Magnetic monopole versus vortex as gauge-invariant topological objects for quark confinement

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Quark confinement: dual superconductor picture based on a non-Abelian Stokes theorem and reformulations of Yang-Mills theory
v1 (277 pages including 59 figures and 13 tables) v2: available soon.

§ Introduction

Numerical simulations of the static quark potential

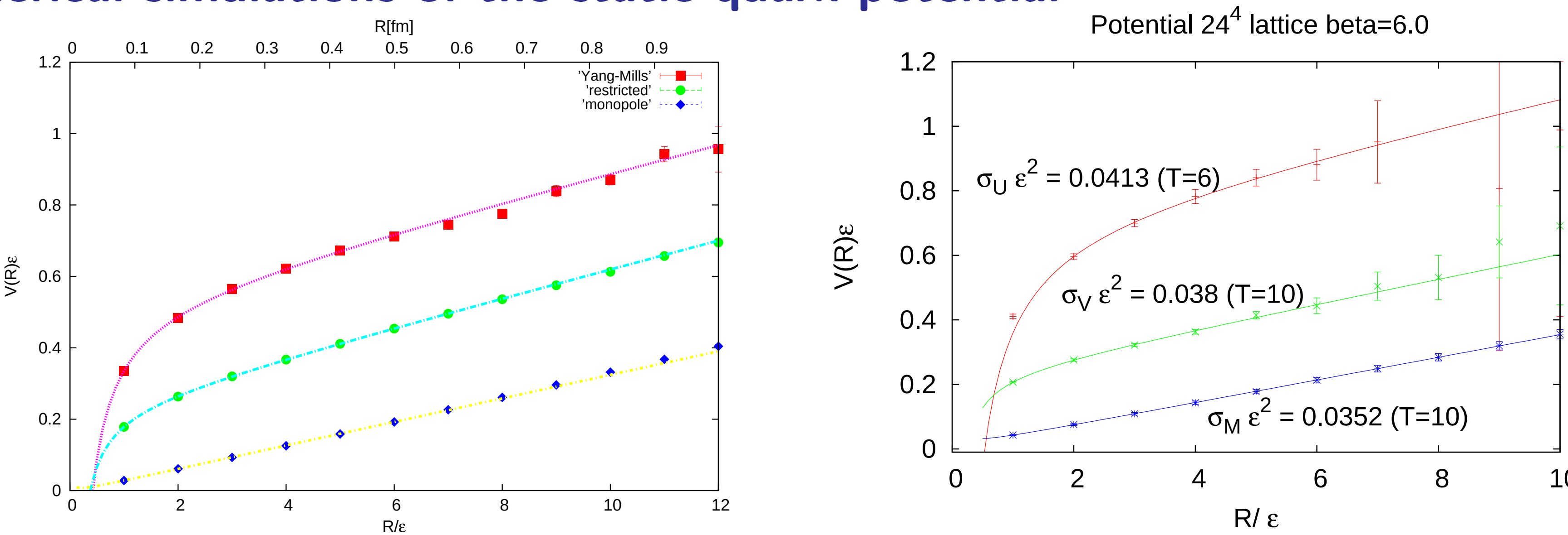


Figure 1: The static quark–antiquark potential $V(R)$ as a function of the distance R in $SU(N)$ Yang-Mills theory. (from above to below): The full potential $V_{\rm full}(R)$, restricted (or "Abelian") part $V_{\rm rest}(R)$ and magnetic–monopole part $V_{\rm mono}(R)$.
(Left) $SU(2)$ at $\beta = 2.5$ on 24^4 lattice, (Right) $SU(3)$ at $\beta = 6.0$ on 24^4 lattice.

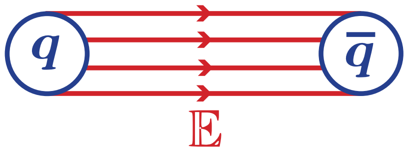
Static quark-antiquark potential $V_{q\bar{q}}(r)$= Coulomb+Linear:

$$V_{q\bar{q}}(r) = -\frac{\alpha}{r} + \sigma r + c \to \infty \ (r \to \infty) \Longrightarrow \text{quark confinement},$$

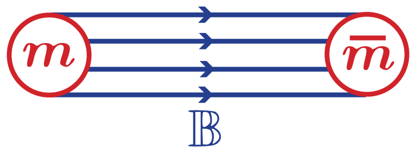
with the parameters, σ: string tension [mass^2], α: dimensionless [mass^0], c: [mass^1].

Dual superconductor hypothesis for quark confinement [Nambu (1974), 't Hooft (1975), Mandelstam (1976), Polyakov (1975,1977) ...]
The key ingredients for the dual superconductivity are as follows.
* dual Meissner effect

In the dual superconductor, the chromoelectric flux must be squeezed into tubes. [$\leftarrow$ In the ordinary superconductor (of the type II), the magnetic flux is squeezed into tubes.]



$\leftarrow$ dual $\rightarrow$



QCD vacuum=dual superconductor $\qquad$ superconductor

* condensation of chromomagnetic monopoles

The dual superconductivity will be caused by condensation of magnetic monopoles. [$\leftarrow$ The ordinary superconductivity is cased by condensation of electric charges into Cooper pairs.]
In order to establish the dual superconductivity, we must answer the following questions:
* How to introduce magnetic monopoles in the Yang-Mills theory without scalar fields?
cf. 't Hooft-Polyakov magnetic monopole
* How to define the electric-magnetic duality in the non-Abelian gauge theory?
* How to preserve the original non-Abelian gauge symmetry?
* How to extract the infrared dominant mode $\mathscr{V}$ for confinement?

In this talk,

- We give a **gauge-invariant definition of (chromo)magnetic monopoles** in the $SU(N)$ Yang-Mills theory (in the absence of the scalar fields) from the non-Abelian Wilson loop operator. This is achieved by using a **non-Abelian Stokes theorem** for the Wilson loop operator. This leads to the **non-Abelian magnetic monopoles**.

  This definition is independent of the gauge fixing. One does not need to use the conventional prescription called the **Abelian projection** proposed by ['t Hooft (1981)] which realizes magnetic monopoles by a partial gauge fixing as **gauge-fixing defects**.

In fact, we have confirmed by numerical simulations on a lattice the following facts.

- The magnetic monopole reproduces the linear potential with almost the same string tension $\sigma_{\rm mono}$ as the original one $\sigma_{\rm full}$: 85% for $SU(2)$, 80% for $SU(3)$. This is called the **magnetic monopole dominance** in the string tension.

- The **dual Meissner effect** occur in $SU(N)$ Yang–Mills theory as signaled by the simultaneous formation of the **chromoelectric flux tube** and the **associated magnetic-monopole current** induced around it.
  See Shibata's talk.

Only the component E_z of the chromoelectric field $(E_x, E_y, E_z) = (F_{14}, F_{24}, F_{34})$ connecting q and $\bar{q}$ has non-zero value. The other components are zero consistently within the numerical errors. Therefore, the chromomagnetic field $(B_x, B_y, B_z) = (F_{23}, F_{31}, F_{12})$ connecting q and $\bar{q}$ does not exist.

The magnitude of the chromoelectric field E_z decreases quickly as the distance y in the direction perpendicular to the line increases. Therefore, we have confirmed the formation of the chromoelectric flux in Yang–Mills theory on a lattice.

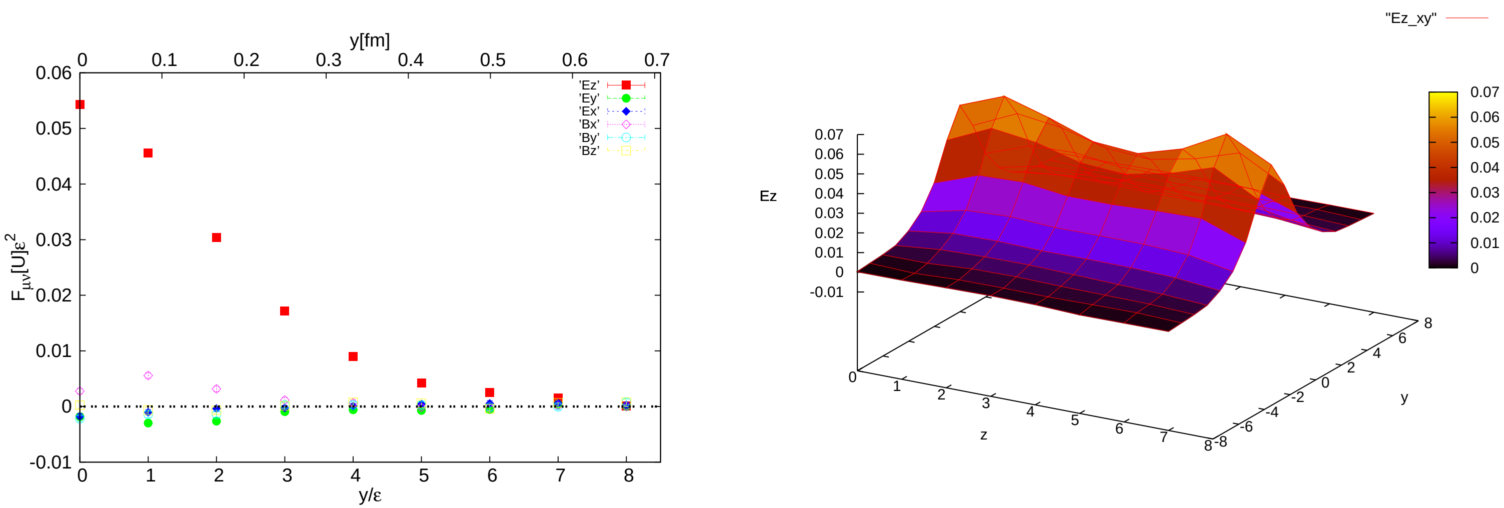


Figure 2: The chromoelectric and chromomagnetic fields obtained from the full field U on 24^4 lattice at $\beta = 2.5$. (Left panel) y dependence of the chromoelectric field $E_i(y) = F_{4i}(y)$ $(i = x, y, z)$ at fixed $z = 4$ (mid-point of $q\bar{q}$). (Right panel) The distribution of $E_z(y, z)$ obtained for the 8×8 Wilson loop with $\bar{q}$ at $(y, z) = (0, 0)$ and q at $(y, z) = (0, 8)$.

