ $\Longrightarrow$ $\boxed{m < 1}$

(4) Stability of the late-time de Sitter point

$$0 < m \equiv Rf''(R)/f'(R) < 1$$

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

- For general relativity, $m = 0$.

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]

$\longrightarrow$ m quantifies the deviation from the Λ CDM model.

(5) Constraints from the violation of the equivalence principle

$$M = M(R)$$

Scale-dependence $\longleftarrow$ **"Chameleon mechanism"**

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

- If the mass of the scalar degree of freedom M is large, namely, the scalar becomes short-ranged, it has no effect at Solar System scales.

⇨ The scalar degree of freedom may acquire a large effective mass at terrestrial and Solar System scales, shielding it from experiments performed there.

(6) Solar-system constraints

[Chiba, Phys. Lett. B 575, 1 (2003)]

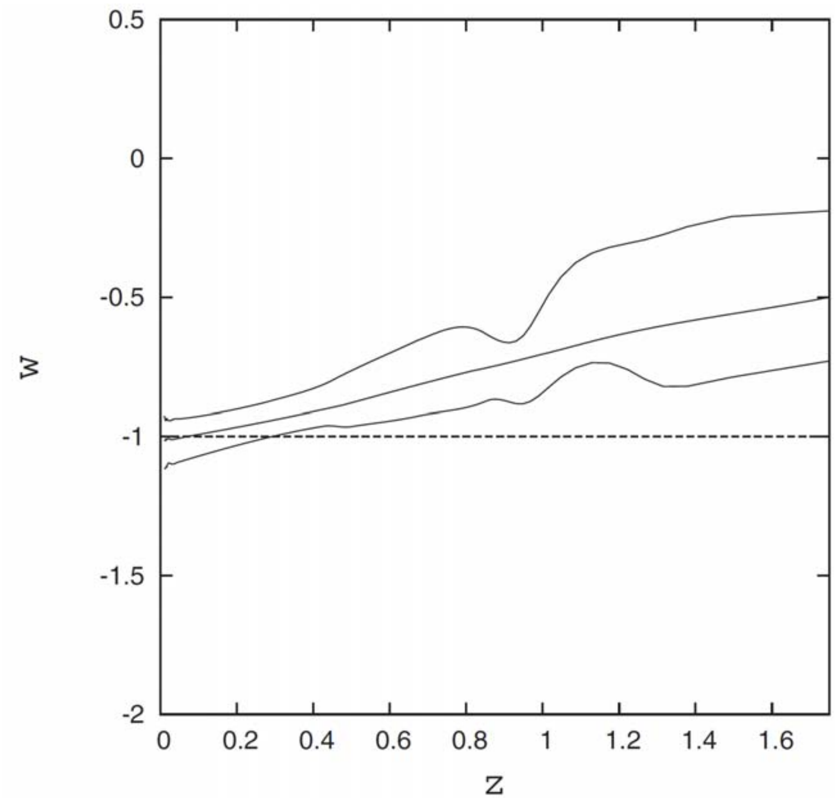
[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

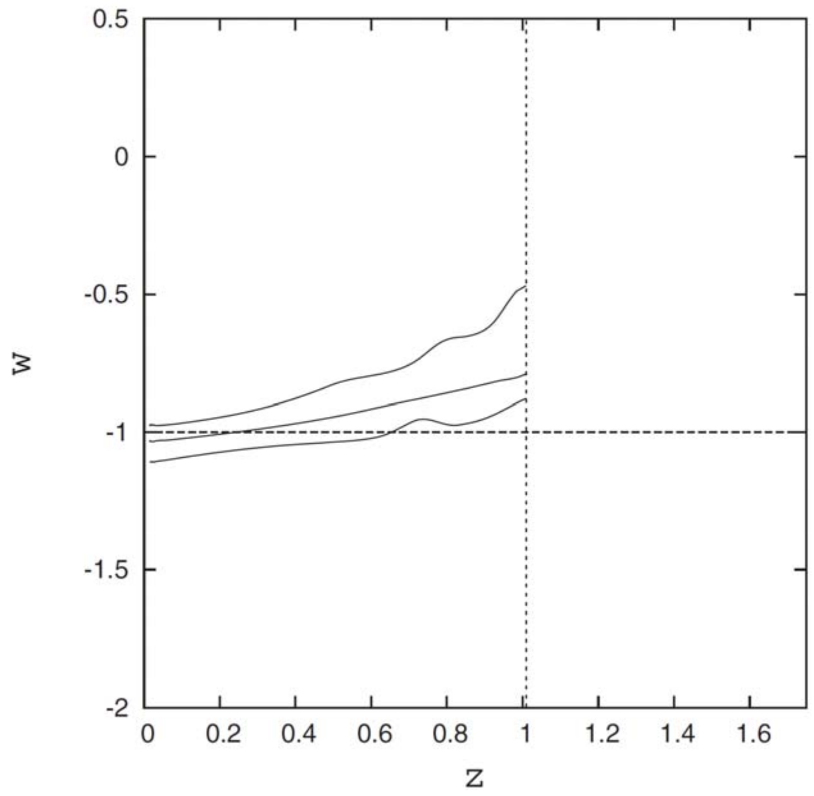
< Data fitting of $w(z)$ (2) >

$$\boxed{w(x) = \frac{(2x/3)\,\mathrm{d}\ln H/\mathrm{d}x - 1}{1 - (H_0/H)^2\Omega_{0m}x^3}}$$

From [Alam, Sahni and Starobinsky, JCAP 0702, 011 (2007)].

$$x = 1 + z$$





SN gold data set+CMB+BAO

SNLS data set+CMB+BAO

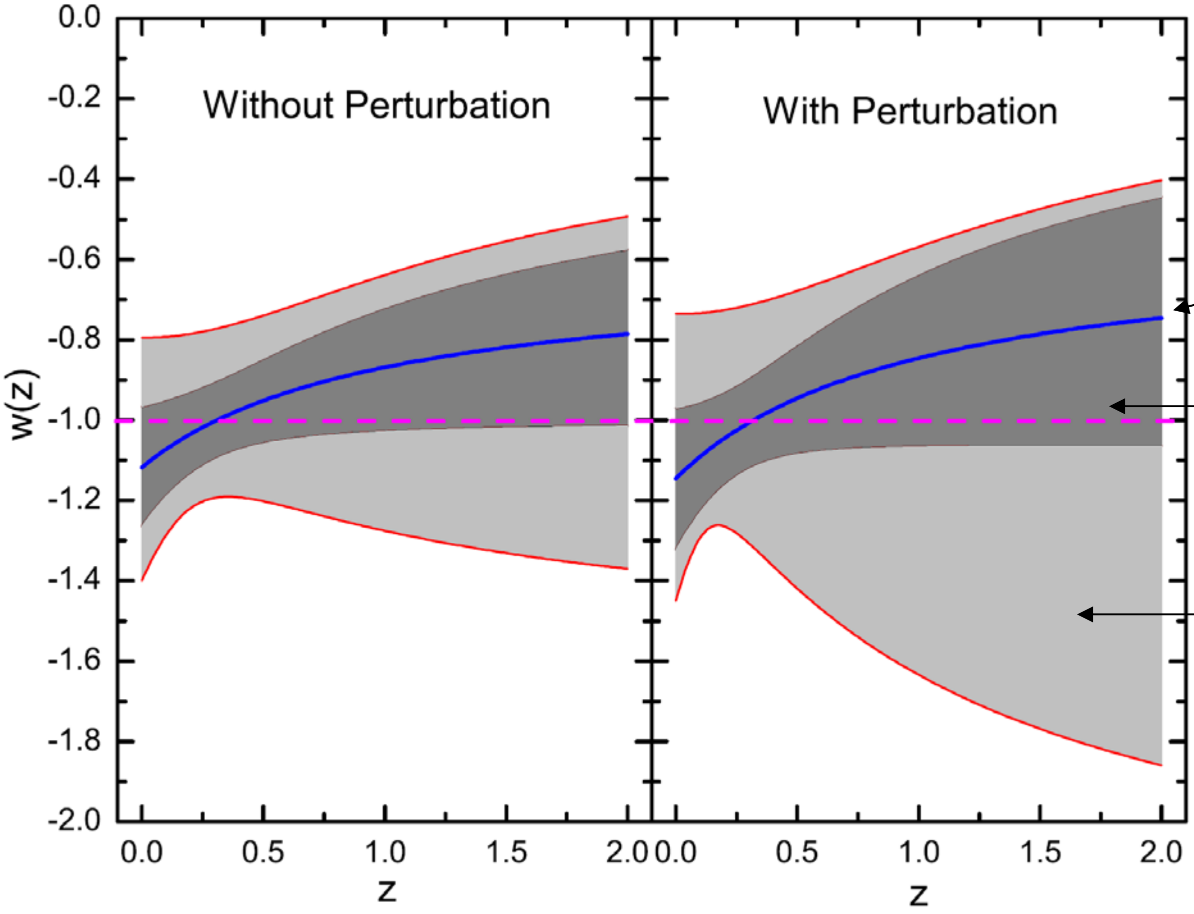
- $\Omega_{0\mathrm{m}} = 0.28 \pm 0.03$

- 2σ **confidence level.**

< Data fitting of $w(z)$ (3) >

$$\boxed{w(z) = w_0 + w_1 \frac{z}{1+z}}$$

From [Zhao, Xia, Feng and Zhang, Int. J. Mod. Phys. D 16, 1229 (2007) [arXiv:astro-ph/0603621]]



Best-fit

68% confidence level

95% confidence level

157 “gold” SN Ia data set+WMAP 3-year data+SDSS with/without dark energy perturbations.

- For most observational probes (except the SNLS data), a low $\Omega_{0\mathrm{m}}$ prior $(0.2 < \Omega_{0\mathrm{m}} < 0.25)$ leads to an increased probability (mild trend) for the crossing of the phantom divide.

