Holography and the conformal window in the Veneziano limit

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1. Brief introduction and motivation

2. Basic properties of V-QCD
   ▶ Definition of the model
   ▶ Conformal transition in V-QCD

3. Results and applications
   ▶ Miransky scaling
   ▶ Hyperscaling
   ▶ Light scalars
   ▶ The S-parameter
   ▶ Four fermion deformations
1. Introduction
Veneziano limit: large $N_f$, $N_c$ with $x = N_f/N_c$ fixed

In the Veneziano limit (discrete) $N_f$ replaced by (continuous) $x = N_f/N_c$

- Transition expected at some $x = x_c$

Computations near the transition difficult

- Schwinger-Dyson approach, ...
- Lattice QCD
- Holography (?) → This talk
A holographic bottom-up model for QCD in the Veneziano limit

- Bottom-up, but trying to follow principles from string theory as closely as possible

More precisely:

- Derive the model from five dimensional noncritical string theory with certain brane configuration
  ⇒ some things do not work (at small coupling)
- Fix model by hand and generalize → arbitrary potentials
- Tune model to match QCD physics and data
- Effective description of QCD

Last steps so far incomplete: model not yet tuned to match any QCD data!
2. V-QCD
Holographic V-QCD: the fusion

The fusion:

1. IHQCD: model for glue inspired by string theory (dilaton gravity)
   [Gursoy, Kiritsis, Nitti; Gubser, Nellore]

2. Adding flavor and chiral symmetry breaking via tachyon brane actions
   [Klebanov, Maldacena; Bigazzi, Casero, Cotrone, Iatrakis, Kiritsis, Paredes]

Consider 1. + 2. in the Veneziano limit with full backreaction
⇒ V-QCD models

Defining V-QCD

Degrees of freedom

- The tachyon $\tau$, and the dilaton $\lambda$
- $\lambda = e^\phi$ is identified as the 't Hooft coupling $g^2 N_c$
- $\tau$ is dual to the $\bar{q}q$ operator

$$S_{V-QCD} = N_c^2 M^3 \int d^5 x \sqrt{g} \left[ R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V_g(\lambda) \right]$$

$$- N_f N_c M^3 \int d^5 x V_f(\lambda, \tau) \sqrt{- \det(g_{ab} + \kappa(\lambda) \partial_a \tau \partial_b \tau)}$$

$$V_f(\lambda, \tau) = V_{f0}(\lambda) \exp(-a(\lambda)\tau^2)$$

$$ds^2 = e^{2A(r)}(dr^2 + \eta_{\mu\nu} x^\mu x^\nu)$$

Need to choose $V_{f0}$, $a$, and $\kappa$ . . .

A simple strategy works (!):

- Match to perturbative QCD in the UV (asymptotic AdS$_5$)
- Logarithmically modified string theory predictions in the IR
Choose reasonable potentials
Ansatz $\tau(r), \lambda(r), A(r)$ in equations of motion
Construct numerically all vacua (various IR geometries)

Desired phase diagram obtained:

- Matching to QCD perturbation theory $\rightarrow$ Banks-Zaks
- Conformal transition (BKT) at $x = x_c \approx 4$

(With tuned potentials, the phase diagram may change)
How does the phase structure arise?

Turning on a tiny tachyon in the conformal window

\[ \tau(r) \sim m_q r^{\gamma_* + 1} + \sigma r^{3-\gamma_*} \quad (IR, \; r \to \infty) \]

Breitenlohner-Freedman (BF) bound for \( \gamma_* \) at the IRFP

\[ (\gamma_* + 1)(3 - \gamma_*) = \Delta_* (4 - \Delta_*) = -m_\tau^2 \ell_*^2 \leq 4 \]

Violation of BF bound \( \Rightarrow \) instability \( \Rightarrow \) tachyon/chiral condensate

- \( \Rightarrow \) bound saturated at the conformal phase transition \((x = x_c)\)
- \( \gamma_* = 1 \) at the transition
- BF bound violation leads to a BKT transition quite in general
- Predictions near the transition to large extent independent of model details
3. Results
Energy scales (at zero quark mass)

V-QCD reproduces the picture with Miransky scaling:

1. **QCD regime**: single energy scale $\Lambda$

2. **Walking regime** ($x_c - x \ll 1$): two scales related by Miransky/BKT scaling law

$$\frac{\Lambda_{UV}}{\Lambda_{IR}} \sim \exp\left(\frac{\kappa}{\sqrt{x_c - x}}\right)$$

