Scale $\Rightarrow$ conformal invariance

RG-cycles, and all that ...

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with

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Two old questions (and the motivation for this work)

1. What are the possible phases of theories (limits as deep IR is approached)? (coulomb, Higgs, confinement, conformal, ...)? Is there, eg, a scale but not conformal phase?

2. RG flows: we know limiting flows can be IRFP (IR fixed points). Wilson speculated one may also have limit cycles or limiting ergodic trajectories. Are there any examples of these?
Quiz

Directions: Select the best answer.

1. Which of the following is false:

A. The trace anomaly is \( T^\mu_\mu = \beta_i \mathcal{O}_i \) (up to equations of motion)

B. A theory is conformal if and only if \( \beta_i = 0 \)

C. Scale invariance does not imply conformal invariance

D. All of the above

E. None of the above
Quiz

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Explanations:
- In $d = 4$
- In flat space
- Classically scale invariant (no dimensional couplings)
- Up to Equations of Motion
Quiz-Solutions

Directions: Select the best answer.

1. Which of the following is false:
   
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   B. A theory is conformal if and only if  \( \beta_i = 0 \)
   
   C. Scale invariance does not imply conformal invariance
   
   ✓ D. All of the above
   
   E. None of the above

In this talk I will explain these answers.
Trace Anomaly

Review derivation in dimensional regularization

Consider QFT with dimensionless couplings
(classically scale invariant) in $d = 4 - \varepsilon$

Compute stress-energy tensor, take trace
\[
T^\mu_\mu = \epsilon \mathcal{L} - (1)\phi \frac{\delta S_0}{\delta \phi}
\]

Lagrangian in terms of renormalized couplings satisfies RGE
\[
\left[ \hat{\beta}(g) \frac{\partial}{\partial g} - \hat{\gamma}\phi \frac{\partial}{\partial \phi} - \varepsilon \right] \mathcal{L} = 0
\]

where \( \hat{\beta} = -\epsilon g + \beta(g) \quad \hat{\gamma} = -\epsilon + \gamma(g) \)

Obtain
\[
T^\mu_\mu = \hat{\beta}(g) \frac{\partial \mathcal{L}}{\partial g} - (1 + \hat{\gamma}(g))\phi \frac{\delta S_0}{\delta \phi}
\]

H. Osborn, NPB 363, 486 (1991)
Notes:

- Presented for scalars, trivially extended for gauge fields, spinors with yukawas
- Presented for single coupling, trivially extended for many couplings, eg

\[
V = \frac{1}{4!} g_{abcd} \phi_a \phi_b \phi_c \phi_d = g_I O_I
\]

\[
T^\mu_\mu = \hat{\beta}_I(g) \frac{\partial L}{\partial g_I} - [(1 + \hat{\gamma}(g)) \phi] \cdot \frac{\delta S_0}{\delta \phi}
\]
So what is the problem with this???
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(This is a rhetorical question, I will now answer :))
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The operator \( O_I = \frac{\partial L}{\partial g_I} \) is not finite in general when inserted in Green functions!

Taking a derivative of the functional integral \( Z = e^{iW} = \int [d\phi] e^{iS_0} \)

\[
\frac{\partial}{i \partial g_I} \int [d\phi] e^{iS_0} = \langle \int d^d x \, O_I \rangle
\]

This is finite, but what about the local operator? It can fail to be finite by a total divergence

\[
\langle [O_I] \rangle \equiv \langle O_I - \partial_\mu J^\mu_I \rangle = \text{finite}
\]

What can this “current” be? For example, for scalars we can have

\[
J^\mu_I = -\frac{1}{2} (N_I)^{ab} (\phi_a \leftrightarrow \partial^\mu \phi_b) = \partial^\mu \phi^T N_I \phi \quad \text{(for short)}
\]

\[
N^T_I = -N_I, \quad N_I = \frac{N^1_I}{\epsilon} + \frac{N^2_I}{\epsilon^2} + \cdots
\]

We do not usually encounter this (need sufficient complexity)
wave-function

2-pt function with one insertion of $O_I$

couterterm:

$$\left( \partial_{\mu} \phi \right)^2$$
Trace anomaly: \[ T^\mu_\mu = \hat{\beta}(g) \frac{\partial L}{\partial g} - (1 + \hat{\gamma}(g)) \phi \frac{\delta S_0}{\delta \phi} \]

really is:

\[ T^\mu_\mu = \beta_I(g)[\mathcal{O}_I] + \partial_\mu J^\mu - (1 + \gamma(g)) \phi \frac{\delta S_0}{\delta \phi} \]

The “hats” have disappeared because the quantities are finite.

The current must be finite by itself.

\[ J^\mu = \hat{\beta}_I J^\mu_I = \partial^\mu \phi^T \hat{\beta}_I N_I \phi = \text{finite} \]

Introducing \( S \), the finite part of \( \hat{\beta}_I N_I \)

\[ S = -g_I N^1_I \]

This only gives the “tree level” current.

