Study of the hyperscaling relation with the Schwinger-Dyson equation
Reference:

arXiv: 1201.4157 [hep-lat]
(to be published in PRD)
Contents

- Motivation
- Hyperscaling relation (review)
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Motivation

Technicolor model is an attractive candidate for the origin of the Electroweak symmetry breaking, though one based on naive scale up of QCD is phenomenologically disfavored...

A theory which has an (approximate) infrared fixed point with large mass anomalous dimension is preferable

Large flavor QCD

\[
\mu \frac{d}{d\mu} \alpha(\mu) = \beta(\alpha) = -b \alpha^2(\mu) - c \alpha^3(\mu)
\]

<table>
<thead>
<tr>
<th>$(N_f = 3)$</th>
<th>$N_f &lt; 8.05$</th>
<th>$8.05 &lt; N_f &lt; 16.5$</th>
<th>$16.5 &lt; N_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = \frac{1}{6\pi}(33 - 2N_f)$</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$c = \frac{1}{16\pi^2}(153 - 19N_f)$</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Indication from the 2-loop analysis

- ...

Fully non-perturbative (lattice) study is desirable

Several ways of checking infrared conformality

- measure the running coupling
- study the spectrum

We discuss this here
Hyperscaling relation (review)

Ref: Miransky, PRD59 105003, 1999
    Del Debbio, Zwicky, PRD82 014502, 2010
Hyperscaling relation (review)

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- Consider a theory which has an IRFP
Hyperscaling relation (review)

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Consider a theory which has an IRFP
Deform it by introducing small fermion bare mass $m$

Hadrons emerge at this scale
Hyperscaling relation (review)

Consider a theory which has an IRFP
- Deform it by introducing small fermion bare mass $m$

Relations between low-energy physical quantities (for example, a hadron mass $M_H$) and the bare fermion mass $m$ can be derived

$$M_H \simeq (\text{const.}) \mu \hat{m}^{1/1+\gamma_*}$$

mass anomalous dimension at the fixed point
(normalized) bare quark mass: $\hat{m}(\mu) = m(\mu)/\mu$
a scale (in the case of lattice, one can consider $\mu = a^{-1}$)

Ref: Miransky, PRD59 105003, 1999
Del Debbio, Zwicky, PRD82 014502, 2010

Hadrons emerge at this scale
Hyperscaling relation (review)

Consider a theory which has an IRFP
Deform it by introducing small fermion bare mass $m$

Relations between low-energy physical quantities (for example, a hadron mass $M_H$) and the bare fermion mass $m$ can be derived

When one considers a theory in a finite volume \( \sim L^4 \)

\[
M_H = L^{-1} f(x)
\]

where \( x = \hat{L} \hat{m}^{1/(1+\gamma_*)} \)

\( \hat{L} \equiv L\mu \)
Hyperscaling relation (review)

Results of lattice simulation with various values of input \((\hat{L}, \hat{m})\) should satisfy the hyperscaling relation (with an appropriate value of \(\gamma^*\)) if the theory has an IRFP.

(Wouldn’t it be interesting to see that all the data you have with different values of \((\hat{L}, \hat{m})\) align in a single curve?)

Many lattice groups use this method to judge whether a theory is conformal, and if it is, to estimate the value of \(\gamma^*\).
Hyperscaling relation (review)

Results of lattice simulation with various values of input $(\hat{L}, \hat{m})$ should satisfy the hyperscaling relation (with an appropriate value of $\gamma^*$) if the theory has an IRFP.

Couple of questions arise here:

- How small $m$ has to be to observe the scaling? (What is the form of correction when it’s not small enough?)
- When the original theory does not have an IRFP, how and how much the scaling relation is violated?