3. **Conformal window** ($x_c \leq x < 11/2$): again one scale $\Lambda$, but slow RG flow
Phase diagram: example at finite $T$

Phases on the $(x, T)$-plane

Loop effects may affect the order of the transition

In the conformal window all low lying masses obey the “hyperscaling” relations

\[ m \sim m_q^{\frac{1}{1+\gamma_*}} \quad (m_q \to 0) \]

\[ \langle \bar{q}q \rangle \sim m_q^{\frac{3-\gamma_*}{1+\gamma_*}} \quad (m_q \to 0) \]


- Appear independently of the details of the Lagrangian
- Also demonstrated in the “dynamic AdS/QCD” models

[Evans, Scott arXiv:1405.5373]
“Phase diagram” on the \((x, m_q)\)-plane:

Hyperscaling seen in “regime B”: extends to \(x < x_c\)
Example: masses for the walking case

\[ x_c - x \ll 1, \] Masses in units of IR (glueball) scale

\[ m/\Lambda_{IR} \]

- All masses have the same behavior at intermediate \( m_q \) (regime B)
- Meson masses enhanced wrt glueballs at large \( m_q \)
Meson mass ratios as a function of $x$

Lowest states of various sectors, normalized to $m_\rho$

All ratios tend to constants as $x \to x_c$: no technidilaton mode

Interpreting the absence of the dilaton

What have we shown?

▶ Violation of BF bound does not automatically yield a light dilaton..

▶ .. while Miransky scaling and hyperscaling relations are reproduced (GMOR and Witten-Veneziano relations also ok)

However . . .

▶ Analytic analysis: scalar fluctuations “critical” in the walking region, suggesting a light state

▶ But criticality not enough: presence of such a light state is sensitive to IR

Could this be a computational error or numerical issue?

▶ Scalar singlet fluctuations are a real mess ..

▶ .. but we did nontrivial checks and all results look reasonable

Notice: easy to obtain light (but not parametrically light) scalars
Discontinuity at $m_q = 0$ in the conformal window

Qualitative agreement with field theory expectations

[Sannino]
Scaling of the S-parameter

As $m_q \to 0$ in the conformal window,

$$S(m_q) \simeq S(0+) + c \left( \frac{m_q}{\Lambda_{\text{UV}}} \right)^{\frac{\Delta_{FF} - 4}{\gamma^* + 1}}$$

- Limiting value $S(0+) = \lim_{m_q \to 0^+} S(m_q)$ is finite and positive (while $S(0) = 0$)
- $\Delta_{FF}$ is the dimension of $\text{tr} F^2$ at the fixed point

\[
\frac{(S(m_q) - S(0+))}{N_c N_f}
\]
The dependence of $\sigma \propto \langle \bar{q}q \rangle$ on the quark mass

- For $x < x_c$ spiral structure

Dots: numerical data
Continuous line: (semi-)analytic prediction

Allows to study the effect of double-trace deformations
Four-fermion operators

Witten’s recipe: modified UV boundary conditions for the tachyon

For interaction term in field theory ($\mathcal{O} = \bar{q}q$)

$$W = -m_q \int d^4x \mathcal{O}(x) + \frac{g_2}{2} \int d^4x \mathcal{O}(x)^2$$

At zero $m_q$:

$g_2$

Chirally broken

Chirally symmetric

Chirally broken

$x_c$

$x_{BZ}$

$x$
> V-QCD agrees with field theory results for QCD at qualitative level
> Most results close to the conformal transition independent of details
> Next step: tuning the model to match quantitatively with experimental/lattice QCD data
Extra slides
V-QCD literature

An ongoing program for studying V-QCD

Exploring the model at qualitative level (good match with QCD!):

- Phase diagram at finite $T$ and $\mu$

- Fluctuation analysis: meson spectra, $S$-parameter, quasi normal modes...
  - [Iatrakis, Zahed arXiv:1410.8540]

- CP-odd terms: axial anomaly
  - [In progress with Arean, Iatrakis, Kiritsis]

- Phase diagram at finite quark mass

This talk: selected results relevant for technicolor

Also just started: quantitative fit to QCD data
The QCD string in the Veneziano limit

Quarks: \( N_f \)