$\Omega_{0\mathrm{m}}$: Current density parameter of matter

[Nesseris and L. Perivolaropoulos, JCAP 0701, 018 (2007)]

< Bekenstein-Hawking entropy on the apparent horizon in the flat FLRW background >

$$\boxed{S = \frac{\pi}{GH^2}}$$

$S = \frac{A}{4G}$: Bekenstein-Hawking entropy

$A = 4\pi\tilde{r}_A^2$: Area of the apparent horizon

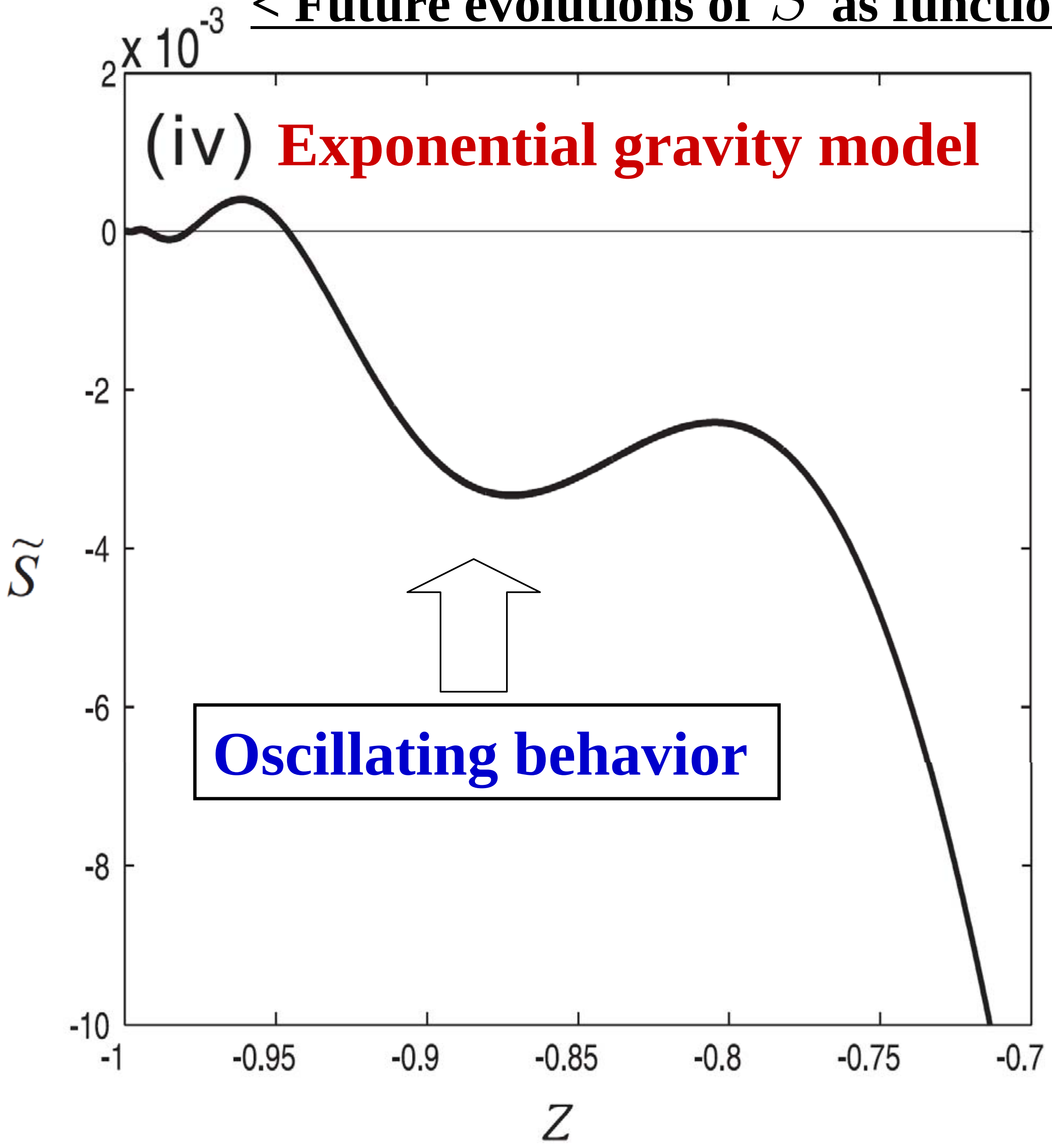
$\tilde{r} = 1/H$: Radius of the apparent horizon in the flat FLRW space-time

- It has been shown that it is possible to obtain a picture of equilibrium thermodynamics on the apparent horizon in the FLRW background for *f*(*R*) gravity due to a suitable redefinition of an energy momentum tensor of the "dark" component that respects a local energy conservation.

[KB, Geng and Tsujikawa, Phys. Lett. B 688, 101 (2010)]

⇨ In this picture, the horizon entropy is simply expressed as $S = \pi / \left(GH^2\right)$.

< Future evolutions of S as functions of z >



Exponential gravity model

Oscillating behavior

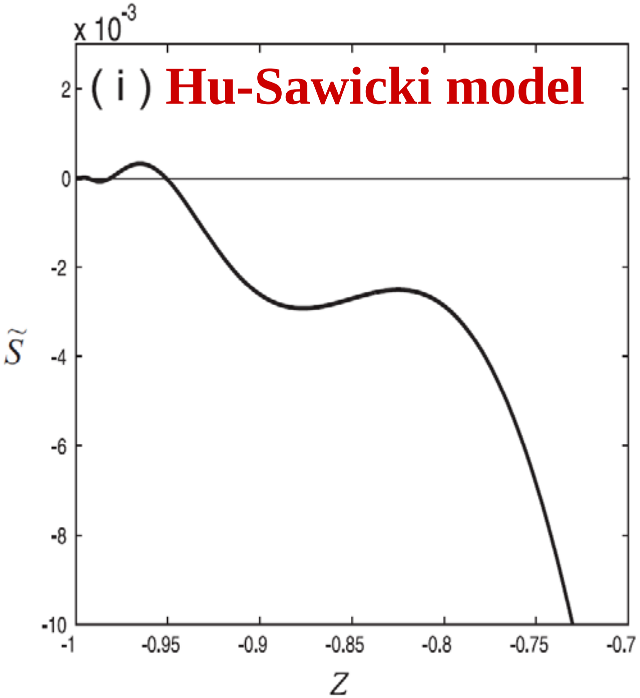
$$\boxed{\tilde{S} \equiv \bar{S} - \bar{S}_{\mathrm{f}}}$$

$$\bar{S} \equiv S/S_0$$

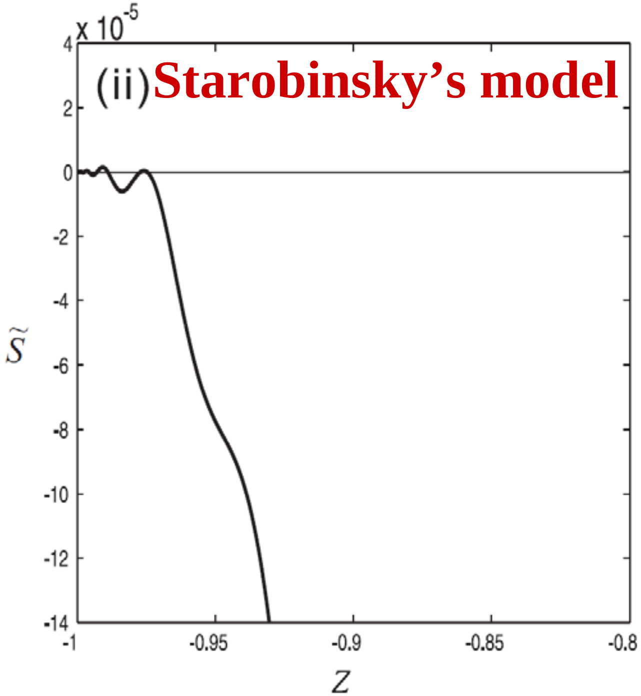
$$\bar{S}_{\mathrm{f}} \equiv \frac{S(z=-1)}{S_0}$$

$$S_0 = \pi/\left(G H_0^2\right)$$

: Present value of the horizon entropy



(i) Hu-Sawicki model

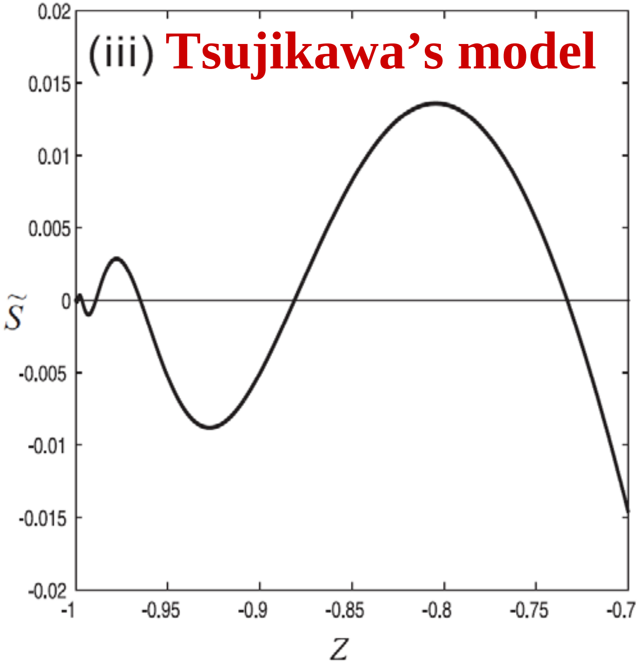


(ii)Starobinsky's model

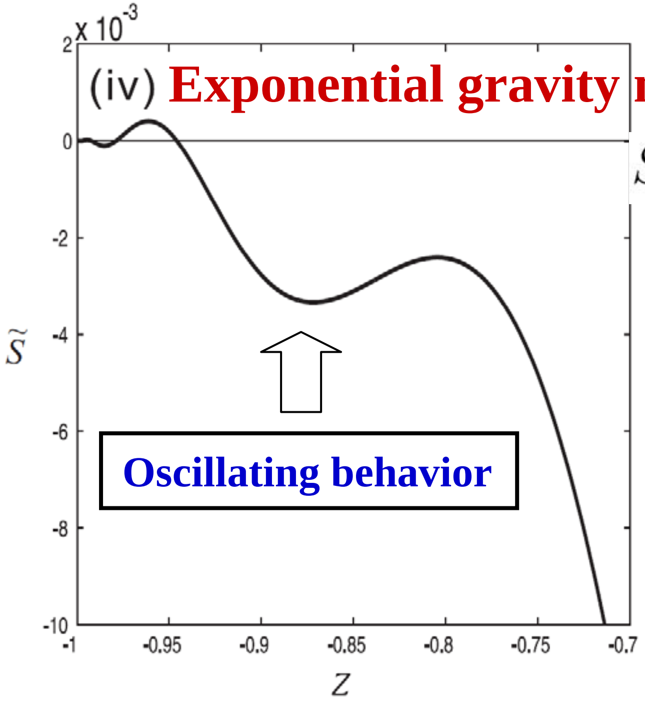
$$\boxed{\tilde{S} \equiv \bar{S} - \bar{S}_{\mathrm{f}}}$$

$$\bar{S} \equiv S/S_0$$

$$\bar{S}_{\mathrm{f}} \equiv \frac{S(z=-1)}{S_0}$$



(iii) Tsujikawa's model



(iv) Exponential gravity model

Oscillating behavior

$$S_0 = \pi / \left(G H_0^2\right)$$

: Present value of the horizon entropy

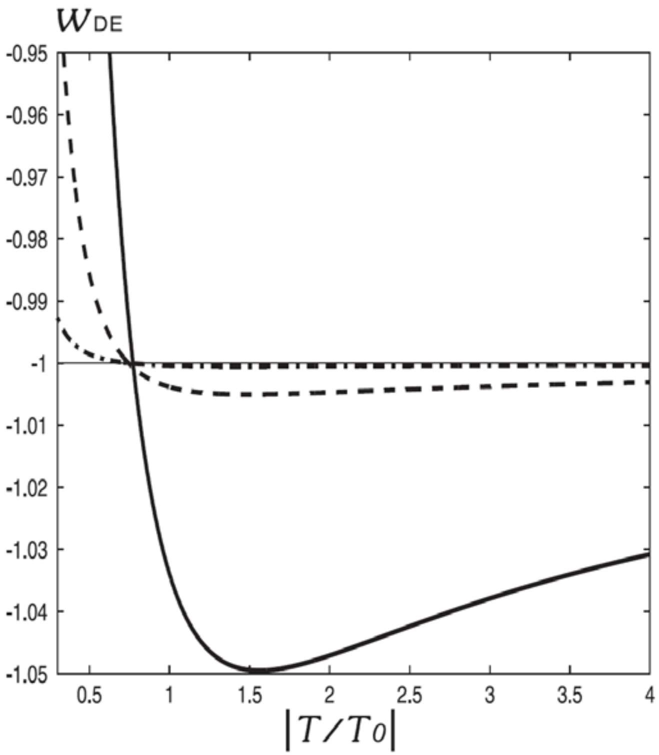
- **Since $S \propto H^{-2}$, the oscillating behavior of S comes from that of H.**

