The Color Glass Condensate Effective Theory

Kobayashi-Maskawa Institute, Nagoya, March 2014

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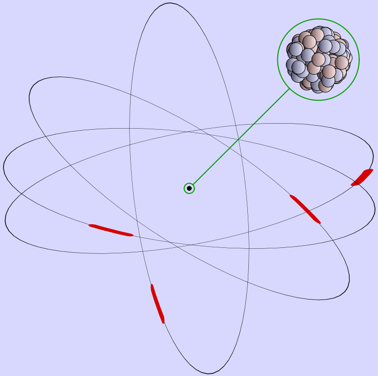
Heavy Ion Collisions

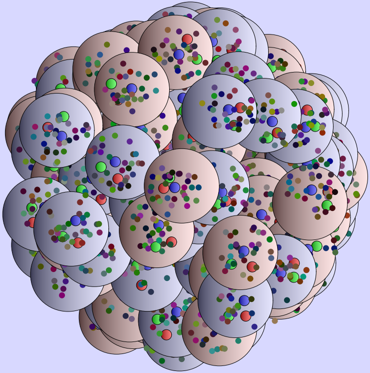
From atoms to nuclei, to quarks and gluons

10^{-10} m : atom (99.98% of the mass is in the nucleus)



From atoms to nuclei, to quarks and gluons

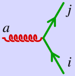
$< 10^{-15}$ m : quarks + gluons



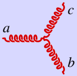
Quarks and gluons

Strong interactions : Quantum Chromo-Dynamics

- Matter : quarks ; Interaction carriers : gluons



$\sim \; g\,(t^a)_{ij}$



$\sim \; g\,(T^a)_{bc}$

- i, j : quark colors ; a, b, c : gluon colors
- $(t^a)_{ij}$: 3×3 SU(3) matrix ; $(T^a)_{bc}$: 8×8 SU(3) matrix

Lagrangian

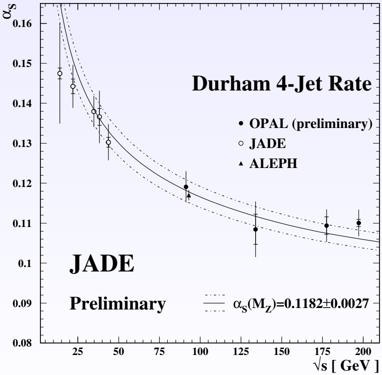
$$\mathcal{L} = -\frac{1}{4}F^2 + \sum_f \overline{\psi}_f(i\not{D} - m_f)\psi_f$$

- Free parameters : quark masses m_f, scale $\Lambda_{\rm QCD}$

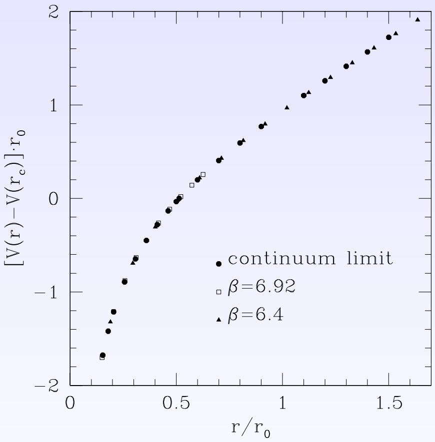
Asymptotic freedom

Running coupling : $\alpha_s = g^2/4\pi$

$$\alpha_s(E) = \frac{2\pi N_c}{(11N_c - 2N_f)\log(E/\Lambda_{QCD})}$$



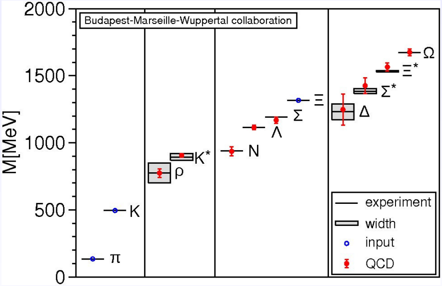
Color confinement



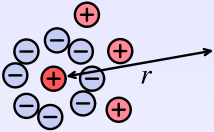
- The quark-antiquark potential increases linearly with distance

Color confinement

- In nature, we do not see free quarks and gluons (the closest we have to actual quarks and gluons are jets)
- Instead, we see hadrons (quark-gluon bound states):



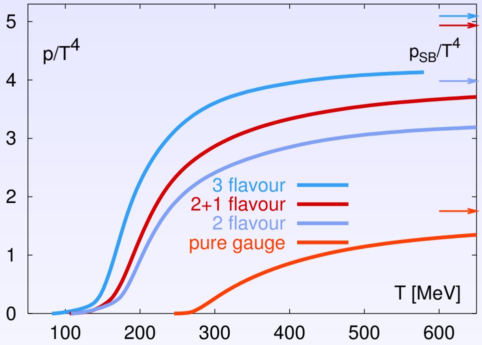
Debye screening at high density



$$V(r) = \frac{\exp(-r/r_{\text{debye}})}{r}$$

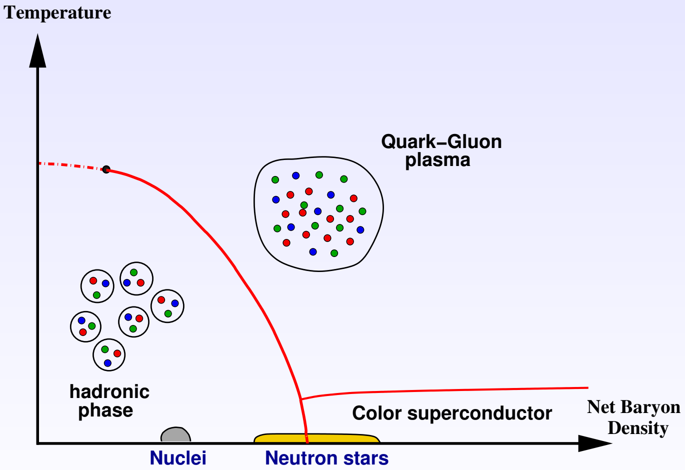
- In a dense environment, color charges are screened by their neighbours
- The Coulomb potential decreases exponentially beyond the Debye radius r_{debye}
- Bound states larger than r_{debye} cannot survive

Deconfinement transition

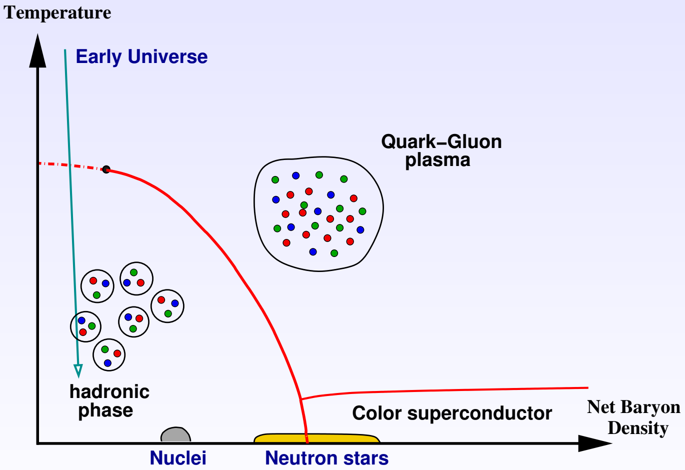


- Fast increase of the pressure :
  - at $T \sim 270$ MeV, if there are only gluons
  - at $T \sim 150$–170 MeV, depending on the number of light quarks