We have also shown that the restricted field V reproduces the dual Meissner effect in the $SU(N)$ Yang–Mills theory on a lattice.

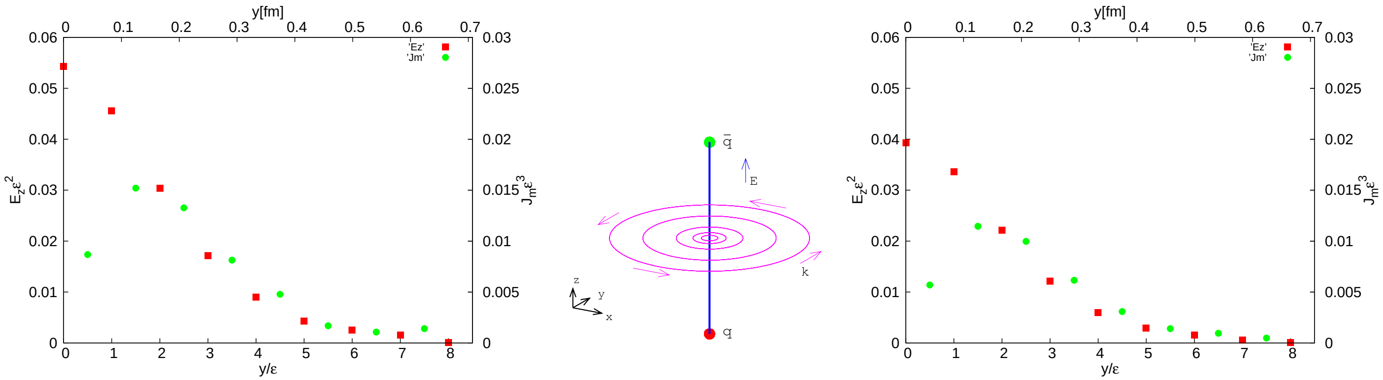


Figure 3: The magnetic-monopole current $\mathbf{k}$ induced around the chromoelectric flux along the z axis connecting a pair of quark and antiquark. (Center panel) The positional relationship between the chromoelectric field E_z and the magnetic current $\mathbf{k}$. (Left panel) The magnitude of the chromoelectric field E_z and the magnetic current $J_m = |\mathbf{k}|$ as functions of the distance y from the z axis calculated from the original full variables. (Right panel) The counterparts of the left graph calculated from the restricted variables.

The superconductor is characterized by δ: the penetration depth, ξ: the coherence length, and κ: the Ginzburg-Landau (GL) parameter defined by

$$\kappa := \frac{\delta}{\xi} < \frac{1}{\sqrt{2}}(\text{type I}), \ = \frac{1}{\sqrt{2}}(\text{BPS}), \ > \frac{1}{\sqrt{2}}(\text{type II}).$$

However, the numerical simulations show that the dual superconductivity of the Yang-Mills vacuum is **type I**, in contrast to the preceding studies which claim the border between type I and type II, i.e., BPS limit. $\kappa_c = 1/\sqrt{2} \simeq 0.707$
For $SU(2)$, [Kato et al.(2014)], the GL parameter, the penetration depth, the coherence length:

$$\kappa = 0.48 \pm 0.07, \ \delta = 0.12\text{fm}, \ \xi = 0.25\text{fm}.$$

For $SU(3)$, [Shibata et al.(2013)] reports

$$\kappa = 0.45 \pm 0.01, \ \delta = 0.12\text{fm}, \ \xi = 0.27\text{fm}, (m_A = 1.64\text{GeV}, \ m_\phi = 1.0\text{GeV})$$

κ: Ginzburg-Landau (GL) parameter δ: penetration depth, ξ: coherence length,

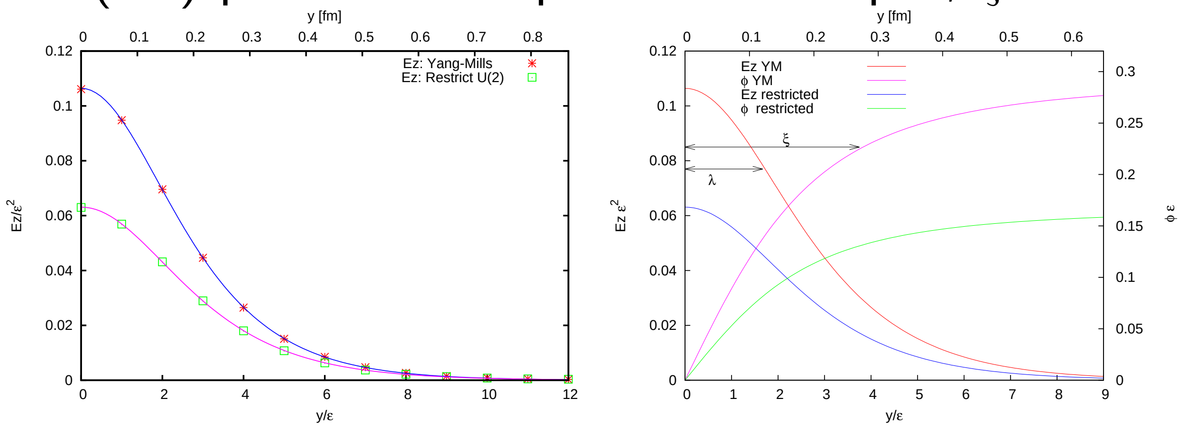
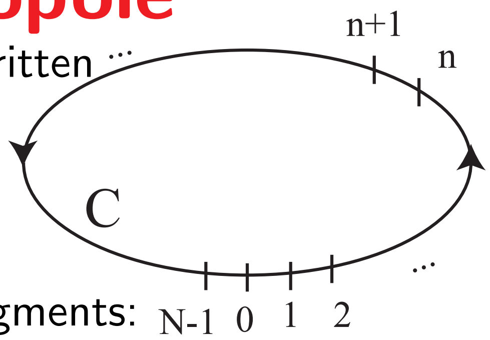


Figure 4: (Left panel) The plot of the chromoelectric field E_z versus the distance y in units of the lattice spacing ϵ and the fitting as a function $E_z(y)$ of y according to (15). The red cross for the original $SU(3)$ field and the green square symbol for the restricted field. (Right panel) The order parameter ϕ reproduced as a function $\phi(y)$ of y according to (15), together with the chromoelectric field $E_z(y)$.

In the type-I superconductor, the attractive force acts between two flux tubes, while the repulsive force in the type-II superconductor. There is no interaction at $\kappa = \frac{1}{\sqrt{2}} \simeq 0.707$.

The **type I dual superconductivity** for the Yang-Mills vacuum yields the attractive force between two non-Abelian vortices. How this result is consistent with the above considerations?
This is a motivation to discuss how the **magnetic monopole condensation** picture are compatible with the **vortex condensation** picture as another promising scenario for quark confinement.
The crucial point is that the **non-Abelian vortices** have internal degrees of freedom, i.e., **orientational moduli**, in addition to the degrees of freedom related to the positions in space. This has a possibility for preventing the vortices from collapse due to the attractive force.
While the Abelian vortices have only the positions and the collapse of the lattice structure for the Abelian vortices in the type I superconductor will not be avoidable.
We must examine the interaction among vortices depending on the orientational moduli in more detail.
This analysis gives an estimate of the string tension based on the vortex condensation picture, and possible interactions between two non-Abelian vortices.

§ Wilson loop operator and magnetic monopole



The non-Abelian **Wilson loop operator** $W_C[\mathscr{A}]$ (in the representation R) is written

$$W_C[\mathscr{A}] := \mathrm{tr}_R \left\{ \mathscr{P} \exp \left[-ig_{\mathrm{YM}} \oint_C \mathscr{A} \right] \right\} / \mathrm{tr}_R(\mathbf{1}).$$

The **path ordering** $\mathscr{P}$ is defined by dividing the path C into N infinitesimal segments:

$$W_C[\mathscr{A}] = \lim_{N\to\infty, \epsilon\to 0} \mathrm{tr}_R \left\{ \mathscr{P} \prod_{n=0}^{N-1} \exp \left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A} \right] \right\} / \mathrm{tr}_R(\mathbf{1}). \qquad (2)$$

The troublesome path ordering in the non-Abelian Wilson loop operator can be removed as first shown for $G = SU(2)$ by [Diakonov and Petrov (1989)], which is the **non-Abelian Stokes theorem** (NAST).

Moreover, the NAST for the Lie group G can be obtained as the path-integral representation of the Wilson loop operator using the **coherent state of the Lie group** G in an unified way. [Kondo (1998), Kondo and Taira (2000), Kondo (2008)].

We follow the standard steps for the path integral: We insert a **complete set** of states at each partition point:

$$\mathbf{1} = \int d\mu(g(x_n)) \, |g(x_n), \Lambda\rangle \, \langle g(x_n), \Lambda| \quad (n = 1, \cdots, N-1). \qquad (3)$$

where $d\mu(g)$ is an invariant measure on G and the state is normalized $\langle g(x_n), \Lambda | g(x_n), \Lambda\rangle = 1$.