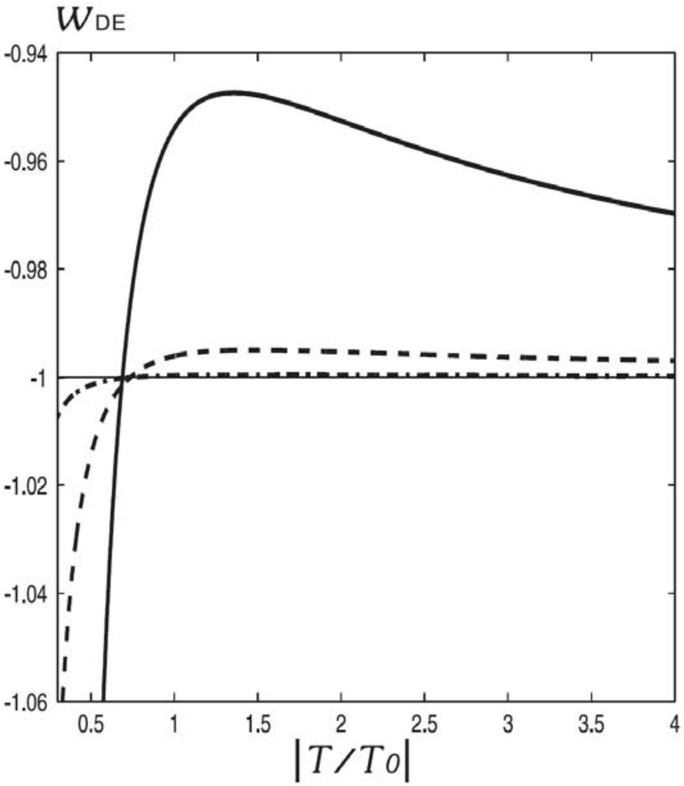
⇨ However, it should be emphasized that although S decreases in some regions, the second law of thermodynamics in *f*(*R*) gravity can be always satisfied.

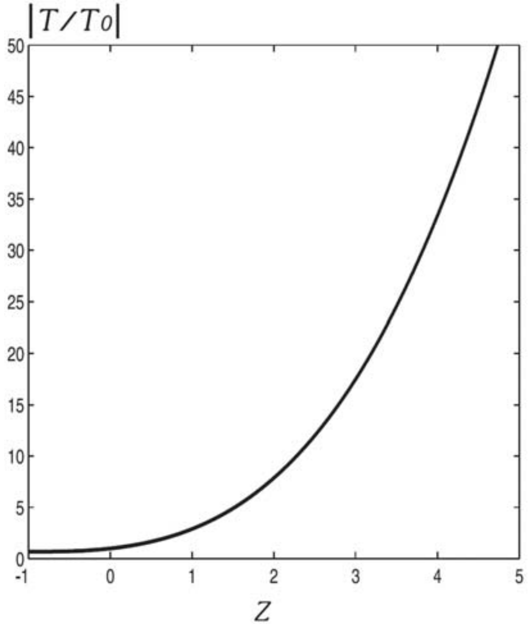
→ This is because S is the cosmological horizon entropy and it is not the total entropy of the universe including the entropy of generic matter.

Cf. It has been shown that the second law of thermodynamics can be verified in both phantom and non-phantom phases for the same temperature of the universe outside and inside the apparent horizon.

[KB and Geng, JCAP 1006, 014 (2010)]