Gluons: \[
\begin{array}{c}
\text{leading diagrams in } \frac{1}{N_c}:\\
gluonic with quark boundaries
\end{array}
\]

\[\text{['t Hooft]}\]

Veneziano limit \( \Rightarrow \) boundaries not suppressed \( \Rightarrow \) open string loops!

\[= \mathcal{O} \left( \frac{N_f}{N_c} \right)\]
“Improved holographic QCD” (IHQCD): well-tested string-inspired bottom-up model for pure Yang-Mills

\[ S_g = M^3 N_c^2 \int d^5x \sqrt{g} \left[ R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V_g(\lambda) \right] \]

with the metric

\[ ds^2 = e^{2A(r)} (dr^2 + \eta_{\mu\nu} x^\mu x^\nu) \]

- \( A \leftrightarrow \log \Lambda \)  energy scale
- \( \lambda = e^\phi \leftrightarrow 't \ Hooft \ coupling \ g^2 N_c \)
- Modify \( V_g \) derived from string theory to match Yang-Mills \( \beta \)-function in the UV (\( \lambda \to 0 \))
Example of fit to lattice data: interaction measure of Yang-Mills

Trace of the energy-momentum tensor
Second building block: Adding flavor

A recipe for adding quarks (in the fundamental of $SU(N_c)$ and in the probe approximation)

- Space-filling probe $D4 - \bar{D}4$ branes in 5D →
  - Tachyon $T \leftrightarrow \bar{q}q$
  - Gauge fields $A_{L/R}^\mu \leftrightarrow \bar{q}\gamma^\mu(1 \pm \gamma_5)q$

- For the vacuum structure only the tachyon is relevant
- Sen-like tachyon DBI action with $V_T \sim \exp(-|T|^2)$
  - Confining IR asymptotics of the geometry triggers ChSB
  - Gell-Mann-Oakes-Renner relation
  - Linear Regge trajectories for mesons
  - A very good fit of the light meson masses

[Klebanov,Maldacena]
Vector correlators and S-parameter

1. Introduce bulk gauge fields dual to vector operators

\[ A^{L/R}_\mu \leftrightarrow \bar{q}\gamma_\mu(1 \pm \gamma_5)q \]

2. Fluctuate full flavor action of V-QCD

\[ S_f = -\frac{1}{2} M^3 N_c \text{Tr} \int d^4x \, dr \left( V_f(\lambda, T^\dagger T) \sqrt{-\det A_L + (L \to R)} \right) \]

\[ A_{L/R \, MN} = g_{MN} + w(\lambda, T) F_{MN}^{(L/R)} + \]

\[ \kappa(\lambda, T) \frac{1}{2} \left[ (D_M T)^\dagger (D_N T) + (D_N T)^\dagger (D_M T) \right] \]

Here \( T \) and \( A^{(L/R)} \) matrices in flavor space.

3. Compute vector-vector correlators using standard recipes

\[ -i \langle J^a_{\mu}(V) J^b_{\nu}(V) \rangle \propto \delta^{ab} \left( q^2 \eta_{\mu\nu} - q_{\mu} q_{\nu} \right) \Pi_V(q^2) \]

\[ -i \langle J^a_{\mu}(A) J^b_{\nu}(A) \rangle \propto \delta^{ab} \left[ (q^2 \eta_{\mu\nu} - q_{\mu} q_{\nu}) \Pi_A(q^2) + q_{\mu} q_{\nu} \Pi_L(q^2) \right] \]
Consequences of the BKT transition

\[ \log(\sigma/\Lambda_{UV}^3) \]

\[ \langle \bar{q}q \rangle \sim \sigma \sim \exp \left( -\frac{\kappa}{\sqrt{x_c - x}} \right) \]

1. Miransky/BKT scaling as \( x \to x_c \) from below
   - E.g., The chiral condensate \( \langle \bar{q}q \rangle \propto \sigma \)

2. Unstable Efimov vacua observed for \( x < x_c \)

3. Turning on the quark mass possible
Turning on finite $m_q$

Quark mass defined through the tachyon boundary conditions in the UV:

\[ \tau(r) \approx m_q (-\log r)^{-\gamma_0/\beta_0} r + \sigma (-\log r)^{\gamma_0/\beta_0} r^3 \]

with $\sigma \sim \langle \bar{q}q \rangle$

- Finite (flavor independent) $m_q$ implies nonzero tachyon and chiral symmetry breaking
- Conformal transition becomes a crossover
- Discontinuous change of IR geometry in the conformal window at $m_q = 0$
Analysis of the tachyon solution $\Rightarrow$ separate different regimes:

Crossover between A and B: $m_q \sim \exp\left[-\frac{2K}{\sqrt{x_c - x}}\right] \sim \langle \bar{q}q \rangle$

- Regimes A and B “model independent”
$U(1)_A$ anomalously broken in QCD

However: axial anomaly is suppressed at large $N_c$ (in the ’t Hooft limit)

- “Solved” in the Veneziano limit, where axial anomaly appears at LO
- $\eta'$ meson (flavor-singlet pseudoscalar) is the corresponding “Goldstone mode”

[Witten, Veneziano]

$$m_{\eta'}^2 \simeq m_{\pi}^2 + x \frac{\chi}{f_{\pi}^2}$$

- $\chi$ is the topological susceptibility (constant term in $F \wedge F$ correlator)
- $f_{\pi}$ is the pion decay constant with $N_{c,f}$ factors divided out
- Good agreement with experimental+lattice values for QCD
The CP-odd term in V-QCD

Bulk axion $a$

- dual to $\text{tr} F \wedge F$
- background value identified as $\theta/N_c$, where $\theta$ is the theta angle of QCD

Tachyon Ansatz $T = \tau e^{i\xi} I$

String motivated CP-odd term added in the action

$$S_a = -\frac{M^3 N_c^2}{2} \int d^5x \sqrt{-\det g} Z(\lambda)$$
$$\times [da - x (2V_a(\lambda, \tau) A - \xi dV_a(\lambda, \tau)) ]^2$$

[Casero, Kiritsis, Paredes]

Symmetry

$$A_\mu \rightarrow A_\mu + \partial_\mu \epsilon, \quad \xi \rightarrow \xi - 2\epsilon, \quad a \rightarrow a + 2x V_a \epsilon$$

reflects the axial anomaly in QCD (with $\epsilon = \epsilon(x_\mu)$)
Analytic derivation by perturbative analysis of the coupled flavor singlet (pseudoscalar meson+glueball) fluctuation equations ⇒

The Witten-Veneziano relation: $\eta'$ becomes light as $x \to 0$

$$m_{\eta'}^2 \simeq m_\pi^2 + x \frac{\chi}{f_\pi^2}$$

PS masses at $m_q = 0$

$\pi$ and $\eta'$ masses at $x = 0.0001$
Four-fermion operators at zero mass

Example: \( x < x_c \) and \( m_q = 0 \)

Efimov spiral: all sols from holography

Straight lines: boundary condition
\( \alpha = g / \beta \)

\[ \Rightarrow \text{find all intersection points, check stability, …} \]

- Either an instability (typically when \( g < 0 \)) or a smooth deformation of the \( g = 0 \) solution
- Location of conformal window unchanged
Finite $T$ and $\mu$ – definitions

Add gauge field

$$S_{V-QCD} = N_c^2 M^3 \int d^5 x \sqrt{g} \left[ R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V_g(\lambda) \right]$$

$$- N_f N_c M^3 \int d^5 x V_f(\lambda, \tau) \times \sqrt{- \det(g_{ab} + \kappa(\lambda) \partial_a \tau \partial_b \tau + w(\lambda) F_{ab})}$$

$$F_{r0} = \partial_r \Phi \quad \Phi = \mu - nr^2 + \cdots$$

A more general metric ($A$ and $f$ solved from EoMs)

$$ds^2 = e^{2A(r)} \left( \frac{dr^2}{f(r)} - f(r) dt^2 + dx^2 \right)$$

Nontrivial blackening factor $f$: black hole solutions possible
Various solutions

Two classes of IR geometries:

1. Black hole solutions $\rightarrow$ temperature and entropy through BH thermodynamics
   
   $f'(r_h) = -4\pi T$; $s = 4\pi M^3 N_c^2 e^{3A(r_h)}$

2. Thermal gas solutions ($f \equiv 1$)
   
   Any $T$ and $\mu$, zero $s$

Two types of tachyon behavior ($\tau \leftrightarrow \bar{q}q$, quark mass and condensate from UV boundary behavior):

1. Vanishing tachyon – chirally symmetric
2. Nontrivial tachyon – chirally broken

$\Rightarrow$ four possible types of background solutions
Computation of pressure