One can show the fully renormalized current is

\[ J^\mu = \partial^\mu \phi^T (S + N_I (Sg)_I) \phi = \text{finite} \]

where \((Sg)_{abcd} = S_{ae}g_{ebcd} + \cdots + S_{de}g_{abce}\)
Compute $S$

Recall $S = -g_I N_I^1$

For theory of $n_s$ real scalars, $n_f$ Weyl spinors (symmetry group $G_F = \text{SO}(n_s) \times \text{SU}(n_f)$, any subgroup possibly gauged), with

$$V = \frac{1}{4!} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d + \left( \frac{1}{2} y_{a|ij} \phi_a \psi_i \psi_j + \text{h.c.} \right).$$

From definition, $S^r = -S$ (in the Lie algebra of $\text{SO}(n_s)$)

$S = 0$ at 1- and 2-loops because individual topologies are symmetric under $a \leftrightarrow b$
Contributions to $N_I$ still symmetric at 2-loops

Fig. 4: At three loops there are many diagrams that contribute to $u_{16}$ in $\pi^{a}$ and $\pi^{b}$.

JO: $S = 0$ to 2-loops in fermion-scalar model, and to 3-loops in pure scalar model

Contributions to $N_I$ not symmetric at 3-loops;
Be mesmerized:

\[(16\pi^2)^3 (N_f^1)_{ab} \partial \mu g_I \supset -\frac{1}{2} \text{tr}(y_a \partial \mu y_c^* y_d y_e^*) \lambda_{bcde} - \frac{1}{3} \text{tr}(y_a y_c^* \partial \mu y_d y_e^*) \lambda_{bcde} - \frac{1}{2} \text{tr}(y_a y_c^* y_d \partial \mu y_e^*) \lambda_{bcde} \]

- \frac{5}{24} \text{tr}(y_a y_c^* y_d y_e^*) \partial \mu \lambda_{bcde} - \frac{1}{24} \text{tr}(y_b \partial \mu y_c^* y_d y_e^*) \lambda_{acde} - \frac{5}{24} \text{tr}(y_b y_c^* \partial \mu y_d y_e^*) \lambda_{acde}

- \frac{1}{24} \text{tr}(y_b y_c^* y_d \partial \mu y_e^*) \lambda_{acde} - \frac{5}{24} \text{tr}(\partial \mu y_b y_c^* y_d y_e^*) \lambda_{acde} - \frac{7}{32} \text{tr}(y_a \partial \mu y_c^* y_d y_d y_b y_e^*)

- \frac{7}{96} \text{tr}(y_a y_c^* \partial \mu y_d y_d y_b y_e^*) - \frac{23}{96} \text{tr}(y_a y_c^* y_d \partial \mu y_d y_b y_e^*) - \frac{7}{96} \text{tr}(y_a y_c^* y_d y_d \partial \mu y_b y_e^*)

- \frac{7}{32} \text{tr}(y_a y_c^* y_d y_d y_b \partial \mu y_e^*) + \frac{1}{16} \text{tr}(y_a \partial \mu y_c^* y_d y_d y_b y_e^*) - \frac{5}{48} \text{tr}(y_a y_c^* \partial \mu y_c^* y_d y_d y_b y_e^*)

- \frac{1}{48} \text{tr}(y_a y_c^* y_c \partial \mu y_d y_b y_e^*) - \frac{7}{96} \text{tr}(y_a y_c y_c y_d y_d \partial \mu y_b y_e^*) + \frac{1}{16} \text{tr}(y_a y_c y_c y_d y_d \partial \mu y_b y_e^*)

+ h.c. - \{a \leftrightarrow b\},

First ever computation of non-vanishing $S$:

\[(16\pi^2)^3 S_{ab} = \frac{5}{8} \text{tr}(y_a y_c^* y_d y_e^*) \lambda_{bcde} + \frac{3}{8} \text{tr}(y_a y_c^* y_d y_d y_b y_e^*) + h.c. - \{a \leftrightarrow b\}\]
SUSY result, all orders in perturbation theory:
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\[ S = 0 \]
• Condition for Scale Invariance?

\[ \partial_\mu D^\mu = 0 \]

where the dilatation (scale) current is given in terms of the improved stress-energy tensor

\[ D^\mu = x_\nu T^{\mu \nu} \]

so that

\[ \partial_\mu D^\mu = T^\mu_\mu \]

• Condition for Conformal Invariance?

\[ \partial_\mu K^{\mu \nu} = -x^\nu T^\mu_\mu = 0 \]

• It appears that in both cases the condition is

\[ T^\mu_\mu = 0 \]
• Improvements? If

\[ T_\mu^\nu = \partial_\mu \partial_\nu L^{\mu\nu} \]

one can improve \( T^{\mu\nu} \) so that scale and conformal still conserved.