Here the **coherent state** $|g, \Lambda\rangle$ is constructed by operating a group element $g \in G$ to a **reference state** $|\Lambda\rangle$ (e.g., the highest- or lowest-weight state) for a given representation R of the Wilson loop we consider:

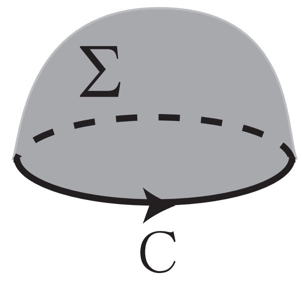
$$|g, \Lambda\rangle = g \, |\Lambda\rangle \, , \quad g \in G. \qquad (4)$$

Finally, we obtain the NAST: the $SU(N)$ Wilson loop operator in the **fundamental representation** can be rewritten into the surface integral form:

$$W_C[\mathscr{A}] = \int [d\mu(g)]_\Sigma \exp\left[- i g_{\mathrm{YM}} \frac{1}{2}\sqrt{\frac{2(N-1)}{N}} \int_{\Sigma:\partial\Sigma=C} f \right], \tag{5}$$

where $[d\mu(g)]_\Sigma$ is the invariant integration measure on the surface Σ bounded by the loop C:

$$[d\mu(g)]_\Sigma := \prod_{x\in\Sigma:\partial\Sigma=C} d\mu(g(x)), \tag{6}$$



the field strength $f_{\mu\nu}$ is given by

$$\begin{aligned} f_{\mu\nu} =& \partial_\mu \mathrm{tr}\{\boldsymbol{n}(x)\mathscr{A}_\nu(x)\} - \partial_\nu \mathrm{tr}\{\boldsymbol{n}(x)\mathscr{A}_\mu(x)\} \\ &+ \frac{2(N-1)}{N} i g_{\mathrm{YM}}^{-1} \mathrm{tr}\{\boldsymbol{n}(x)[\partial_\mu \boldsymbol{n}(x), \partial_\nu \boldsymbol{n}(x)]\}, \end{aligned} \tag{7}$$

with the normalized and traceless Lie algebra $\mathscr{G}$ valued field $\boldsymbol{n}(x)$ called the **color direction field**:

$$\boldsymbol{n}(x) = \sqrt{\frac{N}{2(N-1)}} g(x) \left[\rho - \frac{\mathbf{1}}{\mathrm{tr}(\mathbf{1})}\right] g^\dagger(x) \in \mathscr{G}, \quad g(x) \in G. \tag{8}$$

Here the matrix ρ with the matrix element ρ_{ab} is defined for the **highest- or lowest-weight state** $|\Lambda\rangle = (\lambda_a)$ of a representation R of a group G by

$$\rho := |\Lambda\rangle \langle\Lambda| \,. \tag{9}$$

Non-Abelian magnetic monopoles in Yang-Mills theory Using the Hodge decomposition, the $SU(N)$ Wilson loop operator in the **fundamental representation** is cast into the volume-integral form:

$$W_C[\mathscr{A}] = \int [d\mu(g)] \exp\left\{ -ig_{\rm YM}\frac{1}{2}\sqrt{\frac{2(N-1)}{N}}[(N_\Sigma, j) + (\omega_\Sigma, k)] \right\}, \tag{1}$$

where we have defined the gauge-invariant $(D-3)$-form k and one-form j by

$$k := \delta^* f, \quad j := \delta f, \quad f := 2{\rm tr}\{\boldsymbol{n}\mathscr{F}[\mathscr{V}]\}, \tag{2}$$

and the $(D-3)$-form ω_Σ and one-form N_Σ by (ω_Σ is the D-dim. solid angle)

$$\omega_\Sigma := {}^*d\Delta^{-1}\Theta_\Sigma = \delta\Delta^{-1*}\Theta_\Sigma, \quad N_\Sigma := \delta\Delta^{-1}\Theta_\Sigma, \tag{3}$$

with the inner product for the two forms defined by

$$(\omega_\Sigma, k) = \frac{1}{(D-3)!}\int d^Dx k^{\mu_1\cdots\mu_{D-3}}(x)\omega_\Sigma^{\mu_1\cdots\mu_{D-3}}(x), \quad (N_\Sigma, j) = \int d^Dx j^\mu(x) N_\Sigma^\mu(x). \tag{4}$$

Thus the Wilson loop operator can be expressed by the electric current j and the magnetic current k. The magnetic monopole described by the current k is a topological object of **co-dimension** 3:

- $D=3$: 0-dimensional point defect $\to$ point-like magnetic monopole (cf. Wu-Yang type)
- $D=4$: 1-dimensional line defect $\to$ magnetic monopole loop (closed loop) due to $\delta k = 0$

⊙ **SU(2) case**:[Kondo (1998)]

For $SU(2)$, the gauge-invariant magnetic-monopole current $(D-3)$-form k is obtained

$$k=\delta^* f,\ f_{\mu\nu}=2\mathrm{tr}\{\boldsymbol{n}\mathscr{F}_{\mu\nu}[\mathscr{V}]\}=\partial_\mu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\nu\}-\partial_\nu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\mu\}+ig_{\mathrm{YM}}^{-1}2\mathrm{tr}\{\boldsymbol{n}[\partial_\mu\boldsymbol{n},\partial_\nu\boldsymbol{n}]\}. \tag{5}$$

For the fundamental representation of $SU(2)$, the highest-weight state $|\Lambda\rangle$ yields the color field:

$$|\Lambda\rangle=\begin{pmatrix}1\\0\end{pmatrix}\Longrightarrow\rho:=|\Lambda\rangle\langle\Lambda|=\begin{pmatrix}1\\0\end{pmatrix}(1,0)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\Longrightarrow\rho-\frac{1}{2}\mathbf{1}=\frac{\sigma_3}{2},$$
$$\Longrightarrow\boldsymbol{n}(x)=g(x)\frac{\sigma_3}{2}g(x)^\dagger\in SU(2)/U(1)\simeq S^2\simeq\mathbb{C}P^1. \tag{6}$$

The magnetic charge q_m obeys the **quantization condition** a la Dirac:

$$q_m:=\int d^3x k^0=4\pi g_{\mathrm{YM}}^{-1}\ell,\ \ell\in\mathbb{Z}. \tag{7}$$

This is suggested from a nontrivial Homotopy group of the map $\boldsymbol{n}:S^2\to SU(2)/U(1)$:

$$\pi_2(SU(2)/U(1))=\pi_1(U(1))=\mathbb{Z}. \tag{8}$$

cf. the Abelian magnetic monopole due to 't Hooft-Polyakov associated with the spontaneous breaking $G=SU(2)\to H=U(1)$:

$$\boldsymbol{n}^A\leftrightarrow\hat{\phi}^A(x)/|\hat{\phi}(x)|. \tag{9}$$

⊙ **SU(3) case**:[Kondo (2008)]

For $SU(3)$, the gauge-invariant magnetic-monopole current $(D-3)$-form k is given by

$$k = \delta^* f, \quad f_{\mu\nu} := \partial_\mu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\nu\} - \partial_\nu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\mu\} + \frac{4}{3} i g_{\mathrm{YM}}^{-1} 2\mathrm{tr}\{\boldsymbol{n}[\partial_\mu \boldsymbol{n}, \partial_\nu \boldsymbol{n}]\}. \tag{10}$$

For the fundamental representation of $SU(3)$, the highest-weight state $|\Lambda\rangle$ yields the color field:

$$|\Lambda\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \rho := |\Lambda\rangle\langle\Lambda| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \rho - \frac{1}{3}\mathbf{1} = \frac{-1}{3}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

$$\Longrightarrow \boldsymbol{n}(x) = g(x)\frac{-1}{2\sqrt{3}}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g(x)^\dagger \in SU(3)/U(2) \simeq \mathbb{C}P^2. \tag{12}$$

The matrix $\mathrm{diag.}(-2,1,1)$ is degenerate. Using the Weyl symmetry (global symmetry as a discrete subgroup of color symmetry), it is changed into λ_8. This color field describes a **non-Abelian magnetic monopole**, which corresponds to the spontaneous symmetry breaking $SU(3) \to U(2)$ in the gauge-Higgs model. The magnetic charge obeys the quantization condition:

$$q_m' := \int d^3x k^0 = 2\pi\sqrt{3} g_{\mathrm{YM}}^{-1} n', \ n' \in \mathbb{Z}. \tag{13}$$

This is suggested from a nontrivial Homotopy group of the map $\boldsymbol{n} : S^2 \to SU(3)/U(2)$

$$\pi_2(SU(3)/[SU(2)\times U(1)]) = \pi_1(SU(2)\times U(1)) = \pi_1(U(1)) = \mathbb{Z}. \tag{14}$$

For a **reference state** $|\Lambda\rangle$ of a given representation of a Lie group G, the **maximal stability subgroup** $\tilde{H}$ is defined to be a subgroup leaving $|\Lambda\rangle$ invariant (up to a phase $\phi(h)$):

$$h \in \tilde{H} \Longleftrightarrow h|\Lambda\rangle = |\Lambda\rangle e^{i\phi(h)}. \tag{15}$$

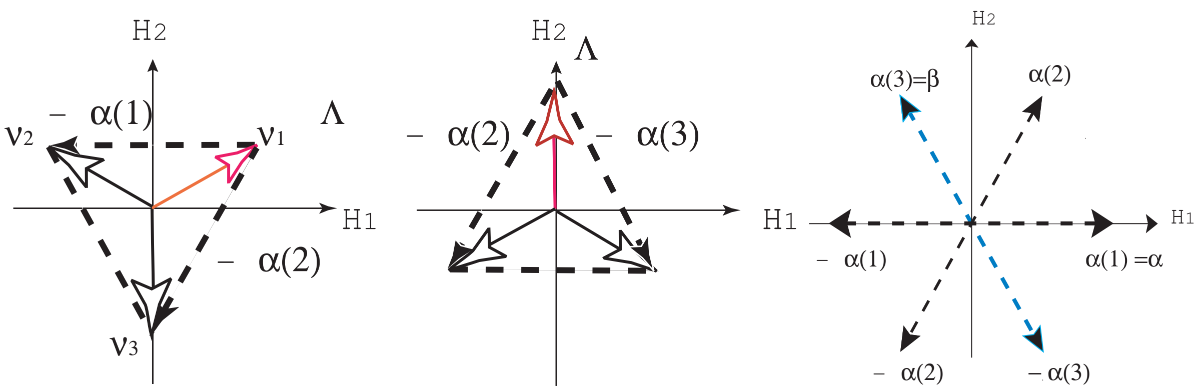
Then a group element g of G is decomposed as

$$g = \xi h \in G, \xi \in G/\tilde{H}, h \in \tilde{H} \Longrightarrow |g,\Lambda\rangle = g|\Lambda\rangle = \xi h|\Lambda\rangle = \xi|\Lambda\rangle e^{i\phi(h)} = |\xi,\Lambda\rangle e^{i\phi(h)}. \tag{16}$$