QCD phase diagram



QGP in the early universe



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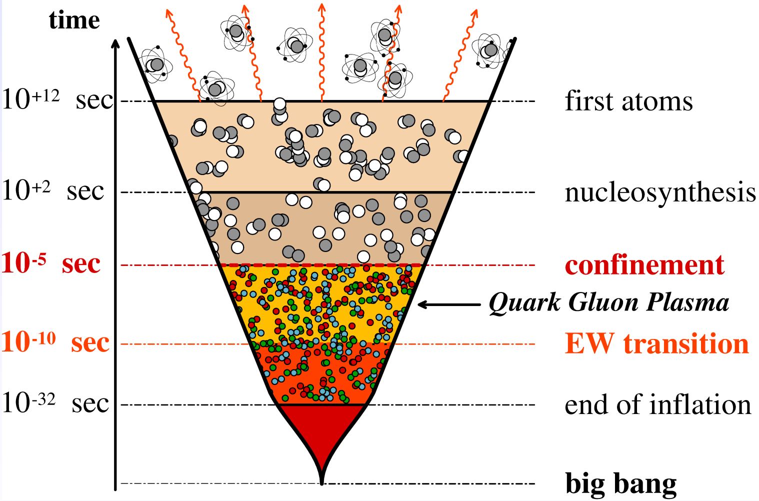
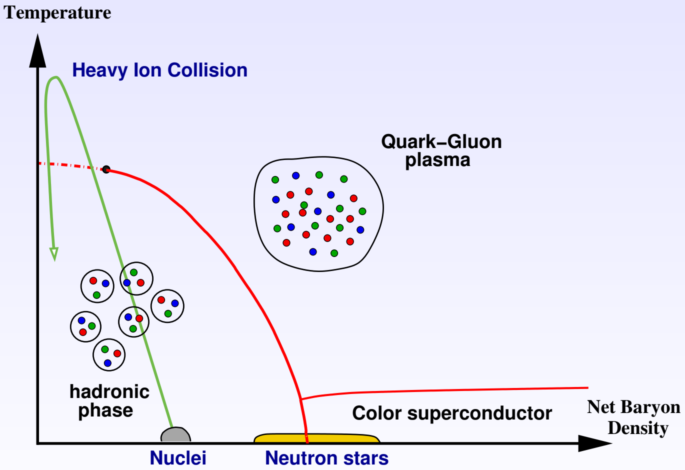
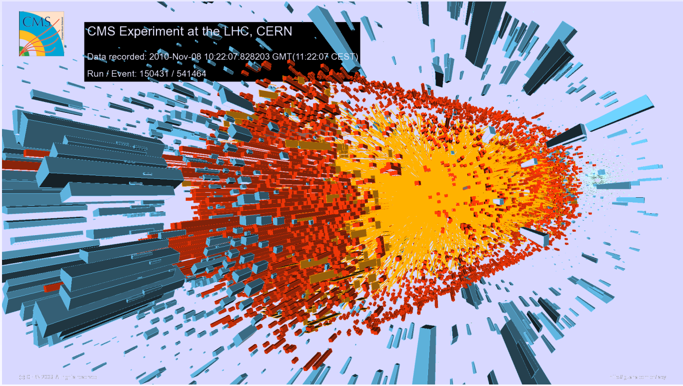


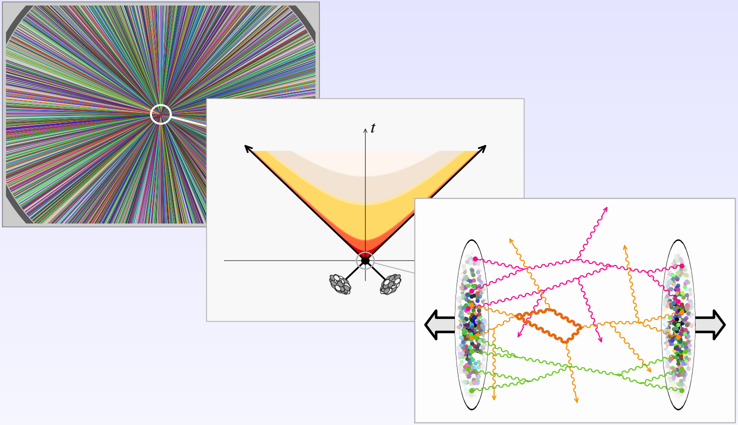
Heavy ion collisions



Experimental facilities : RHIC and LHC

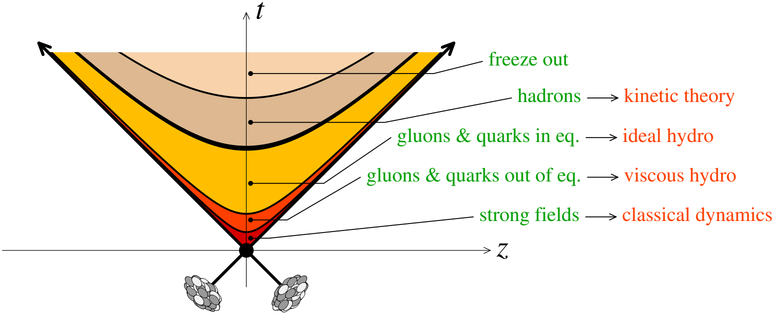
Heavy ion collision at the LHC



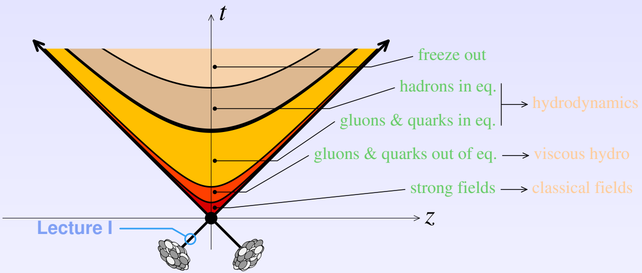


What can be said about hadronic and nuclear collisions in terms of the underlying quarks and gluons degrees of freedom?

Stages of a nucleus-nucleus collision

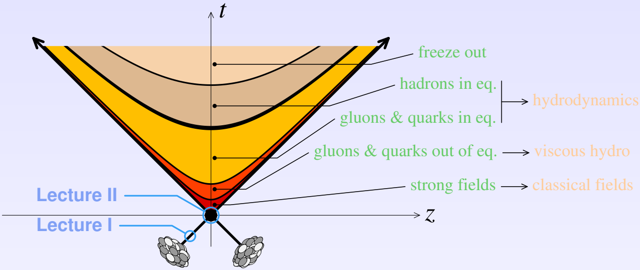


Stages of a nucleus-nucleus collision



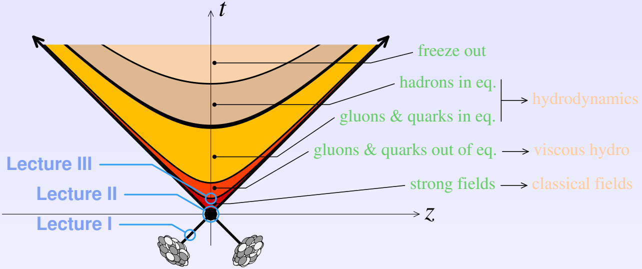
- Lecture I : Nucleon structure, Parton model, Dipole model

Stages of a nucleus-nucleus collision



- Lecture I : Nucleon structure, Parton model, Dipole model
- Lecture II : BK equation, Saturation, Color Glass Condensate