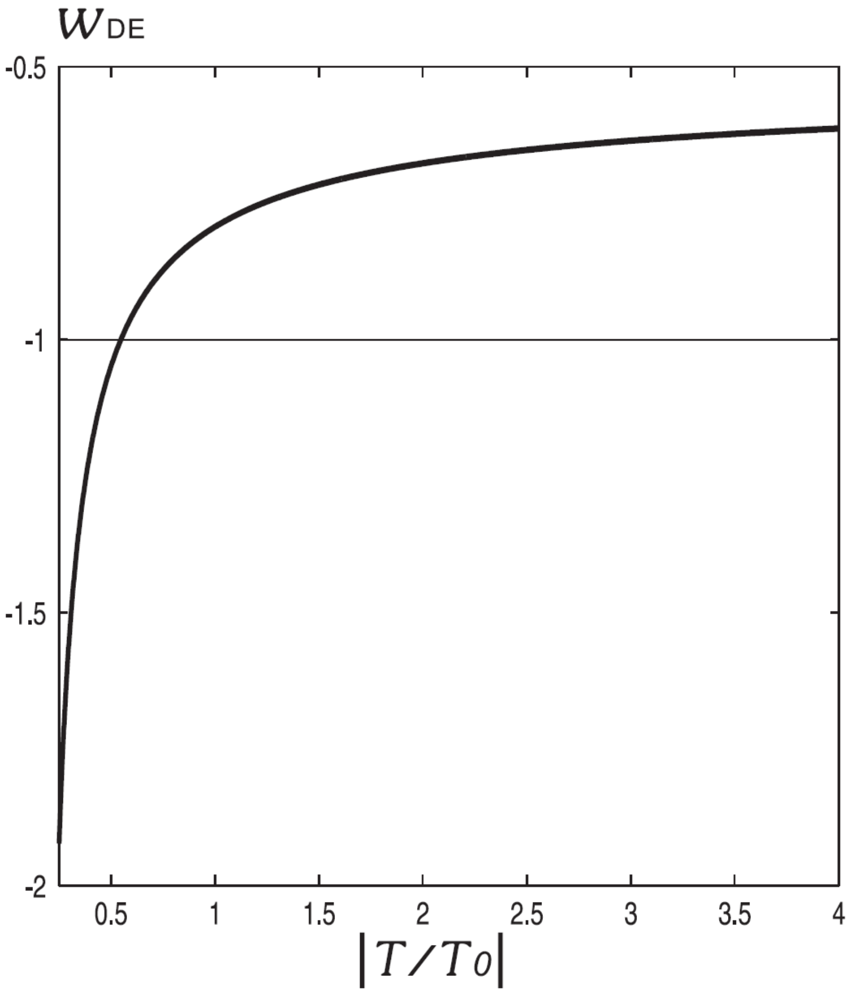






< (a). Exponential $f(T)$ theory >

$$\boxed{f(T) = \alpha T \left(1 - e^{pT_0/T}\right)}$$

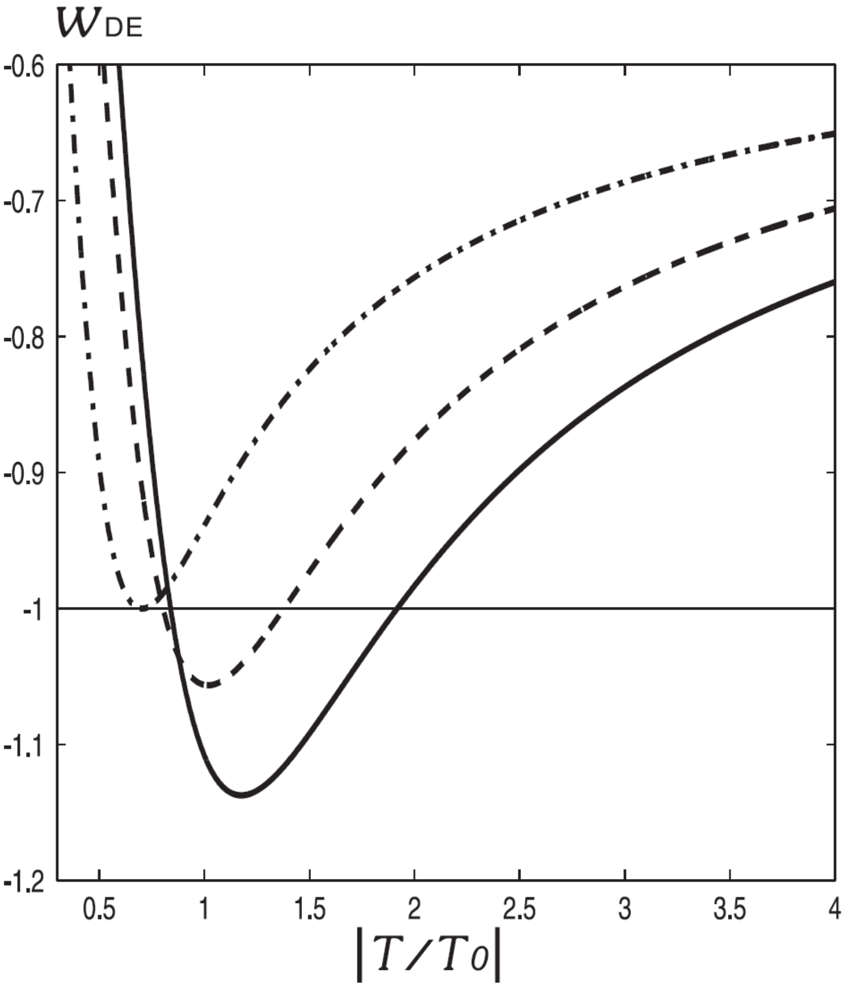


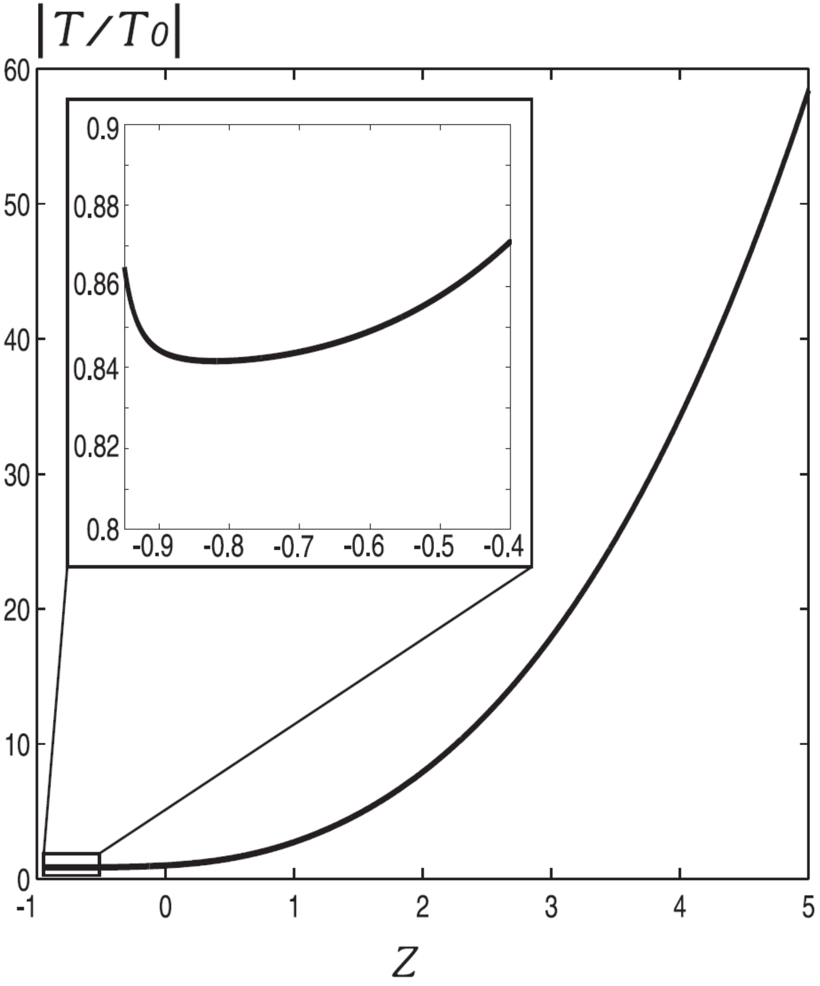
< (b). Logarithmic *f*(*T*) theory >

$$f(T) = \beta T_0 \left(\frac{qT_0}{T} \right)^{-1/2} \ln \left(\frac{qT_0}{T} \right)$$

< (c). Combined *f(T)* theory >

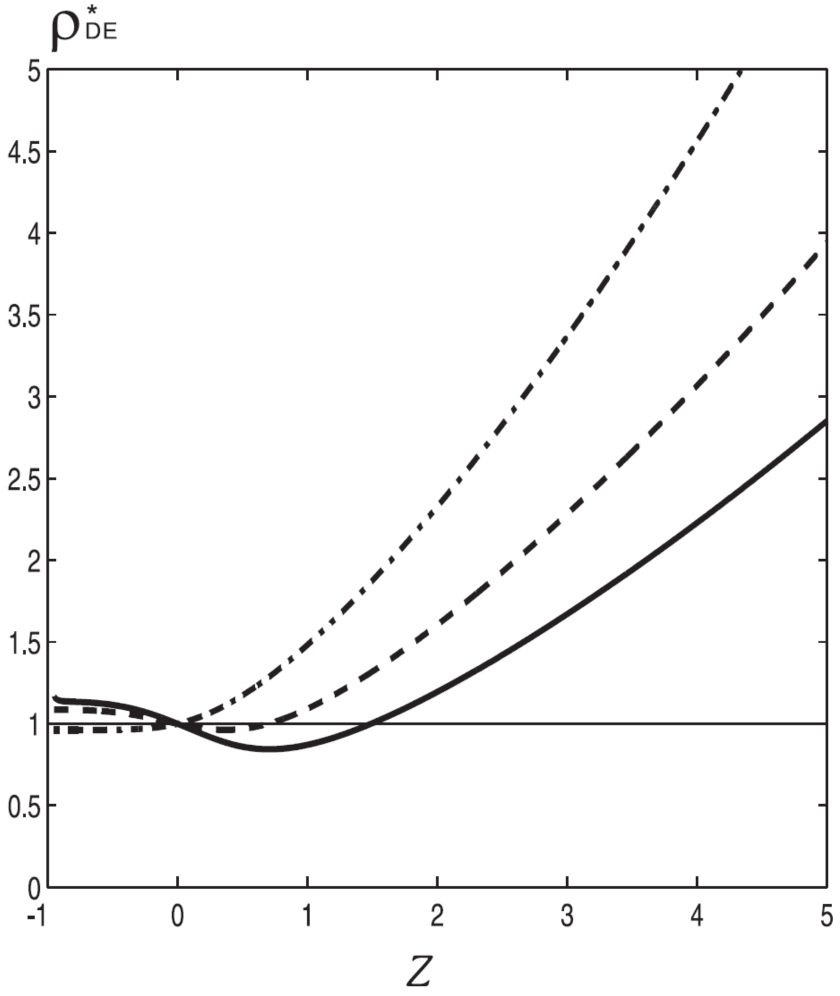
$$f(T) = \gamma \left[T_0 \left(\frac{uT_0}{T} \right)^{-1/2} \ln \left(\frac{uT_0}{T} \right) - T \left(1 - e^{uT_0/T} \right) \right]$$





< (c). Combined *f(T)* theory >

$$f(T) = \gamma \left[T_0 \left(\frac{uT_0}{T} \right)^{-1/2} \ln \left(\frac{uT_0}{T} \right) - T \left(1 - e^{uT_0/T} \right) \right]$$



$$H^2 = \frac{8\pi G}{3}\rho_{\mathrm{M}} - \frac{f}{6} - 2H^2 f_T$$

$$\left(H^2\right)' = \frac{16\pi G P_{\mathrm{M}} + 6H^2 + f + 12H^2 f_T}{24H^2 f_{TT} - 2 - 2f_T}$$

p : Constant

$$\Omega_{\mathrm{m}}^{(0)} \equiv \rho_{\mathrm{m}}^{(0)} / \rho_{\mathrm{crit}}^{(0)}$$

$$\rho_{\mathrm{crit}}^{(0)} = 3H_0^2 / \left(8\pi G\right)$$

$$H^2 = \bar{m}^2 \left(y_H + a^{-3}\right)$$

ρ_{M} and P_{M} : Energy density and pressure of all perfect fluids of generic matter, respectively.

$$\Omega_{\mathrm{m}}^{(0)} = 0.26$$

$$\Omega_{\mathrm{DE}}^{(0)} \equiv \rho_{\mathrm{DE}}^{(0)} / \rho_{\mathrm{crit}}^{(0)} = 1 - \Omega_{\mathrm{m}}^{(0)} = -\alpha \left[1 - \left(1 - 2p\right) e^p\right]$$

$$\bar{m}^2 / T_0 = \left(8\pi G \rho_{\mathrm{m}}^{(0)} / 3\right) / \left(-6H_0^2\right) = -\Omega_{\mathrm{m}}^{(0)} / 6$$

$$y_H(z=0) = H_0^2 / \bar{m}^2 - 1 = 1/\Omega_{\mathrm{m}}^{(0)} - 1$$

- Initial conditions: $y_H(z=0) = 2.8$

< Combined *f(T)* theory >

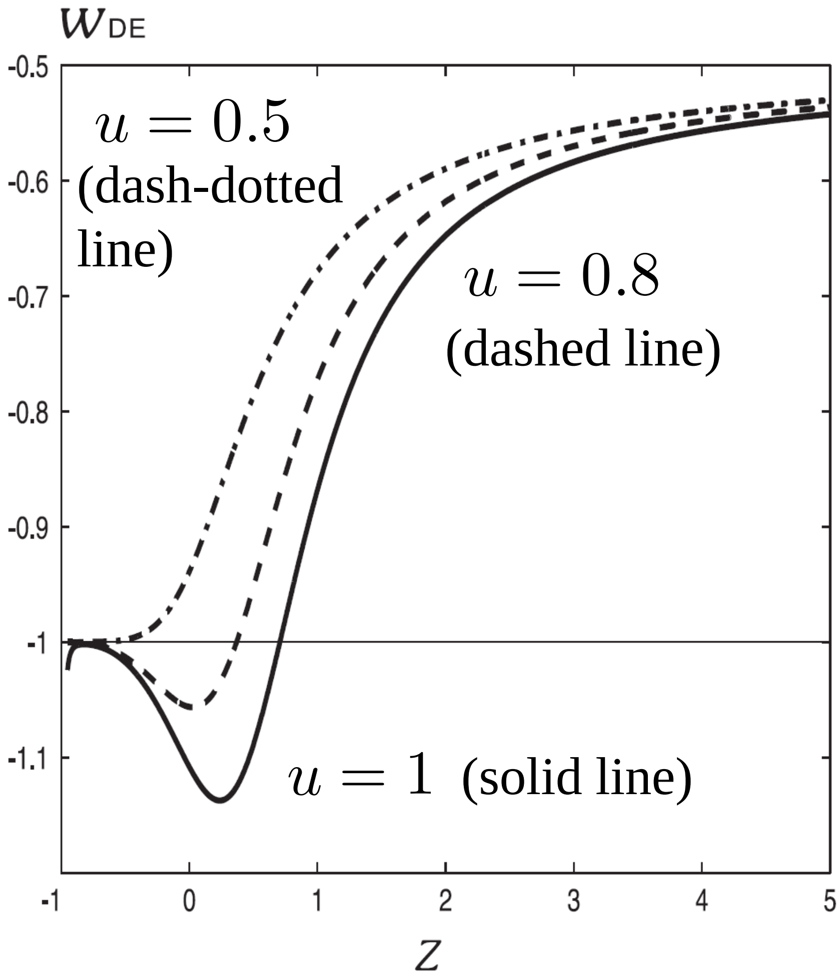
$$f(T) = \gamma \left[\underline{T_0 \left(\frac{uT_0}{T} \right)^{-1/2} \ln \left(\frac{uT_0}{T} \right)} \underline{- T \left(1 - e^{uT_0/T} \right)} \right]$$

u : Constant

($p = q = u > 0$)

Logarithmic term

Exponential term



$$\gamma \equiv \frac{1 - \Omega_{\mathrm{m}}^{(0)}}{2u^{-1/2} + \left[1 - (1 - 2u)\, e^{u} \right]}$$

← $w_{\mathrm{DE}} = -1$

Crossing of the phantom divide

- The model contains only one parameter u if one has the value of $\Omega_{\mathrm{m}}^{(0)}$.

IV. Effective equation of state for the universe and the finite-time future singularities in non-local gravity

Non-local gravity ⇐ **produced by quantum effects**

[Deser and Woodard, Phys. Rev. Lett. 99, 111301 (2007)]

- There was a proposal on the solution of the cosmological constant problem by non-local modification of gravity.

[Arkani-Hamed, Dimopoulos, Dvali and Gabadadze, arXiv:hep-th/0209227]

→ Recently, an explicit mechanism to screen a cosmological constant in non-local gravity has been discussed.

[Nojiri, Odintsov, Sasaki and Zhang, Phys. Lett. B 696, 278 (2011)]

Recent related reference: [Zhang and Sasaki, arXiv:1108.2112 [gr-qc]]

- It is known that so-called matter instability occurs in $F(R)$ gravity.

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

→ This implies that the curvature inside matter sphere becomes very large and hence the curvature singularity could appear.