• But! What if the unbroken symmetry is a combination of two broken symmetries? This happens in other familiar contexts:
  • For spontaneously broken symmetries, as in the SM: \( SU(2) \times U(1) \rightarrow U(1)_{\text{EM}} \)
  • For anomalous currents, as in \( B \) and \( L \) in SM, but not \( B-L \)

• Look for a conserved current of the form

\[ D^\mu = x_\nu T^{\mu\nu} - V^\mu \]

where \( V^\mu \) (the “virial current”) is a non-conserved current that does not depend explicitly on coordinates.

(and which is not of the form \( V^\mu = \partial_\nu L^{\mu\nu} \))
THEN: We can have

$$\partial_\mu D^\mu = T_\mu^\mu - \partial_\mu V^\mu = 0 \quad \text{scale invariance}$$

while

$$T_\mu^\mu = \partial_\mu V^\mu \neq 0 \quad \text{no conformal symmetry}$$

A scale transformation together with a U(1) rotation is still a symmetry.

Now use trace anomaly and explicit form of virial:

$$V^\mu = R_{ab} \partial^\mu \phi_a \phi_b + i P_{ij} \bar{\psi}_i \bar{\sigma}^\mu \psi_j$$

Condition for scale invariance but not conformal is then an algebraic condition:

$$\beta_I - (S\lambda)_I = (R\lambda)_I \neq 0$$

or

$$\beta_I = (Q\lambda)_I \quad \text{with} \quad R = Q - S \neq 0$$

recall notation:

$$(S\lambda)_{abcd} = S_{ae}\lambda_{ebcd} + \cdots + S_{de}\lambda_{abce}$$
\[
\begin{align*}
\beta_{abcd} &= -Q_{a'b'} \lambda_{a'bcd} - Q_{b'b} \lambda_{ab'cd} - Q_{c'c} \lambda_{abc'd} - Q_{d'd} \lambda_{abcd'}, \\
\beta_{aij} &= -Q_{a'a} y_{a'|ij} - P_{i'i} y_{a'|i'j} - P_{j'j} y_{a'|ij'}, 
\end{align*}
\]

These are not functional equations

Solution: specific values of coupling constants (and of \(Q\) and \(P\)) that satisfy these equations

Precisely as in searching for conformal fixed points (with \(Q = P = 0\))

\[
-\frac{d\lambda_I}{dt} = \beta_I = (Q\lambda)_I \quad \lambda_I(t) = (e^{-itQ}\lambda)_I
\]

\[e^{-tQ} \in G_F\] group of internal global transformations ("flavor" symmetry of kinetic terms)

As a function of \(t\): one parameter trajectory in compact space

- Trajectory closes

\[\Rightarrow\]

- Trajectory comes arbitrarily close to initial point (Poincare recurrence)
We look for solutions using perturbation theory:

At 1-loop the form of the beta-functions (ie, which monomials appear) suffices to show that $Q = P = 0$.

At 2-loops this happens by detailed cancellation among terms (and by non-linearity one has a reminder, $Q = \text{order 3-loops}$)

The 3-loop beta functions are not known.

We have computed the necessary terms (not the complete beta-function)
Summary of findings:

- In $d = 4 - \varepsilon$, no gauge fields (like Wilson-Fisher fixed points)
- No non-trivial solution to all orders in perturbation theory for any $n_f$ if $n_s < 2$
- Solutions with $P = 0$ but $Q \neq 0$ at 3-loops in:
  - $n_f = 1$, $n_s = 2$, with unbounded tree-level potential
  - $n_f = 2$, $n_s = 2$, with bounded tree-level potential
- In $d = 4$, SU(3)-YM $n_f = 2 + 2$ (anti-)fundamentals, $n_s = 2$ neutral (Caswell-Banks-Zaks FP)
- Solutions with $P = 0$ but $Q \neq 0$ at 3-loops.
- Unbounded tree-level potential

![Oscillating couplings](image-url)
\( S \) vs \( Q \)

- \( S = S(g) \), defined everywhere in theory space (i.e., space of coupling constants)
- \( Q = \) constant (one for each cycle solution), defined by solving for existence of cycle
- Both appear first at third order in the loop expansion
- Both in Lie algebra of \( SO(n_s) \) in \( G_F \)
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Theorem: In perturbation theory

- On cycles $S = Q$ (up to a symmetry generator)
- On fixed points $S = 0$ (up to a symmetry generator)
S vs Q

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Theorem: In perturbation theory

- On cycles $S = Q$ (up to a symmetry generator)
- On fixed points $S = 0$ (up to a symmetry generator)

Outline of proof:

- On cycle a quantity $A$ remains constant simply because it is $G_F$ invariant
- $A$ satisfies a strong $c$-theorem in perturbation theory, grows like $(\beta - (S\lambda))^2$
- On cycle, $\beta = (Q\lambda)$, the two statements imply $(Q\lambda) - (S\lambda) = 0$
Scale IS Conformal

Legal Disclosure: In $d = 4$, unitary, local, renormalizable, perturbative, interacting QFT with well defined correlators of stress-energy tensor

Recall, the condition for scale but not conformal is $\beta_I = (S\lambda)_I = (R\lambda)_I \neq 0$

or $\beta_I = (Q\lambda)_I$ with $R = Q - S \neq 0$

We just showed $(Rg)_I = 0$ ($R$ may not vanish but then it generates a symmetry)

Hence: scale + Poincare + unitarity + no-nonsense $\Rightarrow$ conformal

See also: Luty, Polchinski, Rattazzi
Quiz-Solutions Explained

Directions: Select the best answer.

1. Which of the following is false:
   
   A. The trace anomaly is \( T^\mu_\mu = \beta_i \mathcal{O}_i \) (up to equations of motion)  
   \[ T^\mu_\mu = \beta_I (g) [\mathcal{O}_I] + \partial_\mu J^\mu \]

   B. A theory is conformal if and only if \( \beta_i = 0 \)
   \[ B_I = \beta_I - (Sg)_I = 0 \]

   C. Scale invariance does not imply conformal invariance

   D. All of the above

   E. None of the above

   Yes it does (at least perturbatively)
After Thoughts, Conclusions, ...

• Is $B_I$ the new beta-function? No. But it does coincide with the beta-function in one particular scheme. This scheme cannot be obtained by a transformation $g \rightarrow f(g)$ alone (not the usual stuff). In this scheme the anomalous dimension matrix of the scalar fields is non-symmetric.

• Perturbative positivity of $\chi^g_{IJ}$ is used for proof of $a$-theorem and that scale implies conformal. New work by Komargodsky and Schwimmer (also Luty, Polchinski, Rattazzi) suggests there is a non-perturbative $a$-theorem. Positivity there is from optical theorem. Suggests a connection that may establish $S = Q$ and “scale implies conformal” non-perturbatively.

• Explicit form of $S$ allows to find examples of $Q$ much more simply:
  • Solve for zeroes of 1-loop beta-function $\beta_I = 0$
  • Plug value of “fixed point” into our 3-loop expression for $S(g)$
  • If $S = 0$ a FP, else $Q = S$

• What is the role of $S$? Are these “cyclic CFT” different form normal, FP CFT? Two point function look completely “normal” in cyclic CFT.

• Many other open questions ($d = 4 - \varepsilon$?, flows between FP and cyclic?, $d = 4$ with bounded potential? ... )
The End
Additional Slides
Explanations

A. The trace anomaly is $T^\mu_\mu = \beta_i \mathcal{O}_i$ (up to equations of motion)

We mean:

- In $d = 4$
- In flat space, else we would have additional terms involving the Riemann tensor quadratically

$$T^\mu_\mu = \beta_i \mathcal{O}_i - \frac{a}{16\pi^2} G + \frac{c}{16\pi^2} F - \frac{b}{16\pi^2} R^2$$

$$F = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - \frac{4}{d-2} R^{\mu\nu} R_{\mu\nu} + \frac{2}{(d-2)(d-1)} R^2,$$

$square \ Weyl$

$$G = \frac{2}{(d-3)(d-2)} (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2),$$

$Euler \ density$

- No dimensional couplings (no masses or cubic scalar couplings)
- There are in fact equations of motion terms on the RHS

$$T^\mu_\mu = \beta_i \mathcal{O}_i + \Delta \phi \frac{\delta}{\delta \phi} S_0$$

where $\Delta$ is the dimension of $\phi$
B. A theory is conformal if and only if \( \beta_i = 0 \)

We mean a theory that is conformal classically (question would be pointless if there are masses). Not identically but at a point in theory space \( \beta_i(g_*) = 0 \)

C. Scale invariance does not imply conformal invariance

Again, for \( d = 4 \). For \( d = 2 \) it was shown (Polchinski, 1987) scale implies conformal.