Schwinger-Dyson equation is a useful tool for such studies:

- we know the phase structure, and a value of $\gamma^*$ for a given theory
- analytic understanding can be obtained (to a certain extent)
- numerical calculations can be easily done in a wide range of parameter space (on your Mac (or PC))
Schwinger-Dyson equation (review)

Self-consistent equation for the full fermion propagator:

\[ iS_F^{-1} \equiv A(p^2)\not{p} - B(p^2) \]

\[ \text{bare fermion propagator} \]

\[ 1 \text{ particle irreducible diagram} \]

\[ \text{in equation...} \]
Schwinger-Dyson equation (review)

Self-consistent equation for the full fermion propagator

\[ iS_F^{-1} \equiv A(p^2)\gamma^\mu - B(p^2) \]

\[ iS_F^{-1}(p) = \gamma^\mu - \frac{m}{2} \]

\[ + C_2 \int \frac{d^4k}{i(2\pi)^4} \bar{g}^2(p,k) \frac{1}{(p-k)^2} \left( g_{\mu\nu} - \frac{(p-k)_\mu(p-k)_\nu}{(p-k)^2} \right) \gamma^\mu \gamma^\nu iS_F(k) \]

We adopted...
Landau gauge
Improved ladder approximation

\[ C_2 = \frac{N_C^2 - 1}{2N_C} \]

\[ \bar{g}(p,k) : \text{running coupling} \]

\text{coupled equation for}
\[ A(p^2), \quad B(p^2) \]

\text{mass function}
\[ \Sigma(p^2) \equiv \frac{B(p^2)}{A(p^2)} \]
Schwinger-Dyson equation (review)

setup for the current study

2-loop running coupling of the SU(Nc) gauge theory

Running coupling is approximated by the step function

Energy scale where the coupling begins to run

Fixed by Nc, Nf
Schwinger-Dyson equation (review)

After several minor approximations... integral equation can be rewritten in a form of linear differential equation with UV and IR boundary conditions:

\[
(x \Sigma(x))'' + \alpha_* \frac{3C_2}{4\pi} \frac{\Sigma(x)}{x + m_P^2} = 0
\]

IRBC : \( \lim_{x \to 0} x^2 \Sigma(x)' = 0 \)

UVBC : \( (x \Sigma(x))'|_{x = \Lambda^2} = m \)

where

\[
x \equiv p_E^2
\]

\[
m_P \equiv \Sigma(x = m_P^2)
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\(m_P\) is sometimes called the “pole mass”, though it’s not. Anyway, we identify this quantity as a typical scale of low-energy physical quantity.
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\( m_P \) is sometimes called the “pole mass”, though it’s not. Anyway, we identify this quantity as a typical scale of low-energy physical quantity.

Note: \( m_P \propto M_H \) is observed from the analysis based on BS equation.
Schwinger-Dyson equation (review)

\[(x \Sigma(x))'' + \alpha * \frac{3C_2}{4\pi} \frac{\Sigma(x)}{x + m_P} = 0\]

IRBC: \[\lim_{x \to 0} x^2 \Sigma(x)' = 0\]

UVBC: \[\left. (x \Sigma(x))' \right|_{x=\Lambda^2} = m\]

By checking whether the solution exists or not for \(\Sigma \neq 0, \ m = 0\), critical coupling is obtained as

\[\alpha_{cr} = \frac{\pi}{3C_2}\]

e.g.) SU(3) fundamental:

\[\alpha_{cr} = \frac{\pi}{4} \iff N_f^{cr} \approx 11.9\]

for \(N_f \geq 12\) an IRFP exists in the chiral limit.
Schwinger-Dyson equation (review)

\[(x\Sigma(x))'' + \alpha_* \frac{3C_2}{4\pi} \frac{\Sigma(x)}{x + m_P^2} = 0\]

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Since we are here interested in hyperscaling of IR conformal theories and correction by bare fermion mass, we consider

\[\alpha_* < \alpha_{cr}, \ m \neq 0\]
The solution of differential equation which satisfies the IRBC looks like

\[ \Sigma(x) = \xi m_P F \left( \frac{1 + \omega}{2}, \frac{1 - \omega}{2}, 2, -\frac{x}{m_P^2} \right), \quad \text{where} \]

\[ \omega \equiv \sqrt{1 - \frac{\alpha_\ast}{\alpha_{cr}}} \]

hypergeometric function

a constant which is determined by \( m_P \equiv \Sigma(x = m_P^2) \)
Schwinger-Dyson equation (review)

The solution of differential equation which satisfies the IRBC looks like

$$\Sigma(x) = \xi m_P F \left( \frac{1 + \omega}{2}, \frac{1 - \omega}{2}, 2, -\frac{x}{m_P^2} \right), \quad \text{where} \quad \omega \equiv \sqrt{1 - \frac{\alpha_\ast}{\alpha_{cr}}}$$