Stages of a nucleus-nucleus collision

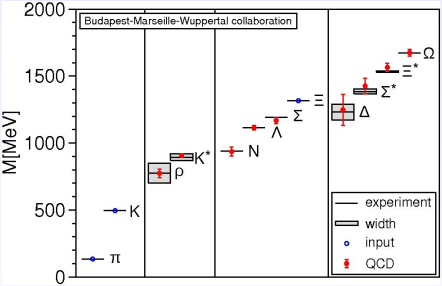


- Lecture I : Nucleon structure, Parton model, Dipole model
- Lecture II : BK equation, Saturation, Color Glass Condensate
- Lecture III : CGC applied to heavy ion collisions

Parton model

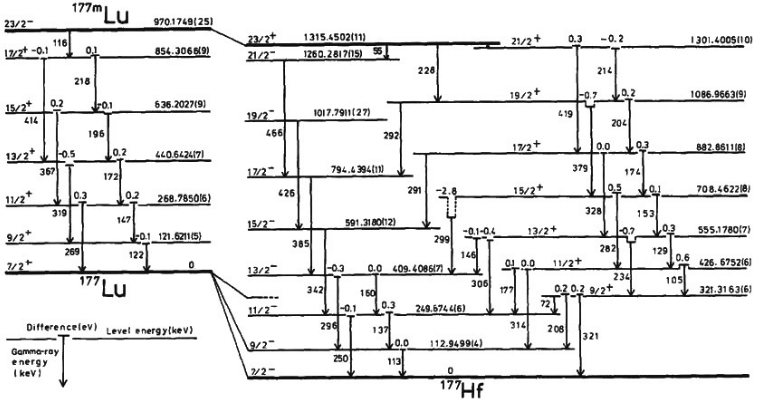
Hadronic spectrum

- In nature, we do not see free quarks and gluons (the closest we have to actual quarks and gluons are jets)
- Instead, we see hadrons (quark-gluon bound states):



- The hadron spectrum is uniquely given by $\Lambda_{\text{QCD}}, m_f$
- But this dependence is non-perturbative (it can now be obtained fairly accurately by lattice simulations)

Nuclear spectrum



- But nuclear spectroscopy is out of reach of lattice QCD, even for the lightest nuclei

Do we need to know all this in order to describe hadronic/nuclear collisions in Quantum–Chromodynamics?

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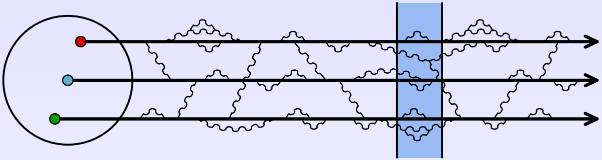
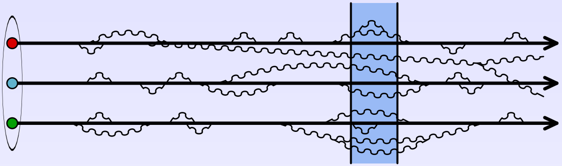
NO !



- A nucleon at rest is a very complicated object...
- Contains valence quarks + fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe



- Dilation of all internal time-scales for a high energy nucleon
- Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe
  ▷ the constituents behave as if they were free
  ▷ the reaction sees a snapshot of the nucleon internals
- Many fluctuations live long enough to be seen by the probe. The nucleon appears denser at high energy (it contains more gluons)

What do we need in order to describe a hadronic collision?

- Provide a snapshot of the two projectiles
  - Flavor and color of each parton
  - Transverse position and momentum
- Since these properties are not know event-by-event, one should aim at a probabilistic description of the parton content of the projectiles

Why is this non trivial?

- In quantum mechanics, the transition probability from some hadronic states to the final state is expressed as :

$$\text{transition probability from hadrons to X} \equiv \left| \sum \text{Amplitudes } h_1 h_2 \to X \right|^2$$

- The parton model assumes that we may be able to write it as :

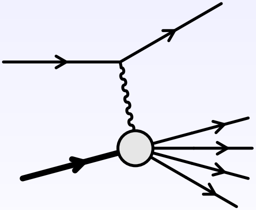
$$\text{transition probability from hadrons to X} \equiv \sum_{\substack{\text{partons}\\ \{q,g\}}} \text{probability to find } \{q,g\} \text{ in } \{h_1,h_2\} \otimes \left| \sum \text{Amplitudes } \{q,g\} \to X \right|^2$$

- This property is known as factorization. It can be justified in QCD, and it is a consequence of the separation between the timescale of confinement and the collision timescale

Deep Inelastic Scattering

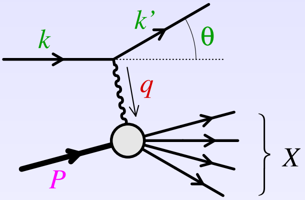
Introduction to DIS

- Basic idea : smash a well known probe on a nucleon or nucleus in order to try to figure out what is inside...
- Photons are very well suited for that purpose because their interactions are well understood
- Deep Inelastic Scattering : collision between an electron and a nucleon or nucleus, by exchange of a virtual photon



- Variant : collision with a neutrino, by exchange of $Z^0, W^{\pm}$

Kinematical variables



- Note : the virtual photon is space-like: $q^2 \leq 0$
- Other invariants of the reaction :

$$\begin{aligned} \nu &\equiv P \cdot q \qquad s \equiv (P+k)^2 \\ M_X^2 &\equiv (P+q)^2 = m_N^2 + 2\nu + q^2 \end{aligned}$$

- One uses commonly : $Q^2 \equiv -q^2$ and $x \equiv Q^2/2\nu$
- In general $M_X^2 \geq m_N^2$, and we have : $0 \leq x \leq 1$
  ($x = 1$ corresponds to the case of elastic scattering)

DIS cross-section

- The inclusive cross-section can be written as :

$$E' \frac{d\sigma_{e-N}}{d^3\vec{k}'} = \frac{1}{32\pi^3(s-m_N^2)} \frac{e^2}{q^4} 4\pi \, L^{\mu\nu} W_{\mu\nu}$$

where $W_{\mu\nu}$ is the hadronic tensor, defined as:

$$4\pi \, W_{\mu\nu} \equiv \sum_{\text{states } X} \int [d\Phi_X](2\pi)^4 \delta(P+q-P_X)$$
$$\times \left\langle \left\langle N(P) \middle| J_\nu(0) \middle| X \right\rangle \left\langle X \middle| J_\mu(0) \middle| N(P) \right\rangle \right\rangle_{\text{spin}}$$

$$4\pi \, W_{\mu\nu} = \int d^4y \; e^{iq\cdot y} \; \left\langle \left\langle N(P) \middle| J_\nu(y) \, J_\mu(0) \middle| N(P) \right\rangle \right\rangle_{\text{spin}}$$

$W_{\mu\nu}$ contains all the informations about the properties of the nucleon under consideration that are relevant to the interaction with the photon

Structure functions

For interactions with a photon :

$$W_{\mu\nu} = -F_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{F_2}{\nu}\left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)$$

- DIS cross-section in the nucleon rest frame :