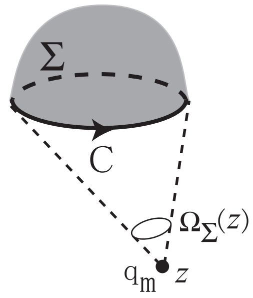
Every representation R of $SU(3)$ which is specified by the Dynkin index [m,n] belongs to (I) or (II):

(I) [Maximal case] $m \neq 0$ and $n \neq 0 \Longrightarrow \tilde{H} = H = U(1) \times U(1)$. maximal torus
e.g., adjoint rep.[1,1], $\{H_1, H_2\} \in u(1) + u(1)$,

(II) [Minimal case] $m = 0$ or $n = 0 \Longrightarrow \tilde{H} = U(2)$.
This case occurs when the weight vector Λ is orthogonal to some of the root vectors.
e.g., fundamental rep. [1,0], $\{H_1, H_2, E_\beta, E_{-\beta}\} \in u(2)$, where $\Lambda \perp \beta, -\beta$.



Magnetic monopole contribution to the Wilson loop



For $D=3$, ω_Σ is the (normalized) solid angle Ω_Σ:

$$\omega_\Sigma(x) = \Omega_\Sigma(x)/(4\pi). \tag{17}$$

For an ensemble of point-like magnetic charges q_m^a located at $x = z_a$ $(a = 1, \cdots, n)$

$$k(x) = \sum_{a=1}^{n} q_m^a \delta^{(3)}(x - z_a), \quad q_m^a = 4\pi g_{\rm YM}^{-1} \ell_a, \quad \ell_a \in \mathbb{Z}, \tag{18}$$

the magnetic-monopole contribution to the $SU(2)$ Wilson loop operator reads

$$W_C^m = \exp\left[i g_{\rm YM} J \int d^3x \omega_\Sigma(x) k(x) \right] = \exp\left\{ iJ \sum_{a=1}^{n} \Omega_\Sigma(z_a)\ell_a \right\}, \quad \ell_a \in \mathbb{Z}. \tag{19}$$

A magnetic monopole located on (or in the neighborhood of) the Wilson surface Σ $\Omega_\Sigma(z) = \pm 2\pi$ for $z \in \Sigma$. contribute to the Wilson loop for a half-integer or integer J:

$$W_C^m = \prod_{a=1}^{n} \exp[\pm iJ(2\pi)\ell_a] = \begin{cases} \prod_{a=1}^{n}(-1)^{\ell_a} & (J = \frac{1}{2}, \frac{3}{2}, \cdots) \\ = 1 & (J = 1, 2, \cdots) \end{cases}. \tag{20}$$

Here, $\exp[ig_{\text{YM}} J(\omega_\Sigma, k)]$ gives a non-trivial contribution, i.e., a center element:

$$\exp[\pm iJ(2\pi)] = (e^{\pm i\pi})^{2J} = (-1)^{2J} = \{1, -1\} \in \mathbb{Z}(2) = \text{Center}(SU(2)), \tag{21}$$

This result does not depend on which surface bounding C is chosen in the non-Abelian Stokes theorem. [This helps us to understand the **N-ality** dependence of the asymptotic string tension.] Here the magnetic flux from a magnetic monopole is assumed to be distributed isotropically in the space. The contribution of a magnetic monopole to the Wilson loop average depends on the location of the monopole relative to the Wilson loop.

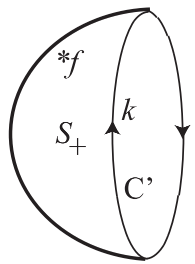
The vortex condensation picture gives an easy way to understand the area law.

§ Vortex from magnetic monopole

We want to find a $(D-1)$-dimensional geometric object V_{D-1} such that
i) V_{D-1} has an intersection with the Wilson loop C as the one-dimensional object,
ii) V_{D-1} has relevance to the magnetic-monopole current as the $(D-3)$-form k.
Suppose that k has the support on the closed $(D-3)$-dimensional subspace $C_{D-3}{}'$:

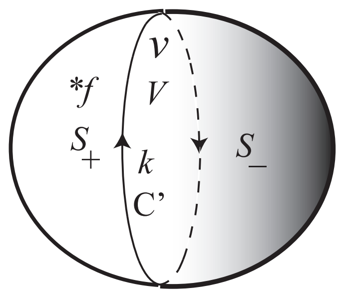
$$k_{\mu_1\ldots\mu_{D-3}}(x; C'_{D-3}) := \Phi \oint_{C'_{D-3}} d^{D-3}\sigma_{\mu_1\ldots\mu_{D-3}}\delta^D(x-\bar{x}(\sigma)). \tag{1}$$

First, we consider the **dual field strength** $({}^*f_2)^{\mu_1\cdots\mu_{D-2}}(x; S_{D-2})$ representing the magnetic flux, which has the support on an open $(D-2)$-dimensional surface S_{D-2} bounded by C'_{D-3}, $\partial S_{D-2} = C'_{D-3}$,



$$k_{D-3}(x, C'_{D-3}) = \delta({}^*f_2)_{D-2}(x, S_{D-2}). \tag{2}$$

Next, we consider the **ideal vortex field** $v^\mu(x; V_{D-1})$ which has the support only on the $(D-1)$-dimensional volume V_{D-1} whose boundary is S_{D-2}: $\partial V_{D-1} = S_{D-2}$ and gives the field strength $f^{\mu\nu}(x; S_{D-2})$:



$$f_2(x, S_{D-2}) = dv_1(x, V_{D-1}). \tag{3}$$

Note that the ideal vortex field v must be singular or **non-orientable**. Otherwise, the magnetic-monopole current becomes trivial $k = 0$, since $k = \delta^* f = \delta^* dv = {}^*ddv = 0$.

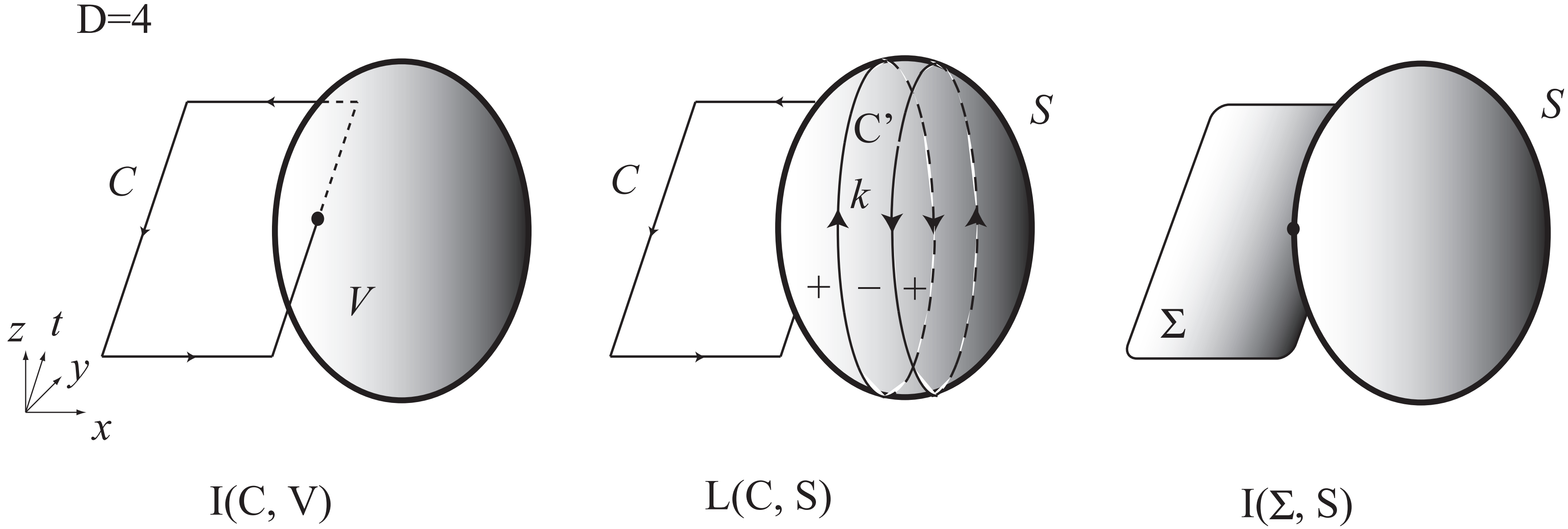


Figure 5: In $D = 4$ dimensional spacetime, it is assumed that the magnetic flux emanating from the magnetic-monopole current k is constrained on a surface to form the vortex surface S. A closed vortex surface S consists of a number of connected pieces S_n $(S = \cup_n S_n)$, and each piece S_n has the magnetic-monopole current k at its boundary ∂S_n.