In the limit of $x \gg m_P^2$

$$\Sigma(x) \simeq \xi m_P \left[ \frac{\Gamma(\omega)}{\Gamma\left(\frac{\omega+1}{2}\right) \Gamma\left(\frac{\omega+3}{2}\right)} \left( \frac{x}{m_P^2} \right)^{\frac{\omega-1}{2}} + (\omega \leftrightarrow -\omega) \right]$$
Schwinger-Dyson equation (review)

The solution of differential equation which satisfies the IRBC looks like

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In the limit of \( x \gg m_P^2 \)

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\]

Inserting this into UVBC, we obtain the relation between \( m \) and \( m_P \)

\[
m = \xi m_P \left[ \frac{\Gamma(\omega)}{\Gamma(\frac{\omega+1}{2})^2} \left( \frac{\Lambda^2}{m_P^2} \right)^{\frac{\omega-1}{2}} + (\omega \leftrightarrow -\omega) \right]
\]
Schwinger-Dyson equation (review)

Rewriting $\omega$ in terms of the mass anomalous dimension,

$$\gamma_* = 1 - \omega \left( = 1 - \sqrt{1 - \frac{\alpha_*}{\alpha_{cr}}} \right),$$

we obtain

$$\frac{m}{\Lambda} = \xi \left[ \frac{\Gamma(1 - \gamma_*)}{\Gamma(\frac{2-\gamma_*}{2})^2} \left( \frac{m_P}{\Lambda} \right)^{1+\gamma_*} + \frac{\Gamma(-1 + \gamma_*)}{\Gamma(\frac{\gamma_*}{2})^2} \left( \frac{m_P}{\Lambda} \right)^{3-\gamma_*} \right]$$

This is a textbook fact,

(Refs: V.A. Miransky, “Dynamical Symmetry Breaking in Quantum Field Theory”)

however, the importance of the second term in the context of hyperscaling relation hasn’t been discussed

Now, let us discuss implications of this
Scaling violation

Rewriting $\omega$ in terms of the mass anomalous dimension,

$$\gamma_* = 1 - \omega \left( = 1 - \sqrt{1 - \frac{\alpha_*}{\alpha_{\text{cr}}}} \right),$$

we obtain

$$\frac{m}{\Lambda} = \xi \left[ \frac{\Gamma(1 - \gamma_*)}{\Gamma(\frac{2-\gamma_*}{2})^2} \left( \frac{m_P}{\Lambda} \right)^{1+\gamma_*} + \frac{\Gamma(-1 + \gamma_*)}{\Gamma(\frac{\gamma_*}{2})^2} \left( \frac{m_P}{\Lambda} \right)^{3-\gamma_*} \right]$$

Hyperscaling and deviation from it
Scaling violation

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Hyperscaling and deviation from it

In what situation the solution approximates well the hyperscaling relation?
Scaling violation

Rewriting $\omega$ in terms of the mass anomalous dimension,

$$\gamma_* = 1 - \omega \left( = 1 - \sqrt{1 - \frac{\alpha_*}{\alpha_{cr}}} \right),$$

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Hyperscaling and deviation from it

In what situation the solution approximates well the hyperscaling relation?

Note

\begin{align*}
\alpha_* & \quad \longleftrightarrow \quad \alpha_{cr} \\
\gamma_* & \quad \longleftrightarrow \quad 1
\end{align*}
Scaling violation

Rewriting $\omega$ in terms of the mass anomalous dimension,

$$\gamma_* = 1 - \omega \left( = 1 - \sqrt{1 - \frac{\alpha_*}{\alpha_{cr}}} \right),$$

we obtain

$$m/\Lambda = \xi \left[ \frac{\Gamma(1 - \gamma_*)}{\Gamma\left(\frac{2 - \gamma_*}{2}\right)^2} \left( \frac{m_P}{\Lambda} \right)^{1+\gamma_*} + \frac{\Gamma(-1 + \gamma_*)}{\Gamma\left(\frac{\gamma_*}{2}\right)^2} \left( \frac{m_P}{\Lambda} \right)^{3-\gamma_*} \right]$$

Hyperscaling and deviation from it

In what situation the solution approximates well the hyperscaling relation?