Unparticle physics uses (except for scalar unparticles) scale invariance without conformal. This gives amplified effects because it allows for

(i) Non-conserved vectors of dimension 3
(ii) Smaller dimensions than allowed by CFT, amplifying the effect of unparticles (eg, vectors of dim<3)
Example: $n_s$ real scalars, $n_f$ Weyl spinors with

$$V = \frac{1}{4!} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d + \left( \frac{1}{2} y_{a|ij} \phi_a \psi_i \psi_j + \text{h.c.} \right).$$

A simple 1-loop computation gives:

$$(N_{c|i j})_{ab} = -\frac{1}{16\pi^2\epsilon} \frac{1}{2} (y^*_{a|i j} \delta_{bc} - y^*_{b|i j} \delta_{ac})$$

(The index “I” now runs over $(abcd)$ and $(a|i j)$)
Trace anomaly:

\[ T_{\mu}^{\mu}(x) = \gamma_{aa'} D^2 \phi_a \phi_{a'} - \gamma_{ii'}^* \bar{\psi}_i i \bar{\sigma}^\mu D_{\mu} \psi_{i'} + \gamma_{ii'} D_{\mu} \bar{\psi}_i i \bar{\sigma}^\mu \psi_{i'} \]
\[ - \frac{1}{4!} (\beta_{abcd} - \gamma_{aa'} \lambda_{a'bcd} - \gamma_{bb'} \lambda_{ab'cd} - \gamma_{cc'} \lambda_{abc'd} - \gamma_{dd'} \lambda_{abcd'}) \phi_a \phi_b \phi_c \phi_d \]
\[ - \frac{1}{2} (\beta_a | ij - \gamma_{aa'} Y_a | ij - \gamma_{ii'} Y_a | ij' - \gamma_{j'j} Y_a | ij') \phi_a \psi_i \psi_j + \text{h.c.} . \]

Using the EOM to eliminate anomalous dimensions is on the same footing as using EOM on the virial current.
Diagrams that can contribute to $Q$ in the $n_s = n_f = 2$ model

2-loops

$D_1^{(2)}$ $D_2^{(2)}$

Fig. 1: Diagrams that contribute to $q$ at two-loop order.

3-loops

$D_1^{(3)}$ $D_2^{(3)}$ (and its symmetric) $D_3^{(3)}$ (and its symmetric)

$D_4^{(3)}$ $D_5^{(3)}$ (and its symmetric) $D_6^{(3)}$

$D_7^{(3)}$ (and its symmetric) $D_8^{(3)}$ $D_9^{(3)}$ (and its symmetric)

$D_{10}^{(3)}$ $D_{11}^{(3)}$ $D_{12}^{(3)}$

Fig. 2: Diagrams that contribute to $q$ at three-loop order.
The proof uses one of the consistency conditions of Jack & Osborn

\[ 8B_I \partial_I \tilde{A} = \chi_{I,J}^g B_I B_J \]

Here

\[ \tilde{A} \equiv a + \frac{1}{8} w_I B_I \]
\[ B_I = \beta_I - (Sg)_I \]
\[ a = \text{coefficient of Euler density in Weyl Anomaly (of “a-theorem” fame)} \]
\[ \chi_{I,J}^g \text{ and } w_I \text{ are defined in } JO \]

\[ \tilde{A} \equiv \text{a } GF \text{ scalar } \Rightarrow (\omega g)_I \partial_I \tilde{A} = 0 \quad \text{for any } \omega \text{ in the Lie algebra of } GF \]

On a cycle or a FP \( B_I \) is of this form:

\[ B_I = \beta_I - (Sg)_I = (Qg)_I - (Sg)_I = ([Q - S]g)_I \]

(\text{for FP use } Q = 0)

\[ \Rightarrow \quad \text{On a cycle or FP} \quad \chi_{I,J}^g B_I B_J = 0 \]

To lowest order in the loop expansion \( \chi_{I,J}^g \text{ is positive definite (a “metric” in theory space)} \)

(\text{only place perturbative assumption enters})

\[ \Rightarrow \quad B_I = 0 \quad \Rightarrow \quad (Sg)_I = (Qg)_I \quad \Rightarrow \quad S = Q + \Delta Q \text{ with } (\Delta Q g)_I = 0, \text{ ie, a symmetry} \]

Q.E.D.

Note: perturbative \( c \)-theorem follows from same consistency condition (Jack & Osborn)
Trace Anomaly II

Follow Jack and Osborn’s derivation of trace anomaly

- Consider QFT in curved background with space-time dependent couplings
  - Curved background: can then
    - Obtain stress-energy tensor by taking derivative w.r.t metric
      \[ T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_0}{\delta g_{\mu\nu}} \]
    - Study Weyl variations: these encode both dilatations and conformal transformations
      \[ g_{\mu\nu}(x) \rightarrow e^{-2\sigma(x)} g_{\mu\nu}(x) \]
  - Space-time dependent couplings
    - Sources for finite composite operators
      \[ [O_I(x)] = \frac{\delta S_0}{\delta g_I(x)} \]
    - Necessary for consistent space-time dependent dilatations (Weyl variations)
      \[ g(\mu) \rightarrow g(e^{-\sigma(x)} \mu) \]
  - Extend renormalized Lagrangian by all possible counterterms (operators of dimension 4) constructed out of metric and couplings, and their derivatives (consistent with diff invariance)
  - Then repeat argument presented in “Trace Anomaly I” but with space-time dependent couplings

\[ T^\mu_\mu = \beta_I(g)[O_I] + \nabla_\mu J^\mu + \cdots \quad (\text{... = stuff that vanishes in flat space, constant couplings}) \]
Bonus: JO consistency conditions (our version)

\[ e^{i\tilde{W}} = \int [d\phi] e^{i(S_0 + S_{\text{c.t.}})} \quad \tilde{W} = W + W_{\text{c.t.}} = W + S_{\text{c.t.}} \]