Three phases turn out to be relevant (at small $x$)

- Tachyonic Thermal gas (chirally broken)
- Tachyonic BH (chirally broken)
- Tachyonless BH (chirally symmetric)

Nontrivial numerical analysis:

1. $T, \mu$ not input parameters, they need to be calculated first
2. Integrate numerically for each phase
   \[ dp = s \, dT + n \, d\mu \]
3. Phase with highest $p$ dominates
Phase diagram at finite $\mu$ (example at fixed $x$)

First attempt: $x = N_f/N_c = 1$, Veneziano limit, zero quark mass

$\text{AdS}_2 \times \mathbb{R}^3$ IR geometry as $T \to 0$

Finite entropy at zero temperature $\Rightarrow$ instability?
1. Meson spectra (at zero temperature and quark mass)
   ▶ Implement (left and right handed) gauge fields in $\mathcal{S}_{V-QCD}$
   ▶ Four towers: scalars, pseudoscalars, vectors, and axial vectors
   ▶ Flavor singlet and nonsinglet ($SU(N_f)$) states

In the region relevant for “walking” technicolor ($x \rightarrow x_c$ from below):
  ▶ Possibly a light “dilaton” (flavor singlet scalar): Goldstone mode due to almost unbroken conformal symmetry. Could the dilaton be the 125 GeV Higgs?
Meson masses

Flavor nonsinglet masses (Example: PotI)

Miransky scaling:

\[ m_n \sim \exp\left( -\frac{\kappa}{\sqrt{x_c - x}} \right) \]

Radial trajectories \( m_n^2 \sim n \) or \( m_n^2 \sim n^2 \) depending on potentials
Scalar singlet masses

Scalar singlet $(0^{++})$ spectrum (PotI):

In log scale

Normalized to the lowest state

No light dilaton state as $x \rightarrow x_c$?
\[ S \sim \frac{d}{dq^2} q^2 \left[ \Pi_V(q^2) - \Pi_A(q^2) \right]_{q^2=0} \]

where (at zero quark mass)

\[ \Pi_{V/A}(q^2) \left( q^2 g^{\mu\nu} - q^\mu q^\nu \right) \delta^{ab} \propto \langle J^\mu_{V/A} J^\nu_{V/A} \rangle \]

in terms of the vector-vector and axial-axial correlators

- The S-parameter might be reduced in the walking regime
Results:

PotI  PotII

\[ S/(N_c N_f) \]

The *S*-parameter increases with \( x \): expected suppression absent

Jumps discontinuously to zero at \( x = x_c \)
QCD at finite $T$ (and $x$)

Expected phase structure at finite temperature (and $x$)

SU(N) gauge theory, massless fermions

$\chi$ symm.

$\chi$ broken

Conformal Window ($\chi$ symm.)
\[ V_g(\lambda) = 12 + \frac{44}{9\pi^2} \lambda + \frac{4619}{3888\pi^4} \frac{\lambda^2}{(1 + \lambda/(8\pi^2))^{2/3}} \sqrt{1 + \log(1 + \lambda/(8\pi^2))} \]

\[ V_f(\lambda, \tau) = V_{f0}(\lambda) e^{-a(\lambda)\tau^2} \]

\[ V_{f0}(\lambda) = \frac{12}{11} + \frac{4(33 - 2x)}{99\pi^2} \lambda + \frac{23473 - 2726x + 92x^2}{42768\pi^4} \lambda^2 \]

\[ a(\lambda) = \frac{3}{22} (11 - x) \]

\[ \kappa(\lambda) = \frac{1}{\left(1 + \frac{115 - 16x}{288\pi^2} \lambda\right)^{4/3}} \]