The ideal vortex field $v^\mu(x; V_{D-1})$ is not unique, and it can be gauge transformed to a **thin vortex field** $s^\mu(x; S_{D-2})$ with the support only on the boundary $S_{D-2} = \partial V_{D-1}$ of V_{D-1}:

$$s(x; S_{D-2}) = v(x; V_{D-1}) + iU(x; V_{D-1})dU^\dagger(x; V_{D-1}), \quad (S_{D-2} = \partial V_{D-1}) \tag{4}$$

It is shown

$$U(x; V_{D-1}) = \exp\left[i\Phi\Omega(x; V_{D-1})\right], \tag{5}$$

where $\Omega(x; V_{D-1})$ is the solid angle taken up by the volume V_{D-1} when viewed from x:

$$\Omega(x; V_{D-1}) := \frac{-1}{A_{D-1}} \int_{V_{D-1}} d^{D-1}\tilde{\sigma}_\mu \, \frac{x^\mu - \bar{x}^\mu(\sigma)}{[(x^\mu - \bar{x}(\sigma))^2]^{D/2}}, \quad A_{D-1} := \frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{6}$$

with A_{D-1} being the area of the unit sphere S^{D-1} in D dimensions, e.g., $A_2 = 4\pi$, $A_3 = 2\pi^2$.
In other words, s^μ and v^μ are gauge equivalent.

Finally, we find that the surface integral of $f(x; S_{D-2})$ over Σ bounded by the Wilson loop C is equivalent to the line integral of $s(x; S_{D-2})$ along the closed loop C:

$$\int_\Sigma f(x; S_{D-2}) = \int_\Sigma dv(x; V_{D-1}) = \int_\Sigma ds(x; S_{D-2}) = \oint_C s(x; S_{D-2}), \tag{7}$$

since the contribution from the last term $iU(x; V_{D-1})dU^\dagger(x; V_{D-1}) = \Phi d\Omega(x; V_{D-1})$ vanishes for any closed loop C. The line integral $\oint_C s$ can be explicitly calculated.

For $D = 3$, the explicit calculation leads to

$$\oint_C dx^\nu s_\nu(x; S_1) = \Phi \int_\Sigma d^2\tilde{\sigma}_\rho(x) \int_{S_1 = \partial V_2} d\sigma_\rho \delta^3(x - \bar{x}(\sigma)) = \Phi I(\Sigma, S_1), \qquad (8)$$

Here I is the intersection number between the Wilson surface Σ and the vortex loop S_1.

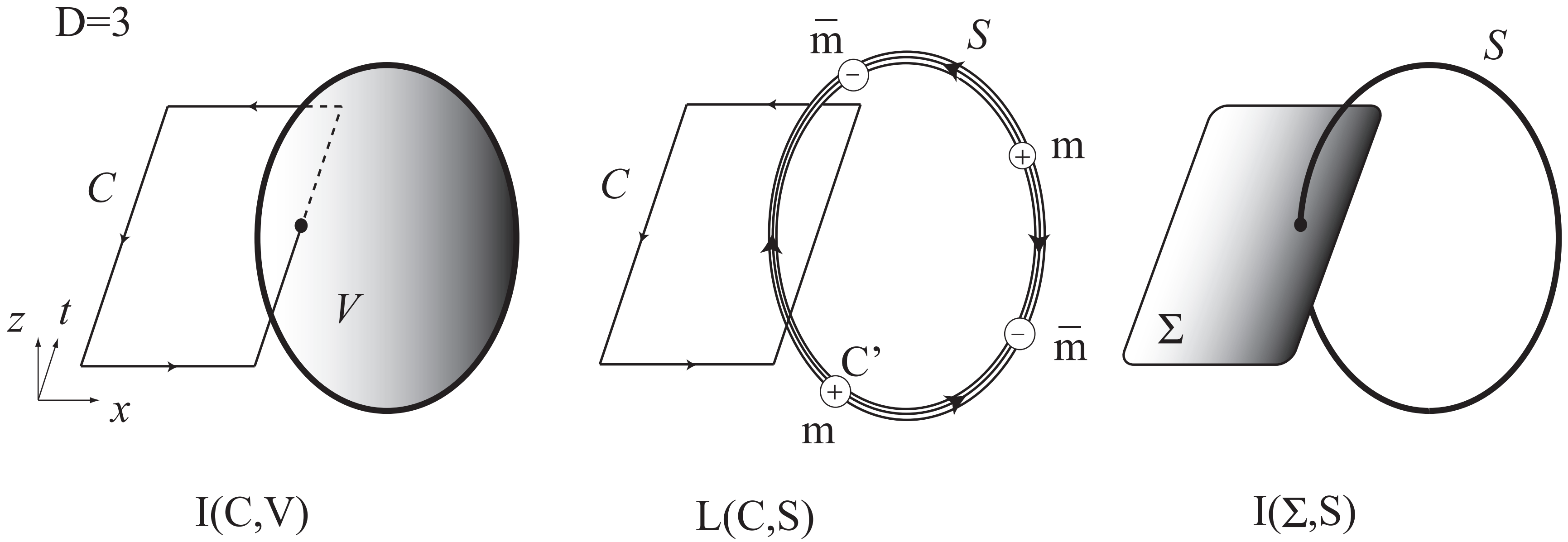


Figure 6: In $D = 3$ dimensional spacetime, it is assumed that the magnetic field emanating from a magnetic monopole m or an antimonopole $\bar{m}$ is squeezed into the flux tube to form the vortex line S.

The intersection numbers and the linking number are integer valued:

$$I(\Sigma_2, S_1) = I(C_1 = \partial\Sigma_2, V_2) = L(C_1, S_1) \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}. \tag{9}$$

For $D = 3$, the contribution to the $SU(2)$ Wilson loop operator from the vortices is

$$W_C^{\mathrm{vor}} := \exp\left[iJg_{\mathrm{YM}} \int_\Sigma f(x, S) \right] = \exp\left[iJg_{\mathrm{YM}} \Phi I(\Sigma, S_1) \right]. \tag{10}$$

If the magnetic flux carries a unit of the magnetic flux according to the quantization condition:

$$\Phi = 2\pi g_{\mathrm{YM}}^{-1}, \tag{11}$$

then a vortex as the tube of the magnetic flux contributes the factor (an element of the center group) to the $SU(2)$ Wilson loop operator:

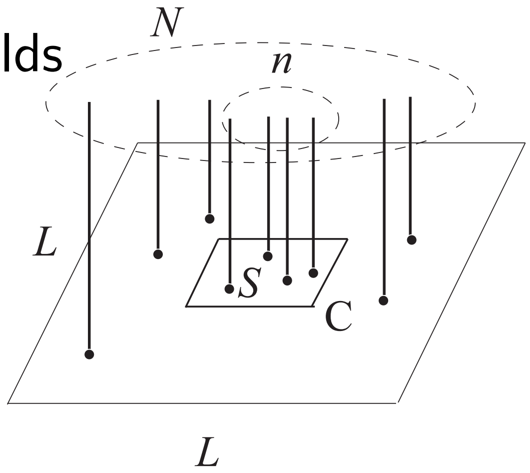
$$W_C^{\mathrm{vor}} = e^{iJ2\pi L} = (-1)^{2JL} \in Z(2), \quad L = I(\Sigma_2, S_1) = I(C_1 = \partial\Sigma_2, V_2) = L(C_1, S_1). \tag{12}$$

This is the most non-trivial contribution, which corresponds to the half of the total magnetic flux emanating from a point magnetic charge.

Vortex picture towards the area law

Let us assume that the vacuum is filled with percolating thin vortices. Suppose that N random vortices intersect a plane of area L^2. Each intersection multiplicatively contributes a factor $(-1)^{2J}$ to the Wilson loop average and n intersections within the loop take the value $(-1)^{2Jn}$. Then the probability that n of the intersections occur within an area S spanned by a Wilson loop is given by $(-1)^{2Jn}\left(\frac{S}{L^2}\right)^n (+1)^{N-n}\left(1-\frac{S}{L^2}\right)^{N-n}$.

Summing over all possibilities with the proper binomial weight yields



$$
\begin{aligned}
W_C &= \sum_{n=0}^{N} \binom{N}{n} (-1)^{2Jn}\left(\frac{S}{L^2}\right)^n (+1)^{N-n}\left(1-\frac{S}{L^2}\right)^{N-n} \\
&= \left(1-\frac{S}{L^2}+(-1)^{2J}\frac{S}{L^2}\right)^N \\
&= \left(1-\frac{[1-(-1)^{2J}]\rho S}{N}\right)^N \to \exp\{-\sigma_J S\} \quad (N\to\infty), \quad \rho := \frac{N}{L^2}, \\
\sigma_J &= [1-(-1)^{2J}]\rho = \begin{cases} \sigma_F = 2\rho & (J=\frac{1}{2},\frac{3}{2},\cdots) \\ 0 & (J=1,2,\cdots) \end{cases}, \qquad (13)
\end{aligned}
$$

where L has been eliminated in favor of the planar vortex density $\rho := N/L^2$.

The limit of a large $N \to \infty$ is taken with a constant ρ. Thus one obtains an area law for the Wilson loop average with the string tension σ_J determined by the vortex density ρ.
The crucial assumption in this argument is the independence of the intersection points. The asymptotic string tensions are zero for all integer-J representations (with N-ality or "biality" equal to 0), while they are nonzero and equal for all half-integer J representations (with N-ality being equal to 1).

We construct the vortex such that the vortex ends at the magnetic monopole, that is to say, the non-Abelian orientational zero modes of the vortex endow the endpoint magnetic monopole and antimonopole with same CP^{N-1} zero modes.

Such a vortex is obtained in the Higgs phase of a $U(N)$ gauge theory with $SU(N)$ flavor symmetry where the vortices with non-Abelian CP^{N-1} orientational zero modes exist. This is a candidate of the dual gauge theory for describing the magnetic flux tube as a vortex. This is different from the dual Abelian-Higgs model with the magnetic gauge symmetry $U(1)^{N-1}$ suggested from the Abelian projection.