- when $\frac{m_P}{\Lambda}$, or equivalently, $\frac{m}{\Lambda}$ is small since $1 + \gamma_* < 3 - \gamma_*$

but, it’s difficult direction for lattice simulations
Scaling violation

Rewriting $\omega$ in terms of the mass anomalous dimension,

$$\gamma_* = 1 - \omega \left(1 - \sqrt{1 - \frac{\alpha_\ast}{\alpha_{cr}}} \right),$$

we obtain

$$\frac{m}{\Lambda} = \xi \left[ \frac{\Gamma(1 - \gamma_*)}{\Gamma\left(\frac{2 - \gamma_*)}{2}\right)^2} \left(\frac{m_P}{\Lambda}\right)^{1+\gamma_*} + \frac{\Gamma(-1 + \gamma_*)}{\Gamma\left(\frac{\gamma_*)}{2}\right)^2} \left(\frac{m_P}{\Lambda}\right)^{3-\gamma_*} \right]$$

Hyperscaling and deviation from it

In what situation the solution approximates well the hyperscaling relation?

when $\gamma_*$ is small since power suppression becomes strong, as well as the coefficient of the second term becomes small

but, our (phenomenological) motivation was large $\gamma_*$ ...
Scaling violation

Rewriting $\omega$ in terms of the mass anomalous dimension,

$$\gamma_* = 1 - \omega \left( = 1 - \sqrt{1 - \frac{\alpha_*)}{\alpha_{cr}} } \right),$$

we obtain

$$\frac{m}{\Lambda} = \xi \left[ \frac{\Gamma(1 - \gamma_*)}{\Gamma(\frac{2-\gamma_*}{2})^2} \left( \frac{m_P}{\Lambda} \right)^{1+\gamma_*} + \frac{\Gamma(-1 + \gamma_*)}{\Gamma(\frac{\gamma_*}{2})^2} \left( \frac{m_P}{\Lambda} \right)^{3-\gamma_*} \right]$$

Hyperscaling and deviation from it

The message is: we should keep the existence of the correction term in mind when we analyze data in practical situations.
Scaling violation

Rewriting $\omega$ in terms of the mass anomalous dimension,

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Hyperscaling and deviation from it

The message is: we should keep the existence of the correction term in mind when we analyze data in practical situations

What happens if we forget it?

Note

<table>
<thead>
<tr>
<th>$\alpha_*$</th>
<th>0 $\longleftrightarrow$ $\alpha_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_*$</td>
<td>0 $\longleftrightarrow$ 1</td>
</tr>
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</table>
Scaling violation

\[
m/\Lambda = \xi \left[ \frac{\Gamma(1-\gamma_*)}{\Gamma(\frac{2-\gamma_*}{2})^2} \left( \frac{m_P}{\Lambda} \right)^{1+\gamma_*} + \frac{\Gamma(-1+\gamma_*)}{\Gamma(\frac{\gamma_*}{2})^2} \left( \frac{m_P}{\Lambda} \right)^{3-\gamma_*} \right]
\]

Hyperscaling correction

Consider \( m_P \frac{\partial}{\partial m_P} (\log m) - 1 \equiv \gamma_{\text{eff}} \)

If there is no correction term, it gives \( \gamma_* \)

The question is: how much it deviates from \( \gamma_* \)
when the correction term exists
Effective anomalous dimension \( \gamma_{\text{eff}} \)

\[
\begin{align*}
\gamma_{\text{eff}} \quad N_f &= 12 \\
\gamma_{\text{eff}} \quad N_f &= 13 \\
\gamma_{\text{eff}} \quad N_f &= 14 \\
\gamma_{\text{eff}} \quad N_f &= 15 \\
\gamma_{\text{eff}} \quad N_f &= 16
\end{align*}
\]

\( m_P / \Lambda \)
Effective anomalous dimension \( SU(3) \) fundamental

\[
\frac{\gamma_{\text{eff}}}{\gamma^*}
\]

\[ N_f = 16 \]
\[ N_f = 15 \]
\[ N_f = 14 \]
\[ N_f = 13 \]
\[ N_f = 12 \]

\( m_P/\Lambda \)
Effective anomalous dimension SU(3) fundamental

\[ \frac{\gamma_{\text{eff}}}{\gamma^*} \]

- \( N_f = 16 \)
- \( N_f = 15 \)
- \( N_f = 14 \)
- \( N_f = 13 \)
- \( N_f = 12 \)