\( S_{\text{c.t.}} \) contains field independent counterterms, dim-4 operators constructed of \( g_{\mu\nu}(x) \) and \( g_I(x) \)

\[ W_{\text{c.t.}} = - \int \sqrt{-\gamma} \mu^{-\epsilon} \lambda \cdot \mathcal{R}, \]

with

\[ \lambda \cdot \mathcal{R} = \lambda_a F + \lambda_b G + \lambda_c H^2 + \varepsilon_i \partial_\mu g^i \partial^\mu H + \frac{1}{2} \mathcal{F}_{ij} \partial_\mu g^i \partial_\nu g^j H + \frac{1}{2} \mathcal{G}_{ij} \partial_\mu g^i \partial_\nu g^j G^{\mu\nu} \]

\[ + \frac{1}{2} \mathcal{A}_{ij} \nabla^2 g^i \nabla^2 g^j + \frac{1}{2} \mathcal{B}_{ijk} \partial_\mu g^i \partial_\nu g^j \nabla^2 g^k + \frac{1}{4} \mathcal{C}_{ijkl} \partial_\mu g^i \partial_\nu g^j \partial_\rho g^k \partial_\sigma g^l. \]

where \( \lambda_a = \lambda_a(g) = \frac{\lambda_a^1}{\epsilon} + \frac{\lambda_a^2}{\epsilon^2} + \cdots \) etc

As usual, for each coupling we can define a beta-function (describing the response of the coupling to changes in the arbitrary renormalization scale)

\[ \left( \epsilon - \hat{\beta}_i^k \frac{\partial}{\partial g^i} \right) \lambda \cdot \mathcal{R} = \beta_\lambda \cdot \mathcal{R} \]

For example:

\[ \chi_{ij}^a = (\epsilon - \hat{\beta}_k^k \partial_k) \mathcal{A}_{ij} - \mathcal{A}_{ik} \partial_j \hat{\beta}_k^k - \mathcal{A}_{jk} \partial_i \hat{\beta}_k^k \]
By changing variables in functional integral

\[ W[e^{-2\tau(x)}\gamma_{\mu\nu}(x), g^i(e^{-\tau(x)}\mu)] = W[\gamma_{\mu\nu}(x), g^i(\mu)] \]

so under infinitesimal \( \tau \rightarrow \tau + \sigma \)

\[ \Delta_\sigma \tilde{W} = \Delta_\sigma W + \Delta_\sigma W_{\text{c.t.}} = \Delta_\sigma W_{\text{c.t.}} = \text{finite} \]

where \( \Delta_\sigma W_{\text{c.t.}} = W_{\text{c.t.}}[(1 - 2\sigma)\gamma_{\mu\nu}, g^i - \sigma \mu \partial dg^i/d\mu] - W_{\text{c.t.}}[\gamma_{\mu\nu}, g^i], \)

This imposes relations among the counterterms ("consistency conditions") and therefore among the corresponding "beta-functions."

For example,

\[ \frac{1}{2} \mathcal{A}_{ij} \nabla^2 g^i \nabla^2 g^j = \frac{1}{2} \mathcal{A}_{ij} \hat{\beta}^i \hat{\beta}^j (\nabla^2 \tau)^2 + \cdots \]

and

\[ \sqrt{-\gamma}(\lambda_a F + \lambda_b G + \lambda_c H^2) = 8\lambda_b \left[(\nabla^2 \tau)^2 - (\partial_\mu \partial_\nu \tau)^2 + \cdots \right] + 4\lambda_c \left[(\nabla^2 \tau)^2 + \cdots \right] \]

gives

\[ 4\lambda_c \sim \frac{1}{2} \mathcal{A}_{ij} \hat{\beta}^i \hat{\beta}^j, \quad \text{where } \sim \text{ means up to finite terms, say } X \]

Then for beta functions one obtains one of the JO consistency conditions:

\[ 8\beta_c = \chi^a_{ij} \hat{\beta}^i \hat{\beta}^j - \beta^i \partial_i X, \]
Where do $N_I$ and $S$ arise in this language? Ans: Renormalized Lagrangian needs additional terms