[Arbuzova and Dolgov, Phys. Lett. B 700, 289 (2011)]

⇒ It is important to examine whether there exists the curvature singularity, i.e., "**the finite-time future singularities**" **in non-local gravity**.

C. Relations between the model parameters and the property of the finite-time future singularities

- $\boxed{f(\eta) = f_{\mathrm{s}} \eta^{\sigma}}$ ⟶ $\underline{f_{\mathrm{s}} \text{ and } \sigma}$ characterize the theory of non-local gravity.
- $\boxed{H \sim \dfrac{h_{\mathrm{s}}}{(t_{\mathrm{s}} - t)^q}}$ ⟶ $\underline{h_{\mathrm{s}}\, ,\, t_{\mathrm{s}} \text{ and } q}$ specify the property of the finite-time future singularity.
- $\underline{\eta_{\mathrm{c}} \text{ and } \xi_{\mathrm{c}}}$ determine a leading-order solution in terms of $(t_{\mathrm{s}} - t)$ for the gravitational field equations in the flat FLRW space-time.
- When $t \to t_{\mathrm{s}}$, for $q > 1$, $a \to \infty$,

  for $-1 < q < 0$ and $0 < q < 1$, $a \to a_{\mathrm{s}}$

for $q > 0$, $H \to \infty$, $\rho_{\mathrm{eff}} = 3H^2/\kappa^2 \to \infty$

for $-1 < q < 0$, H asymptotically becomes finite and also ρ_{eff} asymptotically approaches a finite constant value ρ_{s} .

for $q > -1$, $\dot{H} \sim q h_{\mathrm{s}} (t_{\mathrm{s}} - t)^{-(q+1)} \to \infty$, $P_{\mathrm{eff}} = -\left(2\dot{H} + 3H^2\right)/\kappa^2 \to \infty$

- It is known that the finite-time future singularities can be classified in the following manner:

Type I (“Big Rip”):	In the limit $t \longrightarrow t_{\rm s}$, $\underline{a \longrightarrow \infty \,, \quad \rho_{\rm eff} \longrightarrow \infty \,, \quad \lvert P_{\rm eff} \rvert \longrightarrow \infty}$ $\ast$ The case in which $\rho_{\rm eff}$ and $P_{\rm eff}$ becomes finite values at $t = t_{\rm s}$ is also included.
Type II (“sudden”):	$a \longrightarrow a_{\rm s} \,, \quad \rho_{\rm eff} \longrightarrow \rho_{\rm s} \,, \quad \underline{\lvert P_{\rm eff} \rvert \longrightarrow \infty}$
Type III:	$a \longrightarrow a_{\rm s} \,, \quad \underline{\rho_{\rm eff} \longrightarrow \infty \,, \quad \lvert P_{\rm eff} \rvert \longrightarrow \infty}$
Type IV: **[Nojiri, Odintsov and Tsujikawa, Phys. Rev. D 71, 063004 (2005)]**	$a \longrightarrow a_{\rm s} \,, \quad \rho_{\rm eff} \longrightarrow 0 \,, \quad \lvert P_{\rm eff} \rvert \longrightarrow 0$ $\ast$ Higher derivatives of H diverge. $\ast$ The case in which $\rho_{\rm eff}$ and/or $\lvert P_{\rm eff} \rvert$ asymptotically approach finite values is also included.

Appendix A

< Other models >

- **Appleby-Battye model** **[Appleby and Battye, Phys. Lett. B 654, 7 (2007)]**

$$f_{\rm AB}(R) = \frac{R}{2} + \frac{1}{2b_1} \log\left[\cosh(b_1 R) - \tanh(b_2)\sinh(b_1 R)\right]$$

$b_1(>0), \quad b_2$

: Constant parameters

Cf. **Power-law model**

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

[Li and Barrow, Phys. Rev. D 75, 084010 (2007)]

$$f(R) = R - \mu R^{v}$$

$\mu(>0)$: Constant parameter

$0 < \underline{v < 10^{-10}}$: Constant parameter (close to 0)

**[Capozziello and Tsujikawa,
Phys. Rev. D 77, 107501 (2008)]**

$$\boxed{S = \frac{A}{4G}}$$: **Bekenstein-Hawking horizon entropy in the Einstein gravity**

$$\boxed{\hat{S} = \frac{FA}{4G}}$$: **Wald entropy in modified gravity theories**

$A = 4\pi \tilde{r}_A^2$: Area of the apparent horizon

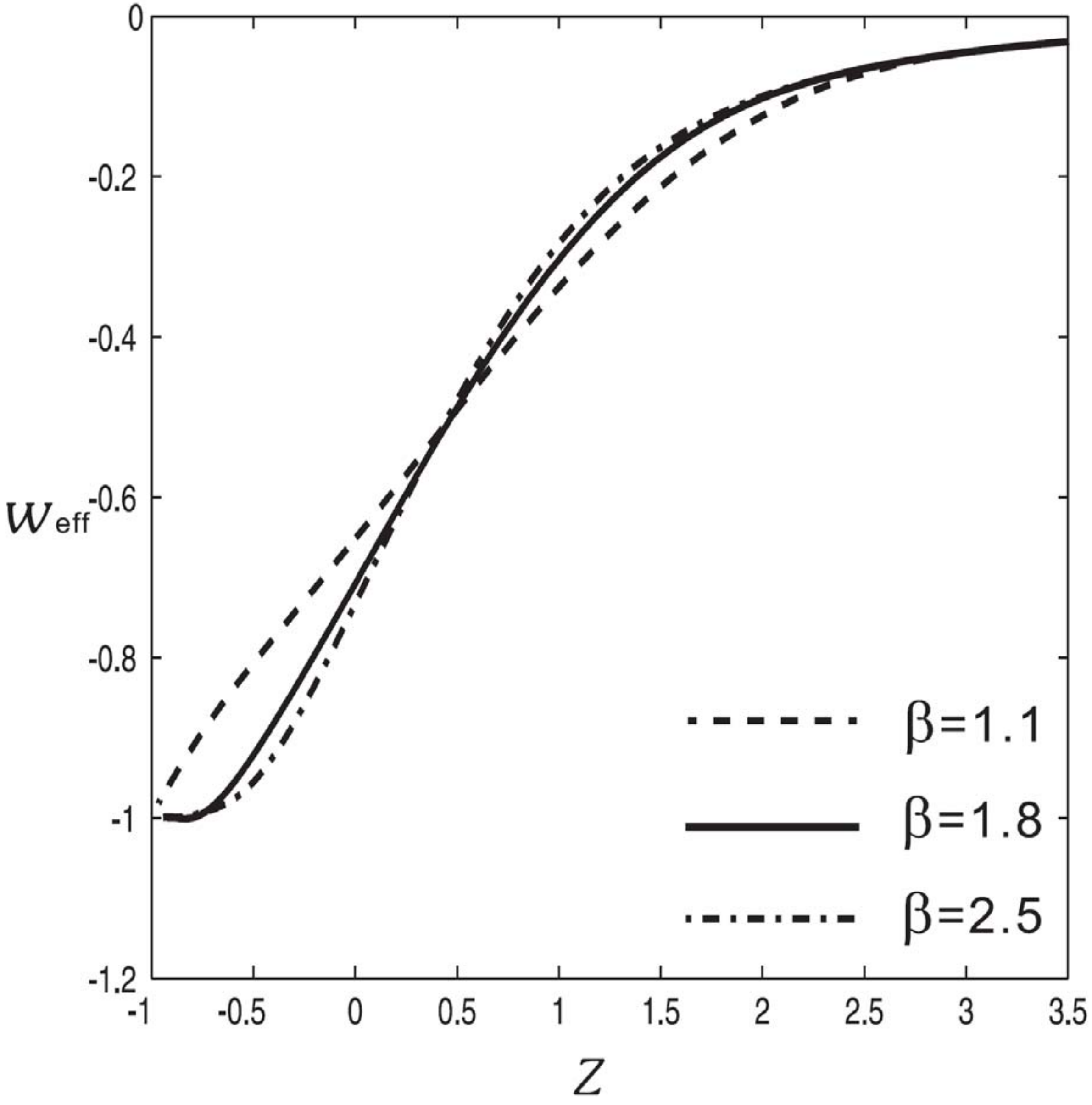
- **Wald introduced a horizon entropy $\hat{S}$ associated with a Noether charge in the context of modified gravity theories.**
- **The Wald entropy $\hat{S}$ is a local quantity defined in terms of quantities on the bifurcate Killing horizon.**

  → **More specifically, it depends on the variation of the Lagrangian density of gravitational theories with respect to the Riemann tensor.**

⇨ **This is equivalent to** $\hat{S} = A/(4G_{\mathrm{eff}})$.

$G_{\mathrm{eff}} = G/F$: Effective gravitational coupling

< Cosmological evolution of $w_{\rm eff}$ in the exponential gravity model >



From [KB, Geng and Lee, JCAP 1008, 021 (2010)].

$$f_{\rm E}(R) = R - \beta R_{\rm E}\left(1 - e^{-R/R_{\rm E}}\right)$$

$$\beta R_{\rm E} \simeq 18 H_0^2 \Omega_{\rm m}^{(0)}$$

< Remarks >

(a) The qualitative results do not strongly depend on the values of the parameters in each model.

(b) The evolutions of the Wald entropy $\hat{S}$ are similar to S in the models of (i)-(iv).

Cf. [KB, Geng and Tsujikawa, Phys. Lett. B 688, 101 (2010)]