§ Non-Abelian vortex in a gauge-Higgs model

We consider a $U(N)$ non-Abelian gauge theory coupled to N_f flavors of the Higgs fields in the fundamental representation of $U(N)$:

$$\mathscr{L} = -\frac{1}{2}\mathrm{tr}[\mathscr{B}_{\mu\nu}\mathscr{B}^{\mu\nu}] + \mathrm{tr}[(D_\mu H)(D^\mu H)^\dagger] - \frac{\lambda}{4}\mathrm{tr}[(HH^\dagger - v^2\mathbf{1}_{N_c})^2], \tag{1}$$

where $D_\mu := \partial_\mu - ig\mathscr{B}_\mu$ and $\mathscr{B}_{\mu\nu} := \partial_\mu\mathscr{B}_\nu - \partial_\nu\mathscr{B}_\mu - ig[\mathscr{B}_\mu, \mathscr{B}_\nu]$. Here the gauge field $\mathscr{B}$ is denoted by an $N \times N$ matrix and the Higgs field H is an $N \times N_f$ matrix. This model has the $U(N)$ gauge symmetry and $SU(N_f)$ flavor symmetry,

$$\begin{aligned} &H(x) \to U(x)H(x)U_f, \quad U(x) \in U(N)^{\mathrm{local}}, U_f \in SU(N_f)^{\mathrm{global}}, \\ &\mathscr{B}_\mu(x) \to U(x)(\mathscr{B}_\mu(x) + ig^{-1}\partial_\mu)U(x)^\dagger. \end{aligned} \tag{2}$$

For $N_f = N$, we choose the vacuum at

$$H = v\mathbf{1}_{N_c} = \mathrm{diag}(v, v, ..., v). \tag{3}$$

Then the color-flavour symmetry is spontaneously broken down to the color-flavour locked symmetry in the vacuum:

$$U(N)^{\mathrm{local}} \times SU(N)^{\mathrm{global}} \to SU(N)^{\mathrm{global}}_{c+f}. \tag{4}$$

Here the Higgs field transforms as

$$H(x) \to UH(x)U_f,\ U_f = U^\dagger, \quad U \in SU(N)^{\text{global}},\ U_f \in SU(N)^{\text{global}}. \tag{5}$$

As the color-flavour symmetry $U(N) \times SU(N)$ is spontaneously broken down to the color-flavor locked symmetry $SU(N)_{c+f}$, the N^2 massless **Nambu-Goldstone particles** associated with the **spontaneous symmetry breaking** $U(N)^{\text{local}} \times SU(N)^{\text{global}} \to SU(N)^{\text{global}}_{c+f}$ are absorbed into the gauge bosons and the gauge bosons acquire the mass $m_{\mathscr{B}} = gv$ due to the **Brout-Englert-Higgs mechanism**, while the other N^2 scalars are also massive $m_H = \sqrt{\lambda}v$.

This model can have the **vortex solution**. This is suggested from the fact that the $U(N)$ group is identified with

$$U(N) \cong \frac{SU(N) \times U(1)}{\mathbb{Z}_N}, \tag{6}$$

and has the nontrivial first Homotopy group:

$$\Pi_1 (U(N)) = \Pi_1 \left(\frac{SU(N) \times U(1)}{\mathbb{Z}_N} \right) = \mathbb{Z}. \tag{7}$$

This is contrast to the fact $\Pi_1 (SU(N)) = 0$. The reason $SU(N) \times U(1)$ is divided by $\mathbb{Z}_N$ is to avoid the double counting, since $\mathbb{Z}_N$ is contained in both $SU(N)$ and $U(1)$. This is notified by e.g., [Auzzi et al. (2004)]In fact, this model has the vortex solution.

In the Abelian case $N = N_f = 1$, the vortex is called the Abrikosov-Nielsen-Olesen (ANO) vortex. The characteristic size of the ANO vortex is estimated to be of the order $(gv)^{-1}$. The gauge symmetry $U(1)$ is completely broken in this model.

In the non-Abelian case $N = N_f \geq 2$, a single vortex solution can be constructed by embedding an ANO vortex solution (B_*, H_*) of the Abelian case into those of the non-Abelian case:

$$H(x) = U(x)\mathrm{diag}(H_*(x), v, ..., v)U(x)^\dagger, \ B(x) = U(x)\mathrm{diag}(B_*(x), 0, ..., 0)U(x)^\dagger. \tag{8}$$

In the presence of vortices, the color-flavour locked symmetry is further broken:

$$SU(N)^{\mathrm{global}}_{c+f} \to SU(N-1)^{\mathrm{global}} \times U(1)^{\mathrm{global}} = U(N-1)^{\mathrm{global}}. \tag{9}$$

Here U takes the value in a coset space, the projective space:

$$\frac{SU(N)_{c+f}}{SU(N-1) \times U(1)} = \frac{SU(N)_{c+f}}{U(N-1)} \cong CP^{N-1}. \tag{10}$$

It parameterizes the orientation of the non-Abelian vortex in the internal space whose moduli are called the **orientational moduli** (**orientational zeromodes**). The orientational moduli space is CP^{N-1}. The broken part is identified with the **Nambu-Goldstone modes** localized in the non-Abelian vortex associated with the spontaneous symmetry breaking $SU(N) \to U(N-1)$.
[Auzzi et al.(2003)] [Hanany & Tong(2003)] [Shifman & Yung(2004)] [Eto et al.(2006)]

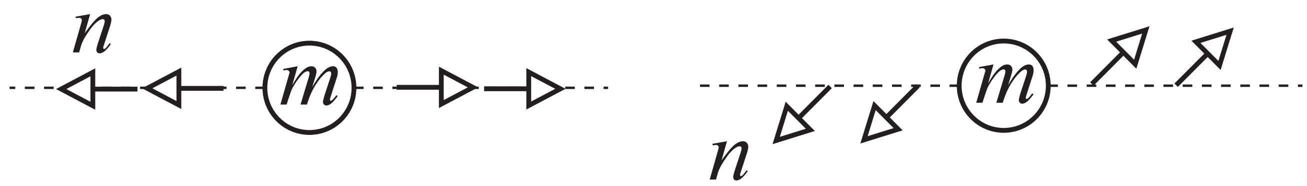


Figure 7: Two vortices with a magnetic monopole m as the junction. Each vortex is described by a different patch discriminated by an internal orientation vector $\vec{\phi}_0$. The color direction field $\boldsymbol{n} = (n^1, n^2, n^3)$ on each vortex has the opposite direction.

§ Conclusion and discussion

1. [Magnetic monopole picture] We have given a gauge-invariant definition of the magnetic monopole in the $SU(N)$ Yang-Mills theory in the absence of the scalar field. This is achieved through a non-Abelian Stokes theorem for the Wilson loop operator.
   This enables one to estimate the magnetic monopole contribution to the Wilson loop average, which confirms the magnetic monopole dominance in the string tension. See Shibata's talk for the numerical simulations on a lattice.

2. As a dual or complementary point of view, we can define a gauge-equivalent thin vortex (non-orientable) which has the magnetic monopole and the anti-magnetic monopole as the boundaries.

3. [Vortex picture] We have discussed the vortex solution in the Higgs phase of a gauge-Higgs model with $U(N)$ gauge fields and N Higgs fields. The model fulfills the requirement that a non-Abelian vortex has the same $\boldsymbol{C}P^{N-1}$ moduli space as those of the non-Abelian magnetic monopole. The non-Abelian magnetic monopole is regarded as a kink making a junction with two non-Abelian vortices with different internal orientation moduli.
   The internal moduli will be important to understand the type I dual superconductivity.

Thank you very much for your attention.

§ String tension and the vortex solution

We assume that the vortices are equally spaced and that the average distance between two vortices is $2R$. Therefore, the **string tension** σ is roughly related to R as

$$\sigma = 2\rho = 2\frac{N}{L^2} \cong 2\frac{1}{\pi R^2}. \tag{11}$$

Therefore, R is estimated as

$$R^{-1} \cong \sqrt{\frac{\pi}{2}}\sqrt{\sigma} = 1.25 \times 440(\mathrm{MeV}) = 550(\mathrm{MeV}) \Longrightarrow R = 0.36(\mathrm{fm}). \tag{12}$$

The value of R is larger than the distance of the onset of the linear potential $r \cong 0.2$(fm). Therefore, this estimation is valid for a relatively large Wilson loop, i.e., the large interquark distancee $r \gg R$.

In the Abelian-Higgs model, two vortices are attractive in the type I, repulsive for the type II, and there are no force between two vortices in the BPS limit. In the Abelian case, therefore, the ordinary **superconductivity** must be type II or BPS. Otherwise, the initial configuration of the vortices will be eventually collapsed by the attractive force among the vortices. In fact, this is consistent with a fact that the vortices of magnetic flux tube form the **Abrikosov lattice** (hexagonal lattice rather than forming a square lattice) in the superconductor as stable configuration, which is verified experimentally.

In the non-Abelian case, recent numerical simulations show that the Yang-Mills vacuum is **type I**, in contrast to the preceding results which claim the border between type I and type II, i.e., BPS limit.

For $SU(2)$, [Kato et al.(2014)] reports

$$\kappa = 0.48 \pm 0.07,\ \delta = 0.12\mathrm{fm},\ \xi = 0.25\mathrm{fm}.$$

For $SU(3)$, [Shibata et al.(2013)] reports

$$\kappa = 0.45 \pm 0.01,\ \delta = 0.12\mathrm{fm},\ \xi = 0.27\mathrm{fm}.$$

The **type I dual superconductivity** for the Yang-Mills vacuum yields the attractive force between two non-Abelian vortices. How this result is consistent with the above considerations? The crucial point is that the non-Abelian vortices have internal degrees of freedom, i.e., orientational moduli, in addition to the degrees of freedom related to the positions in space. This has a possibility for preventing the vortices from collapse due to the attractive force. while the Abelian vortices have only the positions and the collapse of the lattice structure for the Abelian vortices in the type I superconductor will not be avoidable. We must examine the interaction among vortices depending on the orientational moduli in more detail.