In this case the tachyon diverges exponentially:

\[ \tau(r) \sim \tau_0 \exp \left[ \frac{81}{812944} \frac{3^{5/6} (115 - 16x)^{4/3} (11 - x)}{2^{1/6}} \frac{r}{R} \right] \]
Potentials II

\[ V_g(\lambda) = 12 + \frac{44}{9\pi^2}\lambda + \frac{4619}{3888\pi^4}(1 + \lambda/(8\pi^2))^{2/3}\sqrt{1 + \log(1 + \lambda/(8\pi^2))} \]

\[ V_f(\lambda, \tau) = V_{f0}(\lambda)e^{-a(\lambda)\tau^2} \]

\[ V_{f0}(\lambda) = \frac{12}{11} + \frac{4(33 - 2x)}{99\pi^2}\lambda + \frac{23473 - 2726x + 92x^2}{42768\pi^4}\lambda^2 \]

\[ a(\lambda) = \frac{3}{22}(11 - x)\frac{1 + \frac{115 - 16x}{216\pi^2}\lambda + \lambda^2/(8\pi^2)^2}{(1 + \lambda/(8\pi^2))^{4/3}} \]

\[ \kappa(\lambda) = \frac{1}{(1 + \lambda/(8\pi^2))^{4/3}} \]

In this case the tachyon diverges as

\[ \tau(r) \sim \frac{27}{\sqrt{4619}}\frac{2^{3/4}3^{1/4}}{R}\sqrt{r - r_1} \]
Effective potential

For solutions with $\tau = \tau_* = \text{const}$

$$S = M^3 N_c^2 \int d^5 x \sqrt{g} \left[ R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V_g(\lambda) - xV_f(\lambda, \tau_*) \right]$$

IHQCD with an effective potential

$$V_{\text{eff}}(\lambda) = V_g(\lambda) - xV_f(\lambda, \tau_*) = V_g(\lambda) - xV_{f0}(\lambda) \exp(-a(\lambda)\tau_*^2)$$

Minimizing for $\tau_*$ we obtain $\tau_* = 0$ and $\tau_* = \infty$

- $\tau_* = 0$: $V_{\text{eff}}(\lambda) = V_g(\lambda) - xV_{f0}(\lambda)$; fixed point with $V'_{\text{eff}}(\lambda_*) = 0$
- $\tau_* \to \infty$: $V_{\text{eff}}(\lambda) = V_g(\lambda)$ (like YM, no fixed points)
Black hole branches

Example: PotII at $x = 3$, $W_0 = 12/11$

Simple phase structure: 1st order transition at $T = T_h$ from thermal gas to (chirally symmetric) BH
More complicated cases:

PotII at $x = 3$, $W_0$ SB

PotI at $x = 3.5$, $W_0 = 12/11$

- Left: chiral symmetry restored at 2nd order transition with $T = T_{\text{end}} > T_h$
- Right: Additional first order transition between BH phases with broken chiral symmetry

Also other cases ...
Phase diagrams on the \((x, T)\)-plane

**PotI** \( W_0 \) SB

**PotII** \( W_0 \) SB

- No chiral symmetry breaking phase here

\[ \frac{T}{\Lambda} \]

\[ x_f \]
Backgrounds in the walking region

Backgrounds with zero quark mass, $x < x_c \simeq 3.9959 \ (\lambda, A, \tau)$

$x = 3$

$x = 3.5$

$x = 3.9$

$x = 3.97$
Beta functions along the RG flow (evaluated on the background), zero tachyon, YM

\[ x_c \approx 3.9959 \]

\( \beta(\lambda) \) vs. \( \lambda \) for various values of \( x \):
- \( x = 2 \)
- \( x = 3 \)
- \( x = 3.5 \)
- \( x = 3.9 \)
Holographic beta functions

Generalization of the holographic RG flow of IHQCD

\[ \beta(\lambda, \tau) \equiv \frac{d\lambda}{dA}; \quad \gamma(\lambda, \tau) \equiv \frac{d\tau}{dA} \]

linked to

\[ \frac{dg_{QCD}}{d \log \mu}; \quad \frac{dm}{d \log \mu} \]

The full equations of motion boil down to two first order partial non-linear differential equations for $\beta$ and $\gamma$
“Good” solutions numerically (unique)
As $x \to x_c$ from below: walking, dominant solution

- BF-bound for the tachyon violated at the IRFP
- $x_c$ fixed by the BF bound:
  $\Delta = 2 & \gamma_* = 1$
  at the edge of the conformal window

- $\tau(r) \sim r^2 \sin(\kappa \sqrt{x_c - x} \log r + \phi)$ in the walking region
- “0.5 oscillations” $\Rightarrow$ Miransky/BKT scaling,
  amount of walking $\Lambda_{UV}/\Lambda_{IR} \sim \exp(\pi/(\kappa \sqrt{x_c - x}))$
As $x \to x_c$
with known $\kappa$

\[
\langle \bar{q}q \rangle \sim \sigma \sim \exp\left(-\frac{2\pi}{\kappa \sqrt{x_c - x}}\right)
\]

\[
\Lambda_{UV}/\Lambda_{IR} \sim \exp\left(\frac{\pi}{\kappa \sqrt{x_c - x}}\right)
\]
\( \gamma^* \) in the conformal window