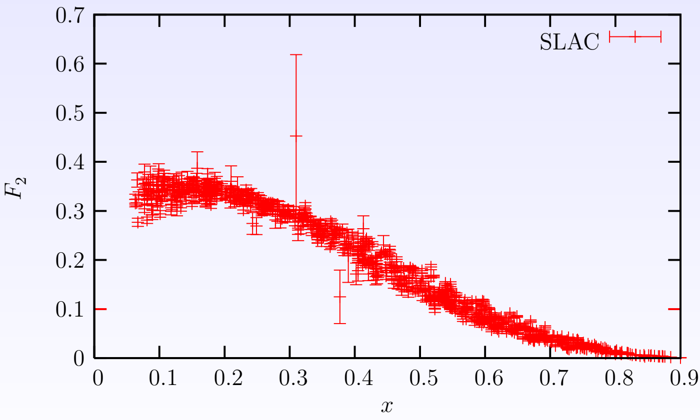
$$\frac{d\sigma_{e-N}}{dE'd\Omega} = \frac{\alpha_{em}^2}{4m_N E^2\sin^4(\theta/2)}\left[2\sin^2(\theta/2)F_1 + \frac{m_N^2}{\nu}\cos^2(\theta/2)F_2\right]$$

where Ω is the solid angle of the scattered electron

- Note: F_1 is proportional to the interaction cross-section between the nucleon and a transverse photon

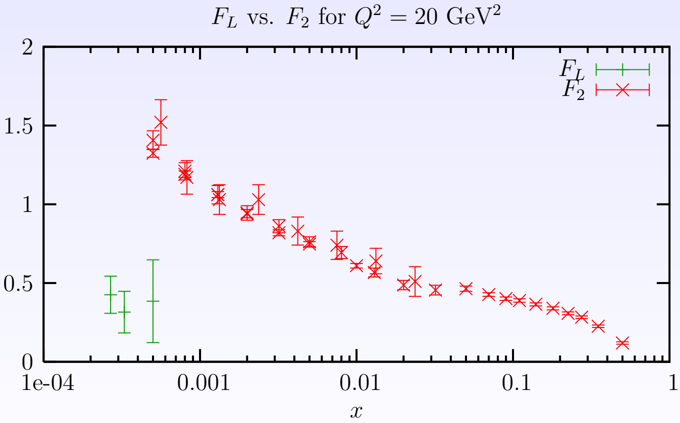
Bjorken scaling

- Bjorken scaling : F_2 depends very weakly on Q^2



Longitudinal structure function

- $F_L \equiv F_2 - 2xF_1$ is quite smaller than F_2 :



Analogy with the e- mu- cross-section

- In terms of F_1 and F_2, the DIS cross-section reads:

$$\frac{d\sigma_{e-N}}{dE'd\Omega} = \frac{\alpha_{em}^2}{4m_N E^2 \sin^4\frac{\theta}{2}}\left[2F_1 \sin^2\frac{\theta}{2} + \frac{m_N^2}{\nu}F_2\cos^2\frac{\theta}{2}\right]$$

- Compare with the $e^-\mu^-$ cross-section:

$$\frac{d\sigma_{e-\mu-}}{dE'd\Omega} = \frac{\alpha_{em}^2\delta(1-x)}{4m_\mu E^2 \sin^4\frac{\theta}{2}}\left[\sin^2\frac{\theta}{2} + \frac{m_\mu^2}{\nu}\cos^2\frac{\theta}{2}\right]$$

  - If the constituents of the nucleon that interact in the DIS process were spin $1/2$ point-like particles, we would have:

$$2F_1 = \frac{m_N}{m_c}\delta(1-x_c) \quad , \quad F_2 = \frac{m_c}{m_N}\delta(1-x_c)$$

    where m_c is some effective mass for the constituent (comparable to m_N because it is trapped inside the nucleon) and $x_c \equiv Q^2/2q\cdot p_c$ with p_c^μ the momentum of the constituent

Analogy with the e- mu- cross-section

- If $p_c^\mu = x_{_F} P^\mu$, then $x_c = x/x_{_F}$, and:

$$2F_1 \sim \delta(x - x_{_F}) \quad , \quad F_2 \sim \delta(x - x_{_F})$$

- The structure functions F_1 and F_2 would therefore not depend on Q^2, but only on x
- Conclusion : Bjorken scaling could be explained if the constituents of the nucleon that are probed in DIS are spin $1/2$ point-like particles

  The variable x measured in DIS would have to be identified with the fraction of momentum carried by the struck constituent

Pre-QCD parton model

- The historical parton model describes the nucleon as a collection of point-like fermions, called partons
- A parton of type i, carrying the fraction x_F of the nucleon momentum, gives the following contribution to the hadronic tensor :

$$4\pi W_i^{\mu\nu} = \int \frac{d^4 p'}{(2\pi)^4} \, 2\pi\,\delta(p'^2)\;(2\pi)^4\,\delta(x_F P + q - p')$$
$$\times \big\langle \big\langle x_F P \big| J^\mu(0) \big| p' \big\rangle \big\langle p' \big| J^\nu(0) \big| x_F P \big\rangle \big\rangle_{\rm spin}$$

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$$\times \big\langle \langle x_F P | J^\mu(0) | p' \rangle \langle p' | J^\nu(0) | x_F P \rangle \big\rangle_{\rm spin}$$

For the scattering on a spin 1/2 elementary constituent, one has:

$$4\pi W_i^{\mu\nu} = 2\pi\, x_F\,\delta(x_F - x)\; e_i^2$$
$$\times\left[-\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \frac{2x_F}{P\cdot q}\left(P^\mu - q^\mu\frac{P\cdot q}{q^2}\right)\left(P^\nu - q^\nu\frac{P\cdot q}{q^2}\right)\right]$$

Pre-QCD parton model

- If there are $f_i(x_F)dx_F$ partons of type i with a momentum fraction between x_F and $x_F + dx_F$, we have

$$W^{\mu\nu} = \sum_i \int_0^1 \frac{dx_F}{x_F}\, f_i(x_F)\, W_i^{\mu\nu}\,, \quad F_1 = \frac{1}{2}\sum_i e_i^2 f_i(x)\,, \quad F_2 = 2xF_1$$

- Callan-Gross relation : $F_2 = 2xF_1$
  - Consequence of spin 1/2 point-like partons
  - **Exercise :** for spin 0 partons, show that

$$W_i^{\mu\nu} \propto (2x_F P^\mu + q^\mu)(2x_F P^\nu + q^\nu) \quad \text{and} \quad F_1 = 0$$

- Caveats and puzzles :
  - The parton model assumes that partons are free inside the nucleon. How does this work in a strongly bound state ?
  - One would like to have a field theoretical description of what is going on, including the effect of interactions, quantum fluctuations...

What have we learned from QCD?

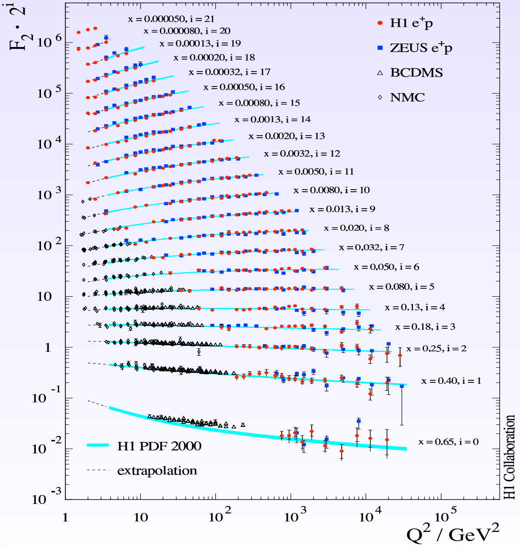
- Asymptotic freedom + time dilation in a high energy hadron explain why the partons appear as almost free at large Q^2