Deviation is large when

\[ \frac{m_P}{\Lambda} \rightarrow \text{large} \]
\[ N_f \rightarrow N_f^{\text{cr}} \]
Finite-volume scaling

When the correction term is important, what happens if we try to judge conformality, and extract the value of mass anomalous dimension from finite-volume hyperscaling relation?

\[ M_H = L^{-1} f(x) \]

where \( x = \hat{L} \hat{m}^{1/(1+\gamma_*)} \)

For the purpose of studying this, we formulate the Schwinger-Dyson equation in a finite size spacetime, and generate data for various values of \((\hat{L}, \hat{m})\)
SD equation

Self-consistent equation for the full fermion propagator

\[ iS_F^{-1} \equiv A(p^2)\phi - B(p^2) \]

\[
A(p^2) = 1 + C_2 \int \frac{d^4 k}{(2\pi)^4} \frac{g^2(p, k)}{k^2 A(k^2)^2 + B(k^2)^2} \cdot \left[ A(k^2) \frac{(p \cdot k)}{p^2(p - k)^2} + 2A(k^2) \frac{p \cdot (p - k)}{p^2(p - k)^4} \right],
\]

\[
B(p^2) = m_0 + 3 \int \frac{d^4 k}{(2\pi)^4} \frac{g^2(p, k)}{k^2 A(k^2)^2 + B(k^2)^2} \frac{B(k^2)}{(p - k)^2}.
\]

In the case of finite volume, we replace continuous momentum by the discrete one:

\[
p \rightarrow p_n = \frac{2\pi n}{L}, \quad \int_{-\infty}^{\infty} dp \ f(p) \rightarrow \frac{2\pi}{L} \sum_{n=-\infty}^{\infty} f(p_n)
\]

Then, do the iteration
SU(3) gauge theory with 12 fundamental fermion

relation between $m_P/\Lambda$ and $m/\Lambda$ for various values of $L\Lambda$

We can obtain data only for $m_p/\Lambda > O(0.1)$
SU(3) gauge theory with 12 fundamental fermion

relation between $m_P/\Lambda$ and $m/\Lambda$ for various values of $\Lambda\Lambda$

Let’s do finite-volume scaling by using these data

$$M_H = L^{-1} f(x)$$

where

$$x = \hat{L} \hat{m}^{1/(1+\gamma_*)}$$
SU(3) gauge theory with 12 fundamental fermion

Note: $\gamma_* \simeq 0.80$ at IRFP

$\gamma_{\text{eff}} = 0.5 \sim 0.6$ for $m_p/\Lambda \gtrsim O(0.1)$
SU(3) gauge theory with 12 fundamental fermion

\[ \gamma = 0.0 \]

\[ m_P L \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ \gamma = 0.1 \]

\[ m_P L \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ \gamma = 0.2 \]

\[ m_P L \]

\[ \mathcal{X} = \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ m_P L \]

\[ \gamma = 0.3 \]

\[ x \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ m_P L \]

\[ \gamma = 0.4 \]

\[ x = \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

$\gamma = 0.5$

$m_P L$

$\mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)}$
SU(3) gauge theory with 12 fundamental fermion

\[ \gamma = 0.6 \]

\[ m_P L \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ \gamma = 0.7 \]

\[ m_P L \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ m_P L \]

\[ \gamma = 0.8 \]

\[ X \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ m_p L \]

\[ \gamma = 0.9 \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ m_P L \]

\[ \gamma = 1.0 \]

\[ \mathcal{C} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ \gamma = 1.5 \]

\[ m_P L \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

\[ m_P L \]

\[ \gamma = 2.0 \]

\[ \chi \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 12 fundamental fermion

Note: $\gamma_* \simeq 0.80$ at IRFP

$$\gamma_{\text{eff}} = 0.5 \sim 0.6 \text{ for } m_p/\Lambda \gtrsim O(0.1)$$

Scaling is good with $\gamma_{\text{eff}}$ rather than $\gamma_*$
SU(3) gauge theory with 12 fundamental fermion

Note: $\gamma_* \simeq 0.80$ at IRFP

$$\gamma_{\text{eff}} = 0.5 \sim 0.6 \quad \text{for} \quad m_p/\Lambda \gtrsim O(0.1)$$