Comparison to other guesses

**V-QCD** (dashed: variation due to \( W_0 \))

**Dyson-Schwinger**

**2-loop PQCD**

**All-orders \( \beta \)**

[Pica, Sannino arXiv:1011.3832]
Understanding the solutions for generic quark masses requires discussing parameters

- YM or QCD with massless quarks: no parameters
- QCD with flavor-independent mass $m$: a single (dimensionless) parameter $m/\Lambda_{\text{QCD}}$
- In this model, after rescalings, this parameter can be mapped to a parameter ($\tau_0$ or $r_1$) that controls the diverging tachyon in the IR
- $x$ has become continuous in the Veneziano limit
Map of all solutions

All “good” solutions ($\tau \neq 0$) obtained varying $x$ and $\tau_0$ or $r_1$
Contouring: quark mass (zero mass is the red contour)
Mass dependence and Efimov vacua

Conformal window \((x > x_c)\)

- For \(m = 0\), unique solution with \(\tau \equiv 0\)
- For \(m > 0\), unique “standard” solution with \(\tau \neq 0\)

Low \(0 < x < x_c\): Efimov vacua

- Unstable solution with \(\tau \equiv 0\) and \(m = 0\)
- “Standard” stable solution, with \(\tau \neq 0\), for all \(m \geq 0\)
- Tower of unstable Efimov vacua (small \(|m|\))
Efimov solutions

- Tachyon oscillates over the walking regime
- $\Lambda_{\text{UV}}/\Lambda_{\text{IR}}$ increased wrt. "standard" solution

![Graph showing $\lambda$, log|$T|$ vs. $r$ with marked $1/\Lambda_{\text{UV}}$ and $1/\Lambda_{\text{IR}}$.]
Effective potential: zero tachyon

Start from Banks-Zaks region, $\tau_* = 0$, chiral symmetry conserved ($\tau \leftrightarrow \bar{q}q$), $V_{\text{eff}}(\lambda) = V_g(\lambda) - xV_{f0}(\lambda)$

- $V_{\text{eff}}$ defines a $\beta$-function as in IHQCD – Fixed point guaranteed in the BZ region, moves to higher $\lambda$ with decreasing $x$
- Fixed point $\lambda_*$ runs to $\infty$ either at finite $x(<x_c)$ or as $x \to 0$

Banks-Zaks $x \to 11/2$  
Conformal Window $x > x_c$  
$x < x_c$ ??
Effective potential: what actually happens

Banks-Zaks
\[ x \rightarrow \frac{11}{2} \]

Conformal Window
\[ x > x_c \]
\[ x < x_c \]

\[ \tau \equiv 0 \]
\[ \tau \equiv 0 \]
\[ \tau \neq 0 \]

- For \( x < x_c \) vacuum has nonzero tachyon (checked by calculating free energies)
- The tachyon screens the fixed point
- In the deep IR \( \tau \) diverges, \( V_{\text{eff}} \rightarrow V_g \) \( \Rightarrow \) dynamics is YM-like
Where is $x_c$?

How is the edge of the conformal window stabilized?
Tachyon IR mass at $\lambda = \lambda_* \leftrightarrow$ quark mass dimension

$$-m^2_{\text{IR}} \ell^2_{\text{IR}} = \Delta_{\text{IR}}(4 - \Delta_{\text{IR}}) = \frac{24a(\lambda_*)}{\kappa(\lambda_*)(V_g(\lambda_*) - xV_0(\lambda_*) )}$$

$$\gamma_* = \Delta_{\text{IR}} - 1$$

Breitenlohner-Freedman (BF) bound (horizontal line)

$$-m^2_{\text{IR}} \ell^2_{\text{IR}} = 4 \iff \gamma_* = 1$$

defines $x_c$
No time to go into details . . . the question boils down to the linearized tachyon solution at the fixed point

- For $\Delta_{\text{IR}}(4 - \Delta_{\text{IR}}) < 4$ \quad ($x > x_c$):

  \[
  \tau(r) \sim m_q r^{\Delta_{\text{IR}}} + \sigma r^{4 - \Delta_{\text{IR}}}
  \]

- For $\Delta_{\text{IR}}(4 - \Delta_{\text{IR}}) > 4$ \quad ($x < x_c$):

  \[
  \tau(r) \sim C r^2 \sin [(\text{Im}\Delta_{\text{IR}}) \log r + \phi]
  \]

Rough analogy:
Tachyon EoM $\leftrightarrow$ Gap equation in Dyson-Schwinger approach

Similar observations have been made in other holographic frameworks

For $m > 0$ the conformal transition disappears.