- QCD loop corrections lead to violations of Bjorken scaling, that are visible as a Q^2 dependence of the structure functions. Physically, $1/Q$ is the spatial resolution at which the hadron is probed

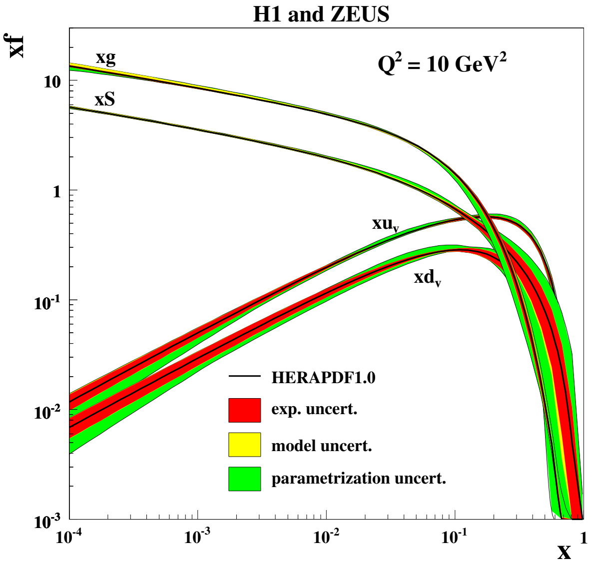
- Parton distributions are non-perturbative in QCD, but their Q^2 and x dependence are governed by equations that are perturbative (DGLAP, BFKL)

- One can prove that the parton distributions are universal, i.e. are the same in all inclusive processes

DIS results for F_2 and DGLAP fit at NLO :

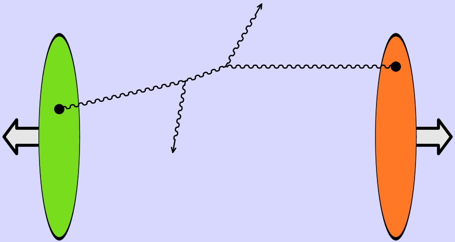


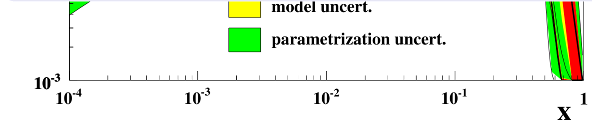
Parton distributions – and possible caveats



Parton distributions – and possible caveats

Large x : dilute, dominated by single parton scattering





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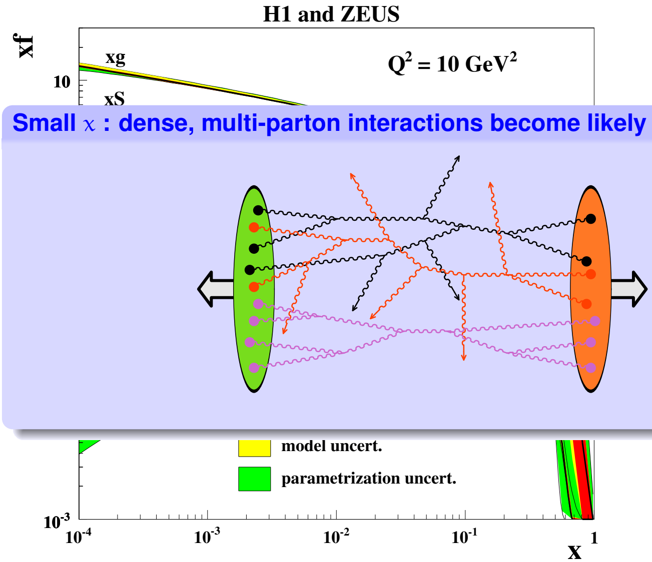
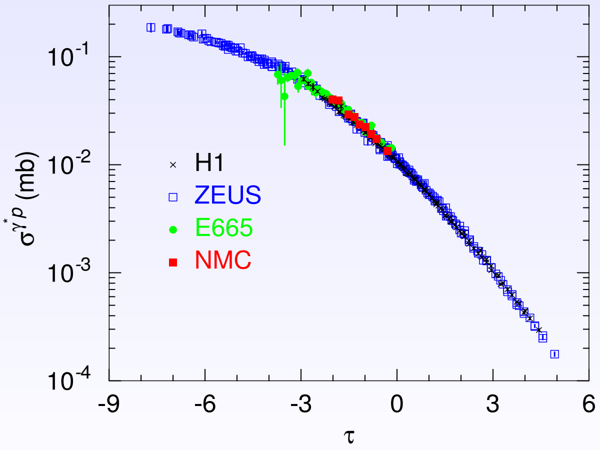


Small x : dense, multi-parton interactions become likely

Small x data displayed differently... (Geometrical scaling)

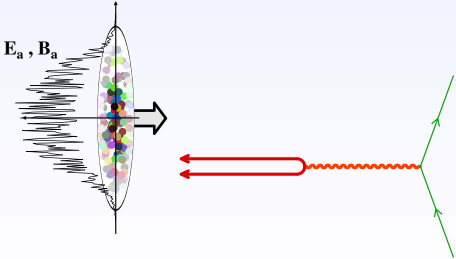
- Small x data ($x \le 10^{-2}$) displayed against $\tau \equiv \log(x^{0.32}\,Q^2)$:



Eikonal Scattering

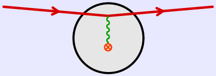
DIS off a highly boosted target

- Note : cross-sections are Lorentz invariant, but the microscopic interpretation may be frame dependent
  Some useful insight can be gained with a frame in which the target proton has a large P^3 momentum
- All the proton internal time scales are Lorentz dilated : its constituents appear frozen to the incoming virtual photon (they behave as if they have a mass $\propto P^3$)
- In the limit $P^3 \to \infty$, the interactions with such a constituent are equivalent to the interactions with its radiated field



Setup

- Consider the scattering amplitude off an external potential :
$S_{\beta\alpha} \equiv \big\langle \beta_{\text{out}} \big| \alpha_{\text{in}} \big\rangle = \big\langle \beta_{\text{in}} \big| U(+\infty, -\infty) \big| \alpha_{\text{in}} \big\rangle$



where $U(+\infty, -\infty)$ is the evolution operator from $t = -\infty$ to $t = +\infty$

$$U(+\infty, -\infty) = T \exp\left[i \int d^4x \, \mathcal{L}_{\text{int}}(\phi_{\text{in}}(x)) \right]$$

Note : $\mathcal{L}_{\text{int}}$ contains the self-interactions of the fields and their interactions with the external potential

- We want to calculate its high energy limit (eikonal limit):

$$S^{(\infty)}_{\beta\alpha} \equiv \lim_{\omega \to +\infty} \big\langle \beta_{\text{in}} \big| e^{-i\omega K^3} \, U(+\infty, -\infty) \, \underbrace{e^{+i\omega K^3} \big| \alpha_{\text{in}} \big\rangle}_{\text{boosted state}}$$

where K^3 is the generator of boosts in the $+z$ direction

Eikonal scattering in a nutshell

- In a scattering at high energy, the collision time goes to zero as E^{-1}
- With scalar interactions, this implies a decrease of the scattering amplitude as E^{-1}
- With vectorial interactions, this decrease is compensated by the growth of the components $J^{0,3}$ of the vector current

$\triangleright$ the eikonal approximation gives the finite limit of the scattering amplitude in the case of vectorial interactions when $E \to +\infty$