[KB, Geng and Lee, JCAP 1008, 021 (2010)]

$$\boxed{S = \frac{A}{4G}}$$

: **Bekenstein-Hawking horizon entropy in the Einstein gravity**

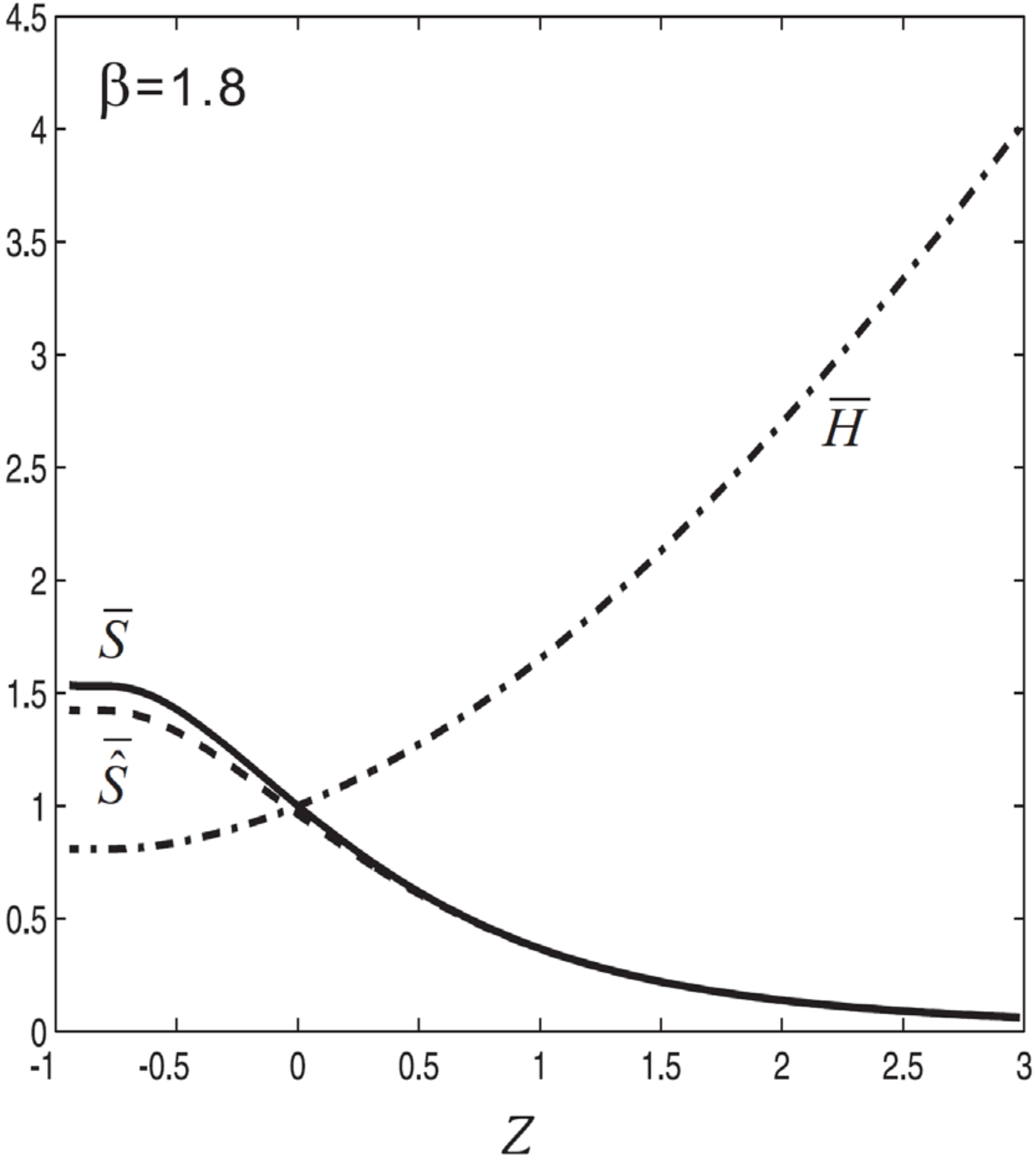
$$\boxed{\hat{S} = \frac{F(R)A}{4G}}$$

: **Wald entropy in modified gravity theories including *f*(*R*) gravity**

(c) The numerical results in the Appleby-Battye model are similar to those in the Starobinsky model of (ii).

< Cosmological evolutions of $\bar{S}$, $\bar{\hat{S}}$ and $\bar{H}$ in the exponential gravity model >

From [KB, Geng and Lee, JCAP 1008, 021 (2010)].



$$f_{\mathrm{E}}(R) =$$

$$R - \beta R_{\mathrm{E}} \left(1 - e^{-R/R_{\mathrm{E}}}\right)$$

$$\beta R_{\mathrm{E}} \simeq 18 H_0^2 \Omega_{\mathrm{m}}^{(0)}$$

$$\bar{S} = S/S_0$$

$$\bar{\hat{S}} = \hat{S}/S_0$$

$$\bar{H} = H/H_0$$

< Numerical results >

Models of (i), (ii), (iii) and (iv)

- $w_{\rm DE}(z=0)=$ -0.92, -0.97, -0.92 and -0.93
- $(\bar{H}_{\rm f}, \bar{S}_{\rm f}) = (0.80, 1.6),\ (0.85, 1.4),\ (0.78, 1.7)$ and $(0.81, 1.5)$
- $(z_{\rm cross}, z_{\rm p}) =$

  (-0.76, -0.82), (-0.83, -0.98), (-0.79, -0.80) and (-0.74, -0.80)

$z_{\rm cross}$ **: Value of z at the first future crossing of the phantom divide**

$z_{\rm p}$ **: Value of z at the first sign change of $\dot{H}$ from negative to positive**

- $(\Xi(z = z_{\rm cross}), \Xi(z = z_{\rm p})) =$

  $(5.2 \times 10^{-3}, 2.1 \times 10^{-3})$, $(1.7 \times 10^{-3}, 4.8 \times 10^{-6})$,

  $(4.1 \times 10^{-3}, 3.1 \times 10^{-3})$ and $(6.2 \times 10^{-3}, 2.8 \times 10^{-3})$

$\boxed{\Xi \equiv \Omega_{\rm m}/\Omega_{\rm DE}}$ $\Omega_{\rm m} \equiv \rho_{\rm m}/\rho_{\rm crit}^{(0)}\,,\quad \Omega_{\rm DE} \equiv \rho_{\rm DE}/\rho_{\rm crit}^{(0)}$

< Initial conditions >

Models of (i), (ii), (iii) and (iv)

- $z_0 = 8.0, \ 8.0, \ 3.0 \ \text{and} \ 3.5$

→ We have taken the initial conditions at $z = z_0$, so that $RF'(z = z_0) \sim 10^{-13}$ with $F' = dF/dR$, to ensure that they can be all close enough to the Λ CDM model with $RF' = 0$.

→ In order to save the calculation time, the different values of z_0 mainly reflect the forms of the models, i.e., the power-law types of (i) and (ii) and the exponential ones of (iii) and (iv).

- At $z = z_0$, $w_{\mathrm{DE}} = -1$.

- $$\boxed{\gamma R_{\mathrm{c}} \simeq 18 H_0^2 \Omega_{\mathrm{m}}^{(0)}}$$

Models of (i), (ii), (iii) and (iv)

$(\gamma, R_{\mathrm{c}}) = (c_1, R_{\mathrm{HS}})$, $(\lambda, R_{\mathrm{S}})$, (μ, R_{T}) and (β, R_{E})

- $$\Omega_{\mathrm{m}}^{(0)} \equiv \rho_{\mathrm{m}}^{(0)} / \rho_{\mathrm{crit}}^{(0)} = 0.26$$

[E. Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]]]

In the high z regime ($z \simeq z_0$), $R/R_{\mathrm{c}} \gg 1$, in which *f*(*R*) gravity has to be very close to the ΛCDM model.

- $$\gamma R_{\mathrm{c}} \simeq 2\Lambda = 2\kappa^2 \left(\rho_{\mathrm{DE}}/\rho_{\mathrm{m}}^{(0)}\right) \left(\rho_{\mathrm{m}}^{(0)}/\rho_{\mathrm{crit}}^{(0)}\right) \rho_{\mathrm{crit}}^{(0)} = 6 \left(\rho_{\mathrm{DE}}/\rho_{\mathrm{m}}^{(0)}\right) H_0^2 \Omega_{\mathrm{m}}^{(0)}$$
- $$\left(\rho_{\mathrm{DE}}/\rho_{\mathrm{m}}^{(0)}\right) = 3.0$$

- $\boxed{y_H\left(z=z_0\right) \simeq \Omega_{\mathrm{DE}}^{(0)}/\Omega_{\mathrm{m}}^{(0)} \simeq 3.0}$ $y_H \equiv \rho_{\mathrm{DE}}/\rho_{\mathrm{m}}^{(0)}$

- By examining the cosmological evolutions of y_H and w_{DE} as functions of the redshift z for the models, we have found that $y_H\left(z=0\right)$ is close to its initial value of $y_H\left(z=z_0\right)$.

- This is because in the higher z regime, the universe is in the phantom phase ($w_{\mathrm{DE}} < -1$) and therefore, ρ_{DE} and y_H increase (since $y_H \propto \rho_{\mathrm{DE}}$), whereas in the lower z regime, the universe is in the non-phantom (quintessence) phase ($w_{\mathrm{DE}} > -1$) and hence they decrease.

⇨ Consequently, the above two effects cancel out.

- Our results are not sensitive to the initial values of z_0 and $y_H\left(z=z_0\right)$.

- The initial condition of $dy_H/d\ln a\left(z=z_0\right)=0$ is due to that the *f*(*R*) gravity models at $z=z_0$ should be very close to the ΛCDM model, in which $dy_H/d\ln a=0$.

$$\rho_{\rm DE}^{(0)}/\rho_{\rm m}^{(0)} = \left(1-\Omega_{\rm m}^{(0)}\right)/\Omega_{\rm m}^{(0)} = 2.85$$

$$R_{\rm c} \simeq 6\gamma^{-1} y_H\,(z = z_0)\bar{m}^2$$

$$\bar{m}^2 = H_0^2\Omega_{\rm m}^{(0)}$$

$$y_H\,(z = z_0) = 2.72$$

$$y_H\,(z = 0) = 2.85$$

< Second law of thermodynamics in equilibrium description >

[KB and Geng, JCAP 1006, 014 (2010)]

We consider the same temperature of the universe outside and inside the apparent horizon.

< Gibbs equation >

$$T\mathrm{d}S_{\mathrm{t}} = \mathrm{d}\left(\rho_{\mathrm{t}}V\right) + P_{\mathrm{t}}\mathrm{d}V = V\mathrm{d}\rho_{\mathrm{t}} + \left(\rho_{\mathrm{t}} + P_{\mathrm{t}}\right)\mathrm{d}V$$

< Second law of thermodynamics >

< Condition >

$$\frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}S_{\mathrm{t}}}{\mathrm{d}t} \geq 0 \quad \Rightarrow \quad \boxed{\frac{12\pi}{G}\frac{H\left(\dot{H} - K/a^2\right)^2}{\left(H^2 + K/a^2\right)^2}\frac{1}{R} \geq 0}$$

$$R = 6\left(\dot{H} + 2H^2 + K/a^2\right)$$

→ **As long as $R > 0$, the second law of thermodynamics can be met in both non-phantom ($\dot{H} < 0$, $w_{\mathrm{eff}} > -1$) and phantom ($\dot{H} > 0$, $w_{\mathrm{eff}} < -1$) phases.**

Cf. [Gong, Wang and Wang, JCAP 0701, 024 (2007)]

[Jamil, Saridakis and Setare, Phys. Rev. D 81, 023007 (2010)]

< Second law of thermodynamics >

[KB and Geng, JCAP 1006, 014 (2010)]

We assume the same temperature between the outside and inside of the apparent horizon.

< Gibbs equation >

$\hat{S}_{\mathrm{t}}$: Entropy of total energy inside the horizon

$$T\mathrm{d}\hat{S}_{\mathrm{t}} = \mathrm{d}\left(\hat{\rho}_{\mathrm{t}} V\right) + \hat{P}_{\mathrm{t}}\mathrm{d}V = V\mathrm{d}\hat{\rho}_{\mathrm{t}} + \left(\hat{\rho}_{\mathrm{t}} + \hat{P}_{\mathrm{t}}\right)\mathrm{d}V$$

< Second law of thermodynamics >

< Condition >

$$\frac{\mathrm{d}\hat{S}}{\mathrm{d}t} + \frac{\mathrm{d}\left(\mathrm{d}_i\hat{S}\right)}{\mathrm{d}t} + \frac{\mathrm{d}\hat{S}_{\mathrm{t}}}{\mathrm{d}t} \geq 0 \quad \Rightarrow \quad \boxed{J \equiv \left(\dot{H} - \frac{K}{a^2}\right)^2 F \geq 0}$$

[Wu, Wang, Yang and Zhang, Class. Quant. Grav. 25, 235018 (2008)]

$F > 0$ because $G_{\mathrm{eff}} = G/F > 0$.