The ratio of typical UV/IR scales $\Lambda_{UV}/\Lambda_{IR}$ varies in a natural way:

$m/\Lambda_{UV} = 10^{-6}, 10^{-5}, \ldots, 10$  \quad  x = 2, 3.5, 3.9, 4.25, 4.5

\[
\frac{\Lambda_{UV}}{\Lambda_{IR}} = \frac{2}{10^{6}}, \frac{3.5}{10^{5}}, \frac{3.9}{10^{5}}, \frac{4.25}{10^{5}}, \frac{4.5}{10^{5}}\]

\[
\frac{m}{\Lambda_{UV}} = 2, 3, 5, 7, 10
\]
The case of $\mathcal{N} = 1 \ SU(N_c)$ superQCD with $N_f$ quark multiplets is known and provides an interesting (and more complex) example for the nonsupersymmetric case. From Seiberg we have learned that:

- $x = 0$ the theory has confinement, a mass gap and $N_c$ distinct vacua associated with a spontaneous breaking of the leftover $R$ symmetry $Z_{N_c}$.
- At $0 < x < 1$, the theory has a runaway ground state.
- At $x = 1$, the theory has a quantum moduli space with no singularity. This reflects confinement with ChSB.
- At $x = 1 + 1/N_c$, the moduli space is classical (and singular). The theory confines, but there is no ChSB.
- At $1 + 2/N_c < x < 3/2$ the theory is in the non-abelian magnetic IR-free phase, with the magnetic gauge group $SU(N_f - N_c)$ IR free.
- At $3/2 < x < 3$, the theory flows to a CFT in the IR. Near $x = 3$ this is the Banks-Zaks region where the original theory has an IR fixed point at weak coupling. Moving to lower values, the coupling of the IR $SU(N_c)$ gauge theory grows. However near $x = 3/2$ the dual magnetic $SU(N_f - N_c)$ is in its Banks-Zaks region, and provides a weakly coupled description of the IR fixed point theory.
- At $x > 3$, the theory is IR free.
Saturating the BF bound (sketch)

Why is the BF bound saturated at the phase transition (massless quarks)?

\[ \Delta_{\text{IR}}(4 - \Delta_{\text{IR}}) = \frac{24a(\lambda_*)}{\kappa(\lambda_*)(V_g(\lambda_*) - xV_0(\lambda_*))} \]

- For \( \Delta_{\text{IR}}(4 - \Delta_{\text{IR}}) < 4 \):
  \[ \tau(r) \sim m_q r^{4-\Delta_{\text{IR}}} + \sigma r^{\Delta_{\text{IR}}} \]

- For \( \Delta_{\text{IR}}(4 - \Delta_{\text{IR}}) > 4 \):
  \[ \tau(r) \sim C r^2 \sin [(\text{Im}\Delta_{\text{IR}}) \log r + \phi] \]

- Saturating the BF bound, the tachyon solutions will entangle
  → required to satisfy boundary conditions

- Nodes in the solution appear through UV → massless solution
Does the nontrivial (ChSB) massless tachyon solution exist? Two possibilities:

- $x > x_c$: BF bound satisfied at the fixed point $\Rightarrow$ only trivial massless solution ($\tau \equiv 0$, ChS intact, fixed point hit)
- $x < x_c$: BF bound violated at the fixed point $\Rightarrow$ a nontrivial massless solution exist, which drives the system away from the fixed point

Conclusion: phase transition at $x = x_c$
As $x \rightarrow x_c$ from below, need to approach the fixed point to satisfy the boundary conditions $\Rightarrow$ nearly conformal, “walking” dynamics
Massless backgrounds: gamma functions \( \frac{\gamma}{\tau} = \frac{d \log \tau}{dA} \)

\( \chi = 2, 3, 3.5, 3.9 \)