Light-cone coordinates

- Light-cone coordinates are defined by choosing a privileged axis (generally the z axis) along which particles have a large momentum. Then, for any 4-vector a^μ, one defines :

$$a^+ \equiv \frac{a^0 + a^3}{\sqrt{2}} \quad , \quad a^- \equiv \frac{a^0 - a^3}{\sqrt{2}}$$

$$a^{1,2} \quad \text{unchanged.} \qquad \text{Notation :} \quad \vec{a}_\perp \equiv (a^1, a^2)$$

- Some useful formulas :

$$x \cdot y = x^+ y^- + x^- y^+ - \vec{x}_\perp \cdot \vec{y}_\perp$$

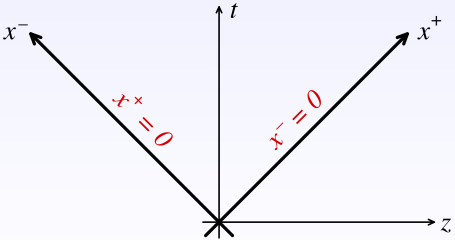
$$d^4x = dx^+ dx^- d^2\vec{x}_\perp$$

$$\square = 2\partial^+\partial^- - \vec{\nabla}_\perp^2 \qquad \text{Notation :} \quad \partial^+ \equiv \frac{\partial}{\partial x^-} \,,\; \partial^- \equiv \frac{\partial}{\partial x^+}$$

Light-cone coordinates

- The Dalembertian is bilinear in the derivatives ∂^+, ∂^-
- The metric tensor is non diagonal :

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



Action of K^- $\quad (K^- = -K^3)$

$$
\begin{aligned}
e^{i\omega K^-}\, P^-\, e^{-i\omega K^-} &= e^{-\omega} P^- \\
e^{i\omega K^-}\, P^+\, e^{-i\omega K^-} &= e^{+\omega} P^+ \\
e^{i\omega K^-}\, P^j\, e^{-i\omega K^-} &= P^j
\end{aligned}
$$

- Simple rescaling of the various operators. This suggests that the light-cone framework is simpler in order to study processes involving highly boosted particles
- These relations play an essential role in the eikonal approximation

Eikonal limit

- Consider an external vector potential, that couples via $e\mathcal{A}_\mu(x)J^\mu(x)$ (J^μ is the Noether current associated to some conserved charge.) Assume that the external potential is non-zero only in a finite range in x^+, $x^+ \in [-L,+L]$

Action of K^- on states and operators

$$e^{-i\omega K^-}\big|\vec{\mathbf{p}}\cdots{}_{\rm in}\big\rangle = \big|(e^{\omega}p^+,\vec{\mathbf{p}}_\perp)\cdots{}_{\rm in}\big\rangle$$
$$e^{-i\omega K^-}a^\dagger_{\rm in}(q)e^{i\omega K^-} = a^\dagger_{\rm in}(e^{\omega}q^+,e^{-\omega}q^-,\vec{\mathbf{q}}_\perp)$$
$$e^{i\omega K^-}\phi_{\rm in}(x)e^{-i\omega K^-} = \phi_{\rm in}(e^{-\omega}x^+,e^{\omega}x^-,\vec{\mathbf{x}}_\perp)$$

Eikonal limit

- Split the S matrix $U(+\infty,-\infty)$ into three factors :

$$U(+\infty,-\infty)=U(+\infty,+L)\times U(+L,-L)\times U(-L,-\infty)$$

Upon application of K^-, this becomes :

$$e^{i\omega K^-}U(+\infty,-\infty)e^{-i\omega K^-}=e^{i\omega K^-}U(+\infty,+L)e^{-i\omega K^-}\times e^{i\omega K^-}U(+L,-L)e^{-i\omega K^-}e^{i\omega K^-}U(-L,-\infty)e^{-i\omega K^-}$$

- The external potential $\mathcal{A}_\mu(x)$ is unaffected by K^-

Action of K^- on $J^\mu(x)$

$$\begin{aligned}
e^{i\omega K^-}J^i(x)e^{-i\omega K^-}&=J^i(e^{-\omega}x^+,e^{\omega}x^-,\vec{x}_\perp)\\
e^{i\omega K^-}J^-(x)e^{-i\omega K^-}&=e^{-\omega}\,J^-(e^{-\omega}x^+,e^{\omega}x^-,\vec{x}_\perp)\\
e^{i\omega K^-}J^+(x)e^{-i\omega K^-}&=e^{\omega}\,J^+(e^{-\omega}x^+,e^{\omega}x^-,\vec{x}_\perp)
\end{aligned}$$

- The factors $U(+\infty,+L)$ and $U(-L,-\infty)$ do not contain the external potential. In order to deal with these factors, it is sufficient to change variables : $e^{-\omega}x^+ \to x^+$, $e^{\omega}x^- \to x^-$. This leads to :

$$\lim_{\omega\to+\infty} e^{i\omega K^-}U(+\infty,+L)e^{-i\omega K^-} = U_0(+\infty,0)$$

$$\lim_{\omega\to+\infty} e^{i\omega K^-}U(-L,-\infty)e^{-i\omega K^-} = U_0(0,-\infty)$$

where U_0 is the same as U, but with the self-interactions only

Eikonal limit

- Therefore, in the limit $\omega \to +\infty$, we have :

$$\lim_{\omega\to+\infty} e^{i\omega K^-} U(+L,-L) e^{-i\omega K^-} = \exp\left[ie \int d^2\vec{x}_\perp \chi(\vec{x}_\perp)\rho(\vec{x}_\perp)\right]$$

with
$$\begin{cases} \chi(\vec{x}_\perp) \equiv \int dx^+ \; \mathcal{A}^-(x^+,0,\vec{x}_\perp) \\ \rho(\vec{x}_\perp) \equiv \int dx^- \; J^+(0,x^-,\vec{x}_\perp) \end{cases}$$

- The high-energy limit of the scattering amplitude is :

$$S^{(\infty)}_{\beta\alpha} = \langle\beta_{in}|U_0(+\infty,0)\; \exp\left[ie\int_{\vec{x}_\perp} \chi(\vec{x}_\perp)\rho(\vec{x}_\perp)\right] U_0(0,-\infty)|\alpha_{in}\rangle$$

  - Only the $-$ component of the vector potential matters
  - The self-interactions and the interactions with the external potential are factorized ▷ parton model
  - This is an exact result in the limit $\omega \to +\infty$

Eikonal limit

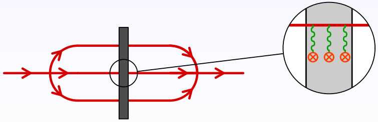
- For each intermediate state $\langle\delta_{in}| \equiv \langle\{k_i^+, \vec{k}_{i\perp}\}|$, define the corresponding light-cone wave function by :

$$\Psi_{\delta\alpha}(\{k_i^+, \vec{x}_{i\perp}\}) \equiv \prod_i \int \frac{d^2\vec{k}_{i\perp}}{(2\pi)^2} e^{-i\vec{k}_{i\perp}\cdot\vec{x}_{i\perp}} \langle\delta_{in}|U(0,-\infty)|\alpha_{in}\rangle$$

- Each charged particle going through the external field acquires a phase proportional to its charge (antiparticles get an opposite phase) :

$$\Psi_{\delta\alpha}(\{k_i^+, \vec{x}_{i\perp}\}) \longrightarrow \Psi_{\delta\alpha}(\{k_i^+, \vec{x}_{i\perp}\}) \prod_i U_i(\vec{x}_\perp)$$

$$U_i(\vec{x}_\perp) \equiv T_+ \exp\left[ig_i \int dx^+ \, \mathcal{A}_a^-(x^+, 0, \vec{x}_\perp) t^a\right]$$