Say, if
\[ \mathcal{L}_0 = \frac{1}{2} g^{\mu \nu} \partial_\mu \phi_0 a \partial_\nu \phi_0 a - \frac{1}{4!} g_{abcd}^0 \phi_0 a \phi_0 b \phi_0 c \phi_0 d \]

need to add a counter-term \[ g^{\mu \nu} (N_I)_{ab} \partial_\nu g_I (\phi_0 b \partial_\mu \phi_0 a) \ (= \partial_\mu \phi_0^T N_I \partial_\mu g_I \phi_0 \text{ in compact form}) \]

In fact JO take
\[ \mathcal{L}_0 = \frac{1}{2} \gamma^{\mu \nu} D_0 \mu \phi_0 a D_0 \nu \phi_0 a + \frac{1}{8} (d - 2) \phi_0 a \phi_0 a H - \frac{1}{4!} g_{abcd}^0 \phi_0 a \phi_0 b \phi_0 c \phi_0 d \]

\[ D_0 \mu \phi_0 = (\partial_\mu + A_0 \mu) \phi_0, \quad A_0 \mu = A_\mu + N_I (D_\mu g)_I, \quad D_\mu = \partial_\mu + A_\mu. \]

The background gauge field:

- Inserts $N_I$ counterterm
- Elevates $G_F$ symmetry (with action on spurions) to local symmetry
- Plays the role of a source for a finite operator, a scalar current

\[ [D_\mu \phi, \omega \phi] = \omega_{ab} \frac{\delta \tilde{S}_0}{\delta A^\mu_{ab}} = D_\mu \phi_0^T (\omega + N_I (\omega g)_I) \phi_0 \]
Is $B_I$ the new beta function?

\[ \mu \frac{\partial g_I}{\partial \mu} = \beta_I(g) \] gives the response of coupling to renormalization re-scaling

$B_I$ does not generally satisfy this. It does satisfy, however

\[ T^\mu_\mu = \beta_I(g)[\mathcal{O}_I] + \nabla_\mu J^\mu = B_I(g)[\mathcal{O}_I] \quad \text{up to EOMs} \]

Moreover, $G_F$ symmetry of the effective action

\[ \left[ (\omega g)_I \frac{\delta}{\delta g_I} + (\omega \phi) \cdot \frac{\delta}{\delta \phi} \right] \Gamma = 0 \]

can be combined with the RGE to give a new RGE but with

\[ \beta_I \rightarrow \beta_I - (\omega g)_I, \quad \gamma \rightarrow \gamma + \omega \]

\[ \left[ \mu \frac{\partial}{\partial \mu} + (\beta_I - (\omega g)_I) \frac{\delta}{\delta g_I} - ((\gamma + \omega)\phi) \cdot \frac{\delta}{\delta \phi} \right] \Gamma = 0 \]

This is not a new beta function. It is a trick for solving equations.
Scheme change

Recall, w.f. renormalization determines only $Z \frac{1}{2} \partial^\mu \phi^T Z \partial_\mu \phi$

Square root is ambiguous: $Z^{\frac{1}{2}} = O \tilde{Z}^{\frac{1}{2}}$, $O^T O = 1$ $Z^{\frac{T}{2}} Z^{\frac{1}{2}} = \tilde{Z}^{\frac{T}{2}} \tilde{Z}^{\frac{1}{2}}$

This allows for different infinite subtractions: $O = 1 + \frac{O^1}{\epsilon} + \cdots$ $(O^1)^T = -O^1$

This induces

$$\hat{\gamma} \rightarrow \hat{\gamma} - \omega, \quad \hat{\beta}_I \rightarrow \hat{\beta}_I + (\omega g)_I, \quad S \rightarrow S + \omega,$$

where $\omega = g_I \partial_I O^1$

Note that $B_I = \beta_I - (S g)_I, \quad \Gamma = \gamma + S$

are unambiguous ("gauge invariant").

There is a gauge in which $S = 0$ and $B_I$ is a beta function.
SUSY

\[ \mathcal{L} = \int d^4 \theta \Phi_a^+ \Phi_a + \left( \int d^2 \theta \frac{1}{3!} y_{abc} \Phi_a \Phi_b \Phi_c + \text{h.c.} \right) \]

Promote coupling to super-space dependent:

\[ Y_{abc}(z) = y_{abc}(z) + \sqrt{2} \theta y_{abc}^\psi(z) + \theta^2 y_{abc}^F(z), \]

\[ z^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta} \]

Only possible counterterm:

\[ \mathcal{L}_{\text{c.t.}} = \int d^4 \theta \Phi_a^+ F_{ab} \Phi_b, \]

\[ F_{ab}(Y, \bar{Y}) = F_{ba}(\bar{Y}, Y) = F_{ba}^*(Y, \bar{Y}). \]