§ Field decomposition a la Cho-Duan-Ge-Faddeev-Niemi

We look for the **gauge covariant decomposition**,

$$\mathscr{A}'_\mu(x) = \mathscr{V}'_\mu(x) + \mathscr{X}'_\mu(x). \tag{1}$$

For the condition (ii) to be gauge covariant, the transformation of the color field $\boldsymbol{n}$ given by

$$g(x) \to U(x)g(x) \Longrightarrow \boldsymbol{n}(x) \to \boldsymbol{n}'(x) = U(x)\boldsymbol{n}(x)U^\dagger(x). \tag{2}$$

requires that $\mathscr{X}_\mu(x)$ transforms as an adjoint (matter) field:

$$\mathscr{X}_\mu(x) \to \mathscr{X}'_\mu(x) = U(x)\mathscr{X}_\mu(x)U^\dagger(x). \tag{3}$$

This immediately means that $\mathscr{V}_\mu(x)$ must transform just like the original gauge field $\mathscr{A}_\mu(x)$:

$$\mathscr{V}_\mu(x) \to \mathscr{V}'_\mu(x) = U(x)\mathscr{V}_\mu(x)U^\dagger(x) + ig_{\mathrm{YM}}^{-1}U(x)\partial_\mu U^\dagger(x), \tag{4}$$

since $\mathscr{A}_\mu(x) \to \mathscr{A}'_\mu(x) = U(x)\mathscr{A}_\mu(x)U^\dagger(x) + ig_{\mathrm{YM}}^{-1}U(x)\partial_\mu U^\dagger(x)$.

These transformation properties impose restrictions on the requirement to be imposed on the restricted field $\mathscr{V}_\mu(x)$. Such a candidate is [covariant constantness of the color field] [which we call the first **defining equation**]:

$$\text{(I)} \quad \mathscr{D}_\mu[\mathscr{V}]\boldsymbol{n} = 0 \quad (\mathscr{D}_\mu[\mathscr{V}] := \partial_\mu - ig_{\mathrm{YM}}[\mathscr{V}_\mu, \cdot]), \tag{5}$$

since the covariant derivative transforms in the adjoint way: $\mathscr{D}_\mu[\mathscr{V}(x)] \to U(x)(\mathscr{D}_\mu[\mathscr{V}](x))U^\dagger(x)$.

For $G = SU(2)$, it is shown that the two conditions (I) and (ii),[the **defining equations** for the decomposition] are compatible and determine the decomposition uniquely:

$$
\begin{aligned}
\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x),& \\
\mathscr{V}_\mu(x) =& c_\mu(x)\boldsymbol{n}(x) + ig_{\rm YM}^{-1}[\boldsymbol{n}(x), \partial_\mu \boldsymbol{n}(x)], \quad c_\mu(x) := \mathscr{A}_\mu(x)\cdot\boldsymbol{n}(x), \\
\mathscr{X}_\mu(x) =& - ig_{\rm YM}^{-1}[\boldsymbol{n}(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{n}(x)].
\end{aligned}
\tag{6}
$$

This is the same as the **Cho–Duan-Ge (CDG) decomposition** or **Cho–Duan-Ge–Faddeev-Niemi (CDGFN) decomposition** [Cho(1980), Duan-Ge (1979), Faddeev-Niemi (1998)].

The condition (I) means that the field strength $\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)$ of the field $\mathscr{V}_\mu(x)$ and $\boldsymbol{n}(x)$ commute:

$$
[\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x), \boldsymbol{n}(x)] = 0. \tag{7}
$$

This follows from the identity:

$$
[\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}, \boldsymbol{n}] = ig_{\rm YM}^{-1}[\mathscr{D}_\mu^{[\mathscr{V}]}, \mathscr{D}_\nu^{[\mathscr{V}]}]\boldsymbol{n}, \tag{8}
$$

which is derived from

$$
\mathscr{F}_{\mu\nu}^{[\mathscr{V}]} = ig_{\rm YM}^{-1}[\mathscr{D}_\mu^{[\mathscr{V}]}, \mathscr{D}_\nu^{[\mathscr{V}]}], \quad \mathscr{D}_\mu^{[\mathscr{V}]} := \partial_\mu - ig_{\rm YM}[\mathscr{V}_\mu, \cdot]. \tag{9}
$$

For $SU(2)$, (7) means that $\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)$ is proportional to $\boldsymbol{n}(x)$:

$$
\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x) = f_{\mu\nu}(x)\boldsymbol{n}(x) \Longrightarrow f_{\mu\nu}(x) = \boldsymbol{n}(x)\cdot\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x) = 2\mathrm{tr}[\boldsymbol{n}(x)\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)], \tag{10}
$$

§ Field decomposition for $SU(N)$: new options

For $G = SU(N)$ $(N \geq 3)$, (I) and (ii) are not sufficient to uniquely determine the decomposition. For $G = SU(N)$ $(N \geq 3)$, the condition (ii) must be modified: [Kondo, Shinohara & Murakami(2008)]
(II) $\mathscr{X}^\mu(x)$ does not have the $\tilde{H}$-commutative part, i.e., $\mathscr{X}^\mu(x)_{\tilde{H}} = 0$:

$$\text{(II)} \quad 0 = \mathscr{X}^\mu(x)_{\tilde{H}} := \mathscr{X}^\mu(x) - \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{X}^\mu(x)]]$$

$$\Longleftrightarrow \mathscr{X}^\mu(x) = \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{X}^\mu(x)]]. \tag{1}$$

This condition is also gauge covariant. Note that the condition (ii) follows from (II). For $G = SU(2)$, i.e., $N = 2$, the condition (II) reduces to (ii). By solving (I) and (II), $\mathscr{X}_\mu(x)$ and $\mathscr{V}_\mu(x)$ are determined

$$\mathscr{X}_\mu(x) = -ig_{\rm YM}^{-1}\frac{2(N-1)}{N}[\boldsymbol{n}(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{n}(x)] \in Lie(G/\tilde{H}) = su(N)/u(N-1), \tag{2}$$

$$\mathscr{V}_\mu(x) = \mathscr{C}_\mu(x) + \mathscr{B}_\mu(x) \in \mathscr{L}ie(G) = su(N),$$

$$[\mathscr{C}_\mu(x), \boldsymbol{n}(x)] = 0 \Longleftrightarrow \mathscr{C}_\mu(x) \times \boldsymbol{n}(x) = 0.$$

$$\mathrm{tr}[\mathscr{B}_\mu(x)\boldsymbol{n}(x)] = 0 \Longleftrightarrow \mathscr{B}_\mu(x) \cdot \boldsymbol{n}(x) = 0.$$

$$\mathscr{C}_\mu(x) = \mathscr{A}_\mu(x) - \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{A}_\mu(x)]] \in \mathscr{L}ie(\tilde{H}) = u(N-1),$$

$$\mathscr{B}_\mu(x) = ig_{\rm YM}^{-1}\frac{2(N-1)}{N}[\boldsymbol{n}(x), \partial_\mu\boldsymbol{n}(x)] \in \mathscr{L}ie(G/\tilde{H}) = su(N)/u(N-1), \tag{3}$$

§ Reformulating Yang-Mills theory using new variables

We consider the change of variables from $\mathscr{A}_\mu$ to new field variables $\mathscr{C}_\mu$, $\mathscr{X}_\mu$ and $\boldsymbol{n}$: (See [Kondo, Murakami and Shinohara (2005)] for $SU(2)$, and [Kondo, Shinohara and Murakami (2008)] for $SU(N)$)

$$\mathscr{A}_\mu^A \Longrightarrow (\boldsymbol{n}^\beta, \mathscr{C}_\mu^k, \mathscr{X}_\mu^b), \tag{1}$$

- $\mathscr{A}_\mu \in Lie(G) \to \#[\mathscr{A}_\mu^A] = D \cdot \mathrm{dim}G = D(N^2-1)$
- $\mathscr{C}_\mu \in Lie(\tilde{H}) = u(N-1) \to \#[\mathscr{C}_\mu^k] = D \cdot \mathrm{dim}\tilde{H} = D(N-1)^2$
- $\mathscr{X}_\mu \in Lie(G/\tilde{H}) \to \#[\mathscr{X}_\mu^b] = D \cdot \mathrm{dim}(G/\tilde{H}) = D(2N-2)$
- $\boldsymbol{n} \in Lie(G/\tilde{H}) \to \#[\boldsymbol{n}^\beta] = \mathrm{dim}(G/\tilde{H}) = 2(N-1)$.

The new theory written in terms of new variables $(\boldsymbol{n}^\beta, \mathscr{C}_\mu^k, \mathscr{X}_\mu^b)$ has the $2(N-1)$ extra degrees of freedom. Therefore, we must give a procedure for eliminating the $2(N-1)$ extra degrees of freedom to obtain the new theory which is equipollent to the original one.
For this purpose, we impose $2(N-1)$ constraints $\boldsymbol{\chi} = 0$, which we call the **reduction condition**:

- $\boldsymbol{\chi} \in Lie(G/\tilde{H}) \to \#[\boldsymbol{\chi}^a] = \mathrm{dim}(G/\tilde{H}) = 2(N-1) = \#[\boldsymbol{n}^\beta]$.