Eikonal limit

- The number and the nature of the particles is unchanged under the action of the eikonal operator. In terms of the transverse coordinates, we simply have

$$\big\langle\gamma_{\rm in}\big|e^{ig\int\rho\chi}\big|\delta_{\rm in}\big\rangle = \delta_{NN'}\prod_i\Big[4\pi k_i^+\delta(k_i^+ - k_i^{+\prime})\delta(\vec{x}_{i\perp} - \vec{x}'_{i\perp})U_{R_i}(\vec{x}_{i\perp})\Big]$$

  where $U_R(\vec{x}_\perp)$ is a Wilson line operator, in the representation R appropriate for the particle going through the target

- Therefore, the high energy scattering amplitude can be written as :

$$S^{(\infty)}_{\beta\alpha} = \sum_\delta\int\Big[\prod_{i\in\delta}d\Phi_i\Big]\Psi^\dagger_{\delta\beta}(\{k_i^+,\vec{x}_{i\perp}\})\Big[\prod_{i\in\delta}U_{R_i}(\vec{x}_{i\perp})\Big]\Psi_{\delta\alpha}(\{k_i^+,\vec{x}_{i\perp}\})$$

DIS in the Eikonal limit

Differential DIS cross-section

- Differential photon-target cross-section ($\gamma^* T \to q\overline{q} + X$) :

$$d\sigma_{\gamma^* T} = \frac{d^3\mathbf{k}}{(2\pi)^2 2E_\mathbf{k}}\frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\;\frac{1}{2q^-}2\pi\delta(q^- - k^- - p^-)$$
$$\times \left<\mathcal{M}^\mu(\mathbf{q}|\mathbf{k},\mathbf{p})\mathcal{M}^{\nu *}(\mathbf{q}|\mathbf{k},\mathbf{p})\right>\;\epsilon_\mu(Q)\,\epsilon^*_\nu(Q)\;,$$

  - $\mathbf{k},\mathbf{p}$: momenta of the quark and antiquark
  - $\mathbf{q}$: momentum of the virtual photon
  - $\epsilon_\mu(Q)$: polarization vector

Total inclusive cross-section

- If we integrate out the final quark and antiquark, we get :

$$\sigma_{\gamma^* T} = \int_0^1 dz \int d^2\vec{r}_\perp \, |\psi(\mathbf{q}|z, \vec{r}_\perp)|^2 \, \sigma_{\text{dipole}}(\vec{r}_\perp)$$

with

$$\sigma_{\text{dipole}}(\vec{r}_\perp) \equiv \frac{2}{N_c} \int d^2\vec{X}_\perp \, \mathrm{Tr} \left\langle 1 - U(\vec{X}_\perp + \frac{\vec{r}_\perp}{2}) U^\dagger(\vec{X}_\perp - \frac{\vec{r}_\perp}{2}) \right\rangle$$

and ψ for the light-cone wave function for a photon that splits into a quark-antiquark intermediate state.

Dipole cross-section

- Computing F_2 requires to know the dipole amplitude

$$\left\langle \mathbf{T}(\vec{x}_\perp, \vec{y}_\perp) \right\rangle_Y \equiv \frac{1}{N_c}\,\mathrm{Tr}\,\left\langle 1 - U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right\rangle$$

as a function of dipole size and rapidity

- This object is often presented in the form of the dipole cross-section :

$$\sigma_{\mathrm{dip}}(\vec{r}_\perp, Y) \equiv 2\int d^2\vec{b}\ \left\langle \mathbf{T}(\vec{b} - \frac{\vec{r}_\perp}{2}, \vec{b} + \frac{\vec{r}_\perp}{2}) \right\rangle_Y$$

Golec-Biernat–Wusthoff model

- GBW modeled the dipole cross-section as a Gaussian, with an energy dependence entirely contained in Q_s

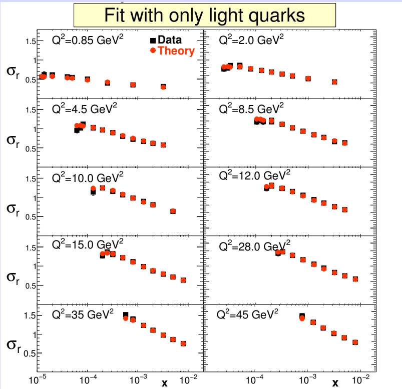
$$\begin{cases} \sigma_{\rm dip}(\vec{r}_\perp, Y) = \sigma_0 \left[1 - e^{-Q_s(Y)^2 r_\perp^2/4}\right] \\ Q_s^2(Y) = Q_0^2\, e^{\lambda(Y-Y_0)} \end{cases}$$

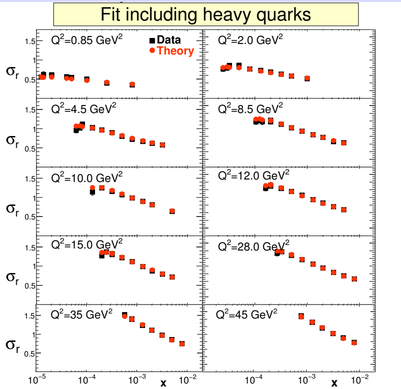
- The exponential form in $\sigma_{\rm dip}$ is inspired of Glauber scattering
- The fit parameters are σ_0, Q_0, λ and possibly an effective quark mass in the photon wave-function
- Quite good for all small-x HERA data, with some problems at large Q^2

State of the art : AAMQS model

- GBW model used only as initial condition
- Evolution with running coupling BK equation

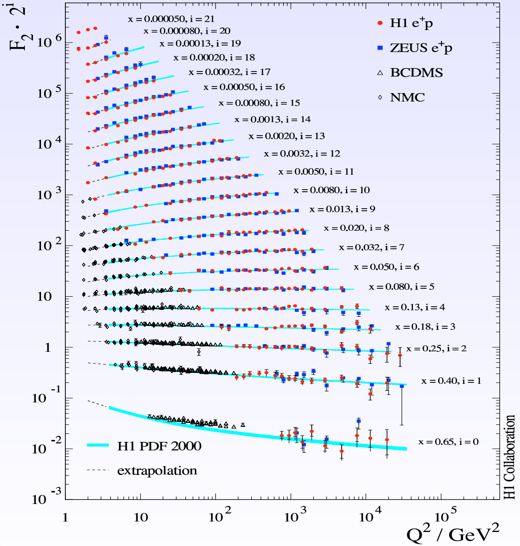
Comparison with HERA data





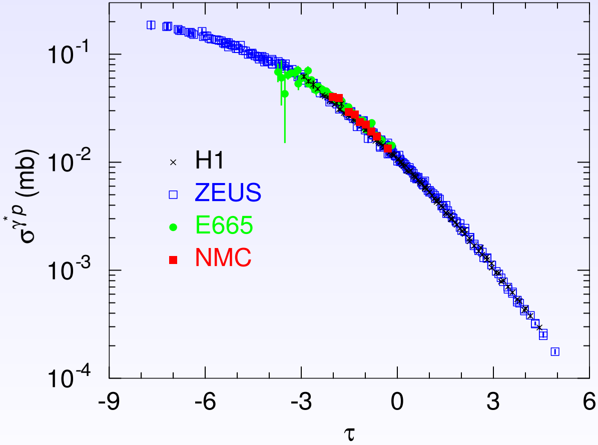
Summary of Lecture I

DIS results for F_2 (DGLAP equation at NLO)