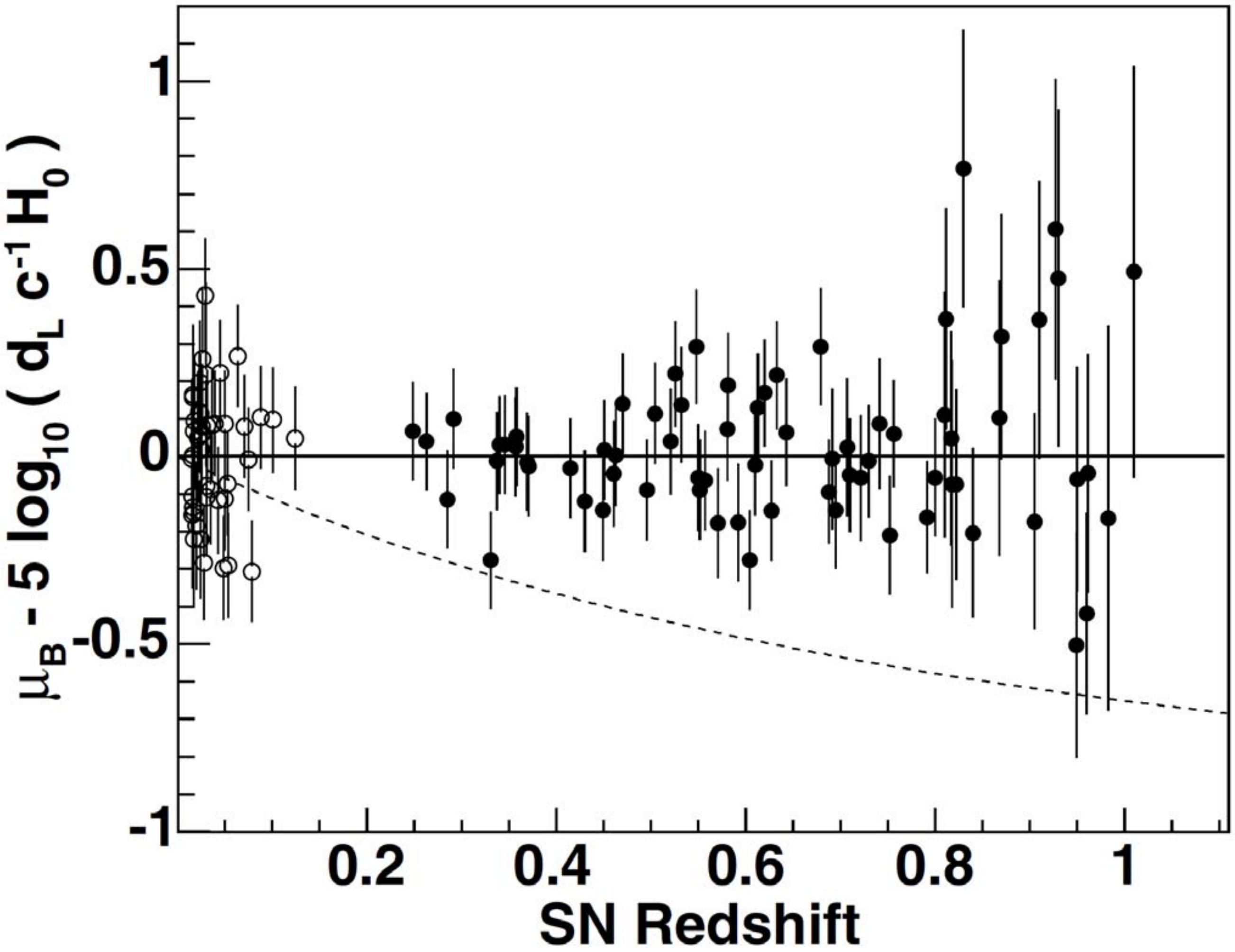
→ **The second law of thermodynamics in *f(R)* gravity can be satisfied in phantom ($\dot{H} > 0$, $w_{\mathrm{eff}} < -1$) as well as non-phantom ($\dot{H} < 0$, $w_{\mathrm{eff}} > -1$) phases.**

$w_{\mathrm{eff}} = -1 - 2\dot{H}/\left(3H^2\right)$: Effective equation of state (EoS)

Backup Slides B

< Residuals for the best fit to a flat Λ cosmology >

$\Delta(m-M)$



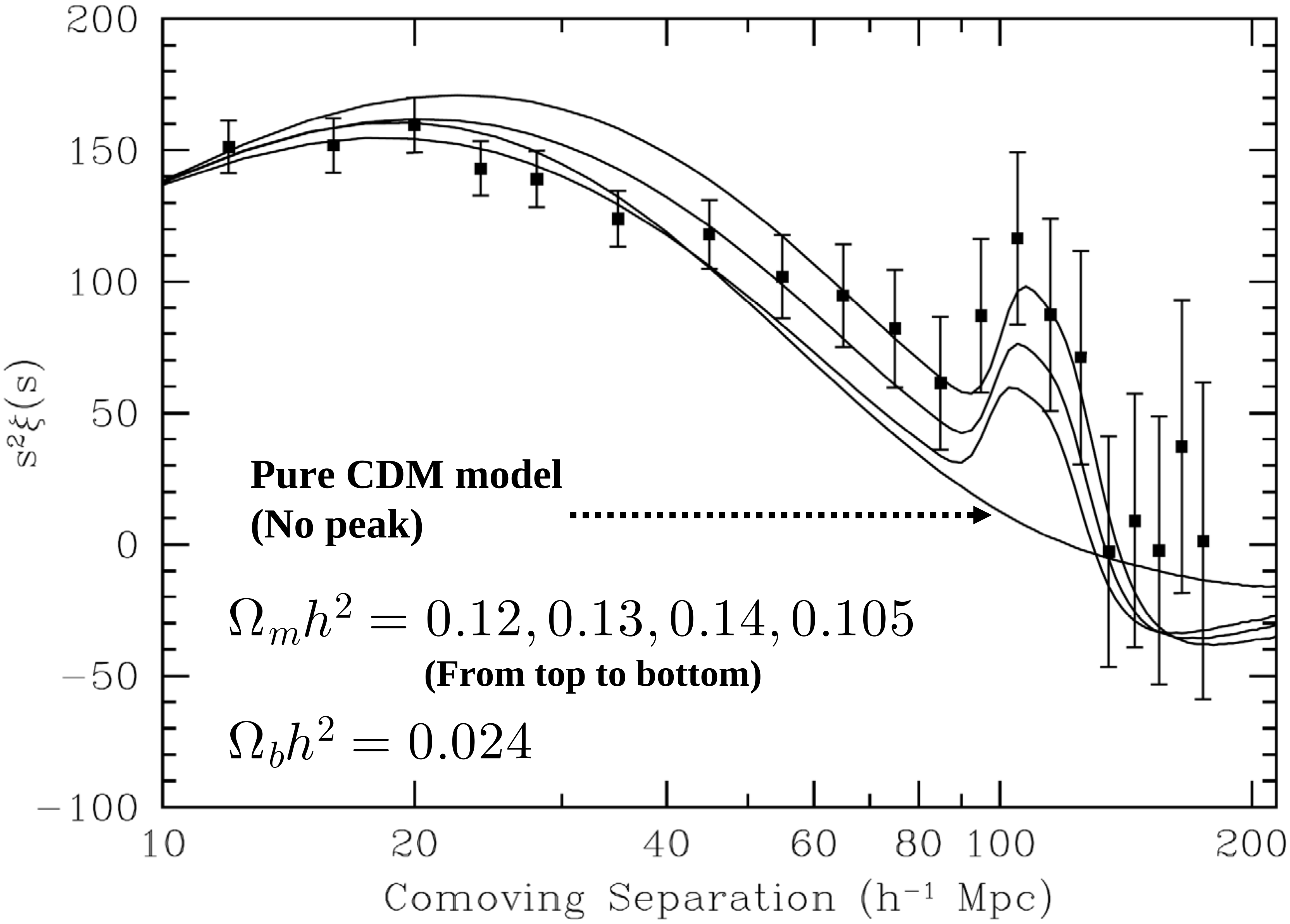
← **Flat Λ cosmology**

← **Pure matter cosmology**

z

From [Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. 447, 31 (2006)]

< Baryon acoustic oscillation (BAO) >



Pure CDM model
(No peak)

$$\Omega_m h^2 = 0.12, 0.13, 0.14, 0.105$$

(From top to bottom)

$$\Omega_b h^2 = 0.024$$

From [Eisenstein *et al.* [SDSS Collaboration], Astrophys. J. 633, 560 (2005)]

< 5-year WMAP data on w >

[Komatsu *et al*. [WMAP Collaboration], Astrophys. J. Suppl. 180, 330 (2009), arXiv:0803.0547 [astro-ph]]

- For the flat universe, constant w : (From WMAP+BAO+SN)

$$-0.14 < 1 + w < 0.12 \quad \text{(95\% CL)}$$

Baryon acoustic oscillation (BAO) : Special pattern in the large-scale correlation function of Sloan Digital Sky Survey (SDSS) luminous red galaxies

- For a variable EoS :

$$-0.33 < 1 + w_0 < 0.21 \quad \text{(95\% CL)} \longleftarrow z_{\rm trans} = 10$$

$$w(a) = \frac{a\tilde{w}(a)}{a + a_{\rm trans}} - \frac{a_{\rm trans}}{a + a_{\rm trans}}$$

$$\tilde{w}(a) = \tilde{w}_0 + (1 - a)\tilde{w}_a \qquad w_0 = w(a = 1)$$

$a < a_{\rm trans}$: Dark energy density tends to a constant value

Cf. Dark Energy : $\Omega_\Lambda = 0.726 \pm 0.015$

Dark Matter : $\Omega_c = 0.228 \pm 0.013$

Baryon : $\Omega_b = 0.0456 \pm 0.0015$ (68% CL)

$$\Omega_i \equiv \frac{\kappa^2 \rho_i^{(0)}}{3H_0^2} = \frac{\rho_i^{(0)}}{\rho_{\rm c}^{(0)}}$$

$$i = \Lambda, c, b$$

$\rho_{\rm c}^{(0)}$: Critical density

- In the flat FLRW background, gravitational field equations read

$$H^2 = \frac{\kappa^2}{3}\rho_{\mathrm{eff}}\,, \quad \dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\mathrm{eff}} + p_{\mathrm{eff}}\right)$$

$\rho_{\mathrm{eff}},\ p_{\mathrm{eff}}$: Effective energy density and pressure from the term $f(R) - R$

$$\rho_{\mathrm{eff}} = \frac{1}{\kappa^2 f'(R)}\left[\frac{1}{2}\left(-f(R) + Rf'(R)\right) - 3H\dot{R}f''(R)\right]$$

$$p_{\mathrm{eff}} = \frac{1}{\kappa^2 f'(R)}\left[\frac{1}{2}\left(f(R) - Rf'(R)\right) + \left(2H\dot{R} + \ddot{R}\right) f''(R) + \dot{R}^2 f'''(R)\right]$$

$$\rightarrow \boxed{w_{\mathrm{eff}} = \frac{p_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = \frac{\left(f(R) - Rf'(R)\right)/2 + \left(2H\dot{R} + \ddot{R}\right) f''(R) + \dot{R}^2 f'''(R)}{\left(-f(R) + Rf'(R)\right)/2 - 3H\dot{R}f''(R)}}$$

- Example : $f(R) \propto R^n \ (n \neq 1)$

$$\longrightarrow a \propto t^q, \quad q = \frac{-2n^2+3n-1}{n-2}$$

$$w_{\mathrm{eff}} = -\frac{6n^2-7n-1}{6n^2-9n+3}$$

[Capozziello, Carloni and Troisi, Recent Res. Dev. Astron. Astrophys. 1, 625 (2003)]

If $q > 1$, accelerated expansion can be realized.