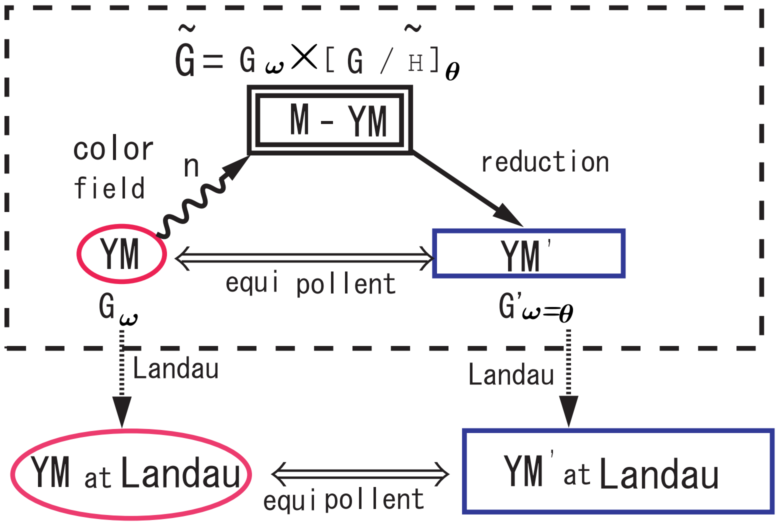


Figure 8: The relationship between the original Yang-Mills (YM) theory and the reformulated Yang-Mills (YM') theory. A single color field $\boldsymbol{n}$ is introduced to enlarge the original Yang-Mills theory with a gauge group G into the master Yang-Mills (M-YM) theory with the enlarged gauge symmetry $\tilde{G} = G \times G/\tilde{H}$. The reduction conditions are imposed to reduce the master Yang-Mills theory to the reformulated Yang-Mills theory with the equipollent gauge symmetry G'. In addition, we can impose any over-all gauge fixing condition, e.g., Landau gauge to both the original YM theory and the reformulated YM' theory.

- Enlarged gauge symmetry by introducing n and the reduction by imposing χ

$$G \overset{\boldsymbol{n}}{\to} G \times G/\tilde{H} \overset{\boldsymbol{\chi}}{\to} G. \tag{2}$$

A choice of the reduction condition in the minimal option is to minimize the functional $F_{\rm red}[\mathscr{A}, \boldsymbol{n}]$:

$$F_{\rm red}[\mathscr{A}, \boldsymbol{n}] = \int d^D x \frac{1}{2} g^2 \mathscr{X}_\mu \cdot \mathscr{X}^\mu = \frac{N-1}{N} \int d^D x (D_\mu[\mathscr{A}]\boldsymbol{n})^2,$$

with respect to the enlarged gauge transformation:

$$\begin{aligned} \delta \mathscr{A}_\mu &= D_\mu[\mathscr{A}]\boldsymbol{\omega} \quad (\boldsymbol{\omega} \in \mathscr{L}ie(G)), \\ \delta \boldsymbol{n} &= ig[\boldsymbol{n}, \boldsymbol{\theta}] = ig[\boldsymbol{n}, \boldsymbol{\theta}_\perp] \quad (\boldsymbol{\theta}_\perp \in \mathscr{L}ie(G/\tilde{H})). \end{aligned} \tag{3}$$

In fact, the enlarged gauge transformation of the functional $F_{\rm red}[\mathscr{A}, \boldsymbol{n}]$ is

$$\delta F_{\rm red}[\mathscr{A}, \boldsymbol{n}] = \delta \int d^D x \frac{1}{2} (D_\mu[\mathscr{A}]\boldsymbol{n})^2 = g \int d^D x (\boldsymbol{\theta}_\perp - \boldsymbol{\omega}_\perp) \cdot i[\boldsymbol{n}, D^\mu[\mathscr{A}] D_\mu[\mathscr{A}]\boldsymbol{n}], \tag{4}$$

where $\boldsymbol{\omega}_\perp$ denotes the component of $\boldsymbol{\omega}$ in the direction $\mathscr{L}(G/\tilde{H})$.
For $\boldsymbol{\omega}_\perp = \boldsymbol{\theta}_\perp$ (diagonal part of $G \times G/\tilde{H}$) $\delta F_{\rm red}[\mathscr{A}, \boldsymbol{n}] = 0$ imposes no condition, while for $\boldsymbol{\omega}_\perp \neq \boldsymbol{\theta}_\perp$ (off-diagonal part of $G \times G/\tilde{H}$) it implies the constraint:

$$\boldsymbol{\chi}[\mathscr{A}, \boldsymbol{n}] := [\boldsymbol{n}, D^\mu[\mathscr{A}] D_\mu[\mathscr{A}]\boldsymbol{n}] \equiv 0. \tag{5}$$

Note that the number of constraint is $\#[\boldsymbol{\chi}] = \dim(G \times G/\tilde{H}) - \dim(G) = \dim(G/\tilde{H})$ as desired.
Finally, we have an equipollent Yang-Mills theory with **the residual local gauge symmetry** $G' := SU(N)^{\rm local}_{\omega'}$ with the gauge transformation parameter:

$$\boldsymbol{\omega}'(x) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\omega}_\perp(x)) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\theta}_\perp(x)), \quad \boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x). \tag{6}$$

	original YM	$\Longrightarrow$ reformulated YM
field variables	$\mathscr{A}_\mu^A \in \mathscr{L}(G)$	$\Longrightarrow \boldsymbol{n}^\beta, \mathscr{C}_\nu^k, \mathscr{X}_\nu^b$
action	$S_{\rm YM}[\mathscr{A}]$	$\Longrightarrow \tilde{S}_{\rm YM}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}]$
integration measure	$\mathcal{D}\mathscr{A}_\mu^A$	$\Longrightarrow \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J}\delta(\tilde{\boldsymbol{\chi}})\Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}]$

At the same time, the color field

$$\boldsymbol{n}(x) \in \mathscr{L}ie(G/\tilde{H})$$

must be obtained by solving the reduction condition $\boldsymbol{\chi} = 0$ for a given $\mathscr{A}$, e.g.,

$$\boldsymbol{\chi}[\mathscr{A}, \boldsymbol{n}] := [\boldsymbol{n}, D^\mu[\mathscr{A}]D_\mu[\mathscr{A}]\boldsymbol{n}] \in \mathscr{L}ie(G/\tilde{H}). \tag{7}$$

Here $\tilde{\boldsymbol{\chi}} = 0$ is the reduction condition written in terms of the new variables:

$$\tilde{\boldsymbol{\chi}} := \tilde{\boldsymbol{\chi}}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}] := D^\mu[\mathscr{V}]\mathscr{X}_\mu, \tag{8}$$

and $\Delta_{\rm FP}^{\rm red}$ is the Faddeev-Popov determinant associated with the reduction condition:

$$\Delta_{\rm FP}^{\rm red} := \det\left(\frac{\delta\boldsymbol{\chi}}{\delta\boldsymbol{\theta}}\right)_{\boldsymbol{\chi}=0} = \det\left(\frac{\delta\boldsymbol{\chi}}{\delta\boldsymbol{n}^\theta}\right)_{\boldsymbol{\chi}=0}. \tag{9}$$

which is obtained by the BRST method as $\Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, c, \mathscr{X}] = \det\{-D_\mu[\mathscr{V}+\mathscr{X}]D_\mu[\mathscr{V}-\mathscr{X}]\}$.
The Jacobian $\tilde{J}$ is very simple, irrespective of the choice of the reduction condition:

$$\tilde{J} = 1. \tag{10}$$

[Kondo, Shinohara & Murakami, Prog.Theor.Phys. **120**, 1–50 (2008). arXiv:0803.0176]

The Wilson loop average in the original theory:

$$W(C) := \langle W_C[\mathscr{A}]\rangle_{\rm YM} = Z_{\rm YM}^{-1}\int \mathcal{D}\mathscr{A}\, e^{-S_{\rm YM}[\mathscr{A}]} W_C[\mathscr{A}]. \tag{11}$$

is defined in the reformulated Yang-Mills theory:

$$\begin{aligned}
\langle W_C[\mathscr{A}]\rangle_{\rm YM'} =& Z_{\rm YM'}^{-1}\int [d\mu(g)]\int \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J}\delta(\tilde{\boldsymbol{\chi}})\Delta_{\rm FP}^{\rm red} e^{-\tilde{S}_{\rm YM}[\boldsymbol{n},\mathscr{C},\mathscr{X}]} \\
& \times \exp\left\{ i g_{\rm YM}\sqrt{\frac{2(N-1)}{N}}[(j, N_\Sigma) + (k, \omega_\Sigma)]\right\}, \\
Z_{\rm YM'} =& \int \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J}\delta(\tilde{\boldsymbol{\chi}})\Delta_{\rm FP}^{\rm red} e^{-S_{\rm YM'}[\boldsymbol{n},\mathscr{C},\mathscr{X}]}.
\end{aligned} \tag{12}$$

Remark:

1. For $SU(2)$, when we fix the color field $\mathbf{n}(x) = (0,0,1)$ or $\boldsymbol{n}(x) = \sigma_3/2$, the reduction condition $D^\mu[\mathscr{V}]\mathscr{X}_\mu = 0$ reduces to the conventional **Maximally Abelian gauge (MA gauge)**.
2. For $SU(3)$, this is not the case: This reduction does not reduce to the conventional Maximally Abelian gauge for $SU(3)$, even if the color field is fixed to be uniform. Therefore, the results to be obtained are nontrivial.
3. For $\Omega_\mu(x) := i g_{\rm YM}^{-1} g(x)\partial_\mu g^\dagger(x)$, $\partial_\mu \boldsymbol{n}(x) = i g_{\rm YM}[\Omega_\mu(x), \boldsymbol{n}(x)]$, i.e., $\mathscr{D}[\Omega]\boldsymbol{n}(x) = 0$.
   It is shown $\Omega_\mu(x) = \mathscr{B}_\mu(x) + a_\mu(x)$ where $[a_\mu(x), \boldsymbol{n}(x)] = 0$. Therefore, $\mathscr{D}[\mathscr{B}]\boldsymbol{n}(x) = 0$. For $\mathscr{V}_\mu(x) = \mathscr{B}_\mu(x) + \mathscr{C}_\mu(x)$, thus, $\mathscr{D}[\mathscr{V}]\boldsymbol{n}(x) = 0$, since $[\mathscr{C}_\mu(x), \boldsymbol{n}(x)] = 0$.