Expanding

\[ \mathcal{L}_{\text{c.t.}} \supset ((N_I)_{ab} \partial^\mu y_I - (N_I)^*_{ba} \partial^\mu y_I^*) (\phi_a^* \partial_\mu \phi_b - \partial_\mu \phi_a^* \phi_b) \]

where

\[ (N_I)_{ab} = \frac{\partial F_{ab}(y, y^*)}{\partial y_I}, \quad (N_I)^*_{ba} = \frac{\partial F_{ab}(y, y^*)}{\partial y_I^*} \]

Finally compute S:

\[ S_{ab} \equiv -\frac{1}{2} (N_I^1)_{ab} y_I - \text{h.c.}, \]

\[ = -\frac{1}{2} \left( y_I \frac{\partial F_{ab}^1(y, y^*)}{\partial y_I} - y_I^* \frac{\partial F_{ab}^1(y, y^*)}{\partial y_I^*} \right) \]

\[ F_{ab}(Y, \bar{Y}) = F_{ab}(e^{-i\alpha Y}, e^{i\alpha \bar{Y}}) \Rightarrow S = 0 \]
Then: We can have

$$\partial_\mu D^\mu = T_\mu^{\mu} - \partial_\mu V^\mu = 0 \quad \text{scale invariance}$$

while

$$T_\mu^{\mu} = \partial_\mu V^\mu \neq 0 \quad \text{no conformal symmetry}$$

A scale transformation together with a U(1) rotation is still a symmetry.

Now use trace anomaly.

Ignore for now divergence of $S$-current, fix later.

Ignore EOM (extra slide if needed)

Then the condition for scale invariance but not conformal can be casted as an algebraic condition on the beta function,

$$\beta_I = (Q\lambda)_I \neq 0$$
Same general model:  \[ V = \frac{1}{4!} \lambda_{abcd} \phi_a \phi_b \phi_c \phi_d + \left( \frac{1}{2} y_{a|ij} \phi_a \psi_i \psi_j + \text{h.c.} \right). \]

Only candidate for virial current:

\[ V^\mu = R_{ab} \partial^\mu \phi_a \phi_b + i P_{ij} \bar{\psi}_i \bar{\sigma}^\mu \psi_j \]

\[ Q^T = -Q, \quad P^\dagger = -P \]

Trace anomaly:

\[ T^\mu_\mu = - \frac{1}{4!} \left( \beta_{abcd} - (S \lambda)_{abcd} \right) \phi_a \phi_b \phi_c \phi_d - \frac{1}{2} \beta_{a|ij} \phi_a \psi_i \psi_j + \text{h.c.} \]

recall:  \((S \lambda)_{abcd} = S_{ae} \lambda_{ebcd} + \cdots + S_{de} \lambda_{abce} \)

Div of virial:

\[ \partial_\mu V^\mu = R_{ab} \partial^2 \phi_a \phi_b - i P^\dagger_{ij} \bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_j + i P_{ij} \partial_\mu \bar{\psi}_i \bar{\sigma}^\mu \psi_j \]

Using EOM and \( T^\mu_\mu = \partial_\mu V^\mu \)  \( \Rightarrow \beta_{abcd} - (S \lambda)_{abcd} = (R \lambda)_{abcd} \)

\[ \beta_{abcd} = -Q_{a'|a} \lambda_{a'bcd} - Q_{b'b} \lambda_{ab'cd} - Q_{c'c} \lambda_{abc'd} - Q_{d'd} \lambda_{abcd'} \]

\[ \beta_{a|ij} = -Q_{a'a} y_{a'|i} - P_{i'} i y_{a|i'} j - P_{j'} j y_{a|i'} \]

For short, 1st equation:  \( \beta_I = (Q \lambda)_I \)
Scale IS Conformal

Legal Disclosure: In $d = 4$, unitary, local, renormalizable, perturbative, interacting QFT with well defined correlators of stress-energy tensor

Recall: scale invariance \( \partial_{\mu} D^{\mu} = T_{\mu}^{\mu} - \partial_{\mu} V^{\mu} = 0 \)

no conformal symmetry \( T_{\mu}^{\mu} = \partial_{\mu} V^{\mu} \neq 0 \)

Use now correct form of Trace Anomaly

\[ T_{\mu}^{\mu} = \beta_I (g)[O_I] + \partial_{\mu} J^{\mu} \quad \left( = (\beta_I - (Sg)_I)[O_I] = B_I [O_I] \right) \]

If the virial is \( V^{\mu} = \partial^{\mu} \phi^T R \phi \) (we reserve \( Q \) for solutions to \( \beta_I = (Q\lambda)_I \))

the condition for scale but not conformal is

\[ B_I = (Rg)_I \neq 0 \]

But this gives \( B_I \) in the Lie algebra of \( G_F \) and by the preceding proof \( B_I = 0 \) and hence \( (Rg)_I = 0 \)

\( R \) may not vanish but then it generates a symmetry

Hence: scale + Poincare + unitarity + no-nonsense \( \Rightarrow \) conformal