(For $n = 3/2$ or $n = -1$, $q = 2$ and $w_{\mathrm{eff}} = -2/3$.)

Appendix B

(7) Free of curvature singularities

[Frolov, Phys. Rev. Lett. 101, 061103 (2008)]

- **Existence of relativistic stars**

[Kobayashi and Maeda, Phys. Rev. D 78, 064019 (2008)]

[Kobayashi and Maeda, Phys. Rev. D 79, 024009 (2009)]

< Model with satisfying the condition (7) >

$$F_{\mathrm{MJWQ}}(R) = R - \alpha R_* \ln\left(1 + \frac{R}{R_*}\right)$$

$\alpha > 0, \quad R_* > 0$: Constants

[Miranda, Joras, Waga and Quartin, Phys. Rev. Lett. 102, 221101 (2009)]

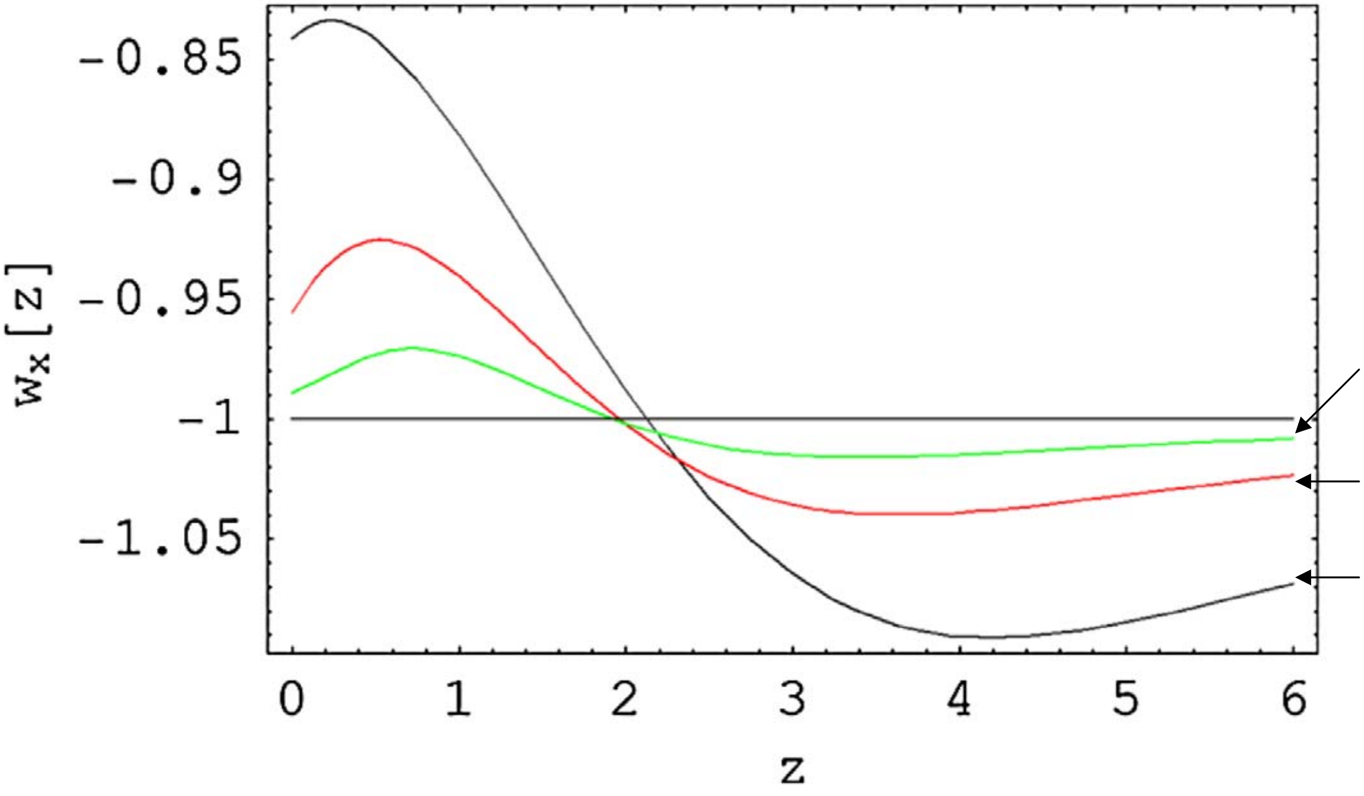
Cf. [de la Cruz-Dombriz, Dobado and Maroto, Phys. Rev. Lett. 103, 179001 (2009)]

< Recent work >

[Martinelli, Melchiorri and Amendola, Phys. Rev. D 79, 123516 (2009)]

Cf. [Nozari and Azizi, Phys. Lett. B 680, 205 (2009)]

- It has been shown that in the Hu-Sawicki model, the transition from the phantom phase to the non-phamtom one can also occur.



From [Martinelli, Melchiorri and Amendola, Phys. Rev. D 79, 123516 (2009)]

$n = 1$

$F'(R_0) = -0.01$

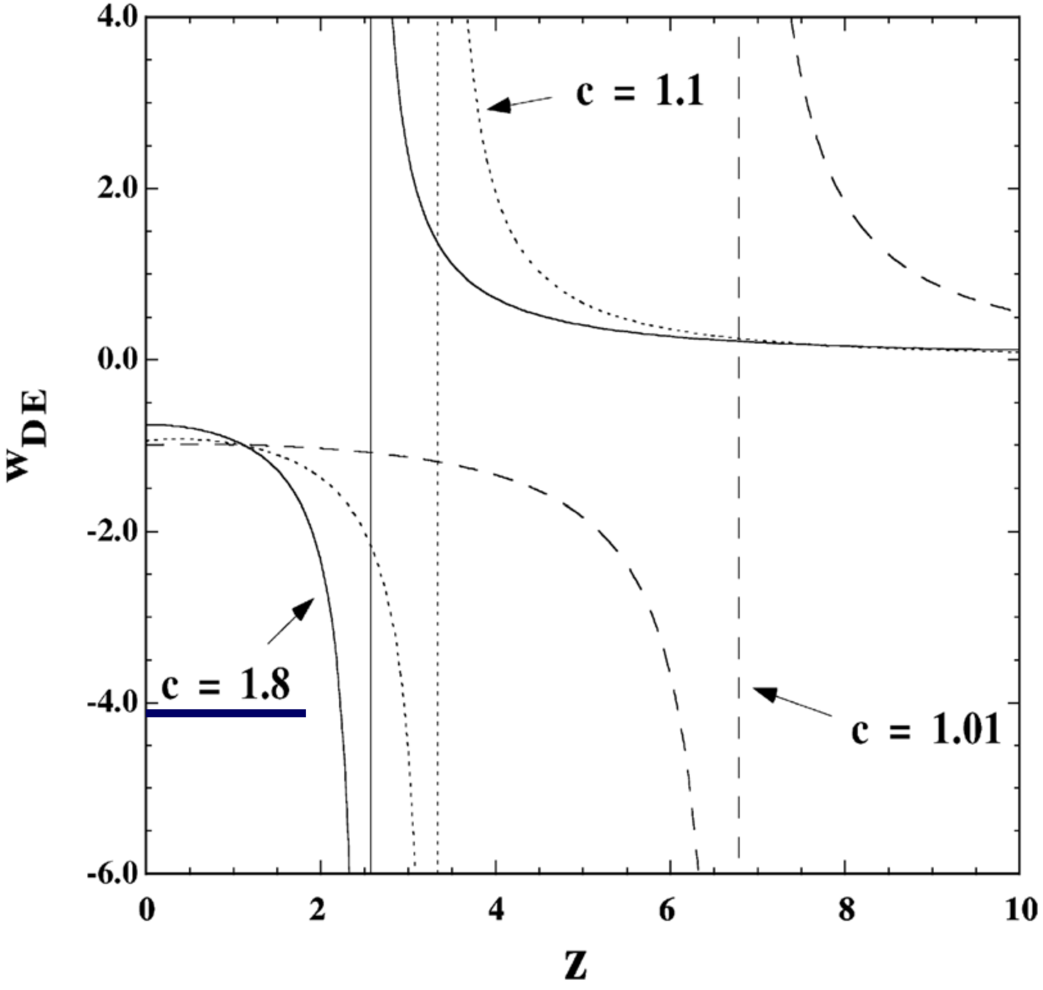
$F'(R_0) = -0.03$

$F'(R_0) = -0.1$

$\Omega_m = 0.24$

⇨ We reconstruct an explicit model of $F(R)$ gravity with realizing the crossing of the phantom divide.

< Preceding work >



From [Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

$$F(R) = \left(R^{1/c} - \Lambda\right)^c$$

$c,\ \Lambda$: Constants

- Example: $c = 1.8$

Phantom phase

⇩

Non-phantom phase

(5) Existence of a matter-dominated stage and that of a late-time cosmic acceleration

⟶ **Analysis of $m(r)$ curve on the (r, m) plane**

$$m \equiv RF''(R)/F'(R), \qquad r \equiv -RF'(R)/F(R)$$

- **Presence of a matter-dominated stage**

$$m(r) \approx +0 \text{ and } \frac{dm}{dr} > -1 \text{ at } r \approx -1$$

- **Presence of a late-time acceleration**

(i) $m(r) = -r - 1, \quad \frac{\sqrt{3}-1}{2} < m \leq 1$ and $\frac{dm}{dr} < -1$

(ii) $0 < m \leq 1$ at $r = -2$

- **Combing local gravity constraints, we obtain**

$$m(r) = C(-r-1)^p \text{ with } p > 1 \text{ as } r \to -1.$$

$C > 0, \ p$: Constants

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]