Scaling is good with $\gamma_{\text{eff}}$ rather than $\gamma_*$

This is also an indication that the correction coming from a finite volume effect is negligible
SU(3) gauge theory with 12 fundamental fermion

Note: $\gamma_* \simeq 0.80$ at IRFP

$\gamma_{\text{eff}} = 0.5 \sim 0.6$ for $m_p/\Lambda \gtrsim O(0.1)$

Scaling is good with $\gamma_{\text{eff}}$ rather than $\gamma_*$

We have also confirmed similar results for $N_f = 14, 16$
Another question:
when the theory does not have an IRFP
(namely, in the chiral symmetry breaking phase ),
how and how much the scaling relation is violated?

We show two examples:
SU(3), 9 flavor: deeply broken
SU(3), 11 flavor: close to the critical flavor
SU(3) gauge theory with 9 fundamental fermion

\[ m_P L \]

\[ \gamma = 0.0 \]

\[ \chi \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 9 fundamental fermions

\[ \gamma = 0.5 \]

\[ m_P L \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 9 fundamental fermion

\[ \gamma = 1.0 \]

\[ m_P L \equiv \frac{L}{1 + \gamma} \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 9 fundamental fermion

$$\gamma = 1.5$$

$$m_P L \equiv \frac{L}{1 + \gamma}$$

$$\mathcal{X} = \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)}$$
SU(3) gauge theory with 9 fundamental fermion

\[ \gamma = 2.0 \]

\[ m_P L \]

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 9 fundamental fermion

\[ \gamma = 2.0 \]

\[ m_P L \]

\[ \mathcal{X} = \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]

totally violated
SU(3) gauge theory with 11 fundamental fermion

\[ m_P L \]

\[ \gamma = 1.0 \]

\[ \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
SU(3) gauge theory with 11 fundamental fermion

\[ \gamma = 1.0 \]

Imagine there are a few percent error bar for each data... You would say it's consistent with IR conformal...

\[ \mathcal{X} \equiv \hat{L} \left( \frac{m}{\Lambda} \right)^{1/(1+\gamma)} \]
Summary

From the analytic form of relation between the bare quark mass and the physical quantity obtained from the SD equation, we showed the importance of the mass correction to the hyperscaling analysis.

SD equation in a finite size spacetime was formulated, and finite-volume scaling was studied by using numerical data in a self-consistent manner.
Summary

Possible scenarios (and possible confusions)

case 1: when the theory is deep in the hadronic phase

There is no confusion
Possible scenarios (and possible confusions)

case 2: when the theory is in the hadronic phase, but close to the edge of the conformal window

One might observe approximate scaling behavior, and conclude that the theory is IR conformal...
Summary

Possible scenarios (and possible confusions)

case 2: when the theory is in the hadronic phase, but close to the edge of the conformal window

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Confusing!!
Summary

Possible scenarios (and possible confusions)

case 3: when the theory is in the conformal window, and data are taken in a wide range of input bare mass

One might observe misalignment in the hyperscaling plot, because effective $\gamma$ is different for different mass regions... and those might look like not consistent with IR conformality
Summary

Possible scenarios (and possible confusions)

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Confusing!!
Summary

Possible scenarios (and possible confusions)

case 4: when the theory is in the conformal window, and data are taken in a rather small range of large bare mass region

One might observe good alignment in the hyperscaling plot, and obtain an effective value of $\gamma$. However, it is very possible that mass corrections to the hyperscaling relations for different physical quantities are different, so one might obtain non-universal values of $\gamma$ for hyperscaling plots with different physical quantities...
Possible scenarios (and possible confusions)

case 4: when the theory is in the conformal window, and data are taken in a rather small range of large bare mass region

One might observe good alignment in the hyperscaling plot, and obtain an effective value of $\gamma$. However, it is very possible that mass corrections to the hyperscaling relations for different physical quantities are different, so one might obtain non-universal values of $\gamma$ for hyperscaling plots with different physical quantities... Confusing!!
Summary

It might be natural that our community is confused by the hyperscaling results.

Of course, the mass correction shown here might be just one of possible sources of confusion, but it is definitely important to keep this in mind especially when one’s simulations are limited in a large mass region.

To judge the conformality, a combination of several analysis (ChPT, running coupling, etc.) is important.