Small x data displayed differently... (Geometrical scaling)

- Small x data ($x \le 10^{-2}$) displayed against $\tau \equiv \log(x^{0.32}\,Q^2)$:



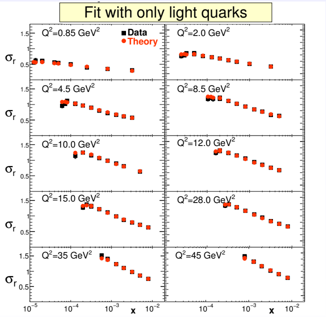
DIS cross-section in the eikonal limit

$$\sigma_{\gamma^* T}(x, Q^2) = \int_0^1 dz \int d^2\vec{r}_\perp \left|\psi(Q^2|z, \vec{r}_\perp)\right|^2 \sigma_{\text{dipole}}(x, \vec{r}_\perp)$$

- Golec-Biernat–Wusthoff model :

$$\sigma_{\text{dip}}(x, \vec{r}_\perp) = \sigma_0 \left[1 - e^{-Q_s^2(x) r_\perp^2/4}\right] \qquad Q_s^2(x) = Q_0^2\,(x/x_0)^\lambda$$

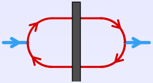
- State of the art : AAMQS model
  - The GBW model is used as input at $x_0 \approx 10^{-2}$
  - Smaller x's are obtained from the Balitsky-Kovchegov equation



BFKL equation

Scattering of a dipole

- Take a virtual photon as initial and final state. At lowest order, the scattering amplitude can be written as :



$$\propto \left|\Psi^{(0)}(\vec{x}_\perp, \vec{y}_\perp)\right|^2 \mathrm{tr}\left[U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$

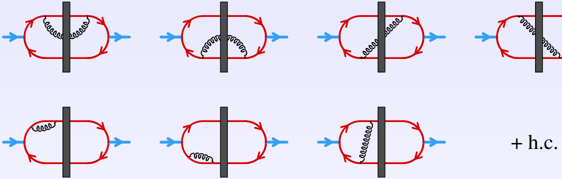
- It turns out that 1-loop corrections to this contribution are enhanced by $\alpha_s \log(p^+)$, which can be large when the quark or antiquark has a large p^+

- In the gauge $A^+ = 0$, the emission of a gluon of momentum k by a quark can be written as :

$$= 2gt^a \frac{\vec{\epsilon}_\lambda \cdot \vec{k}_\perp}{k_\perp^2}$$

Scattering of a dipole

- The following diagrams must be evaluated :



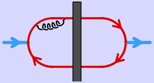
- When connecting two gluons, one must use :

$$\sum_{\lambda} \vec{\epsilon}_{\lambda}^{\,i}\,\vec{\epsilon}_{\lambda}^{\,j} = -g^{ij}$$

Virtual corrections

- Consider first the loop corrections inside the wavefunction of the incoming or outgoing dipole

Example



$$= \left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \mathrm{tr}\left[t^a t^a U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$

$$\times -2\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{z}_\perp}{(2\pi)^2} \frac{(\vec{x}_\perp-\vec{z}_\perp)\cdot(\vec{x}_\perp-\vec{z}_\perp)}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{x}_\perp-\vec{z}_\perp)^2}$$

Reminder : $t^a t^a = (N_c^2-1)/2N_c$ (denoted C_F)

Virtual corrections

- The sum of all virtual corrections is :

$$-\frac{C_F \alpha_s}{\pi^2} \int \frac{dk^+}{k^+} \int d^2\vec{z}_\perp \; \frac{(\vec{x}_\perp - \vec{y}_\perp)^2}{(\vec{x}_\perp - \vec{z}_\perp)^2 (\vec{y}_\perp - \vec{z}_\perp)^2} \times \left| \Psi^{(0)}(\vec{x}_\perp, \vec{y}_\perp) \right|^2 \mathrm{tr}\left[U(\vec{x}_\perp) U^\dagger(\vec{y}_\perp) \right]$$

- The integral over k^+ is divergent. It should have an upper bound at p^+ :

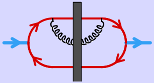
$$\int^{p^+} \frac{dk^+}{k^+} = \ln(p^+) = Y$$

▷ When Y is large, $\alpha_s Y$ may not be small,and these corrections should be resummed

Real corrections

- There are also real corrections, for which the state that interacts with the target has an extra gluon

Example



$$= \left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{x}_\perp) t^b U^\dagger(\vec{y}_\perp)\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{z}_\perp}{(2\pi)^2} \widetilde{U}_{ab}(\vec{z}_\perp) \frac{(\vec{x}_\perp - \vec{z}_\perp)\cdot(\vec{x}_\perp - \vec{z}_\perp)}{(\vec{x}_\perp - \vec{z}_\perp)^2 (\vec{x}_\perp - \vec{z}_\perp)^2}$$

($\widetilde{U}_{ab}(\vec{z}_\perp)$ is a Wilson line in the adjoint representation)

- In order to simplify the color structure, use :

$$t^a \widetilde{U}_{ab}(\vec{z}_\perp) = U(\vec{z}_\perp) t^b U^\dagger(\vec{z}_\perp)$$

- + the $SU(N_c)$ Fierz identity :

$$t^b_{ij} t^b_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta kl$$

Evolution equation

- Denote : $\mathbf{S}(\vec{x}_\perp,\vec{y}_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\left[U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$

- The full LO + NLO scattering amplitude reads :

$$N_c\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2\Bigg[\mathbf{S}(\vec{x}_\perp,\vec{y}_\perp)-\frac{\alpha_s N_c Y}{2\pi^2}\int d^2\vec{z}_\perp\,\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}\times\Big\{\mathbf{S}(\vec{x}_\perp,\vec{y}_\perp)-\mathbf{S}(\vec{x}_\perp,\vec{z}_\perp)\mathbf{S}(\vec{z}_\perp,\vec{y}_\perp)\Big\}\Bigg]$$

$$\frac{\partial \mathbf{S}(\vec{x}_\perp,\vec{y}_\perp)}{\partial Y}=-\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp\,\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}\times\Big\{\mathbf{S}(\vec{x}_\perp,\vec{y}_\perp)-\mathbf{S}(\vec{x}_\perp,\vec{z}_\perp)\mathbf{S}(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

Kuraev, Lipatov, Fadin (1977), Balitsky, Lipatov (1978)

- Write $\mathbf{S}(\vec{x}_\perp, \vec{y}_\perp) \equiv 1 - \mathbf{T}(\vec{x}_\perp, \vec{y}_\perp)$ and assume that we are in the dilute regime, so that the scattering amplitude $\mathbf{T}$ is small

  Drop the terms that are non-linear in $\mathbf{T}$

BFKL equation in coordinate space

$$\frac{\partial\,\mathbf{T}(\vec{x}_\perp, \vec{y}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{z}_\perp \; \frac{(\vec{x}_\perp - \vec{y}_\perp)^2}{(\vec{x}_\perp - \vec{z}_\perp)^2 (\vec{y}_\perp - \vec{z}_\perp)^2}$$
$$\times \Big\{ \mathbf{T}(\vec{x}_\perp, \vec{z}_\perp) + \mathbf{T}(\vec{z}_\perp, \vec{y}_\perp) - \mathbf{T}(\vec{x}_\perp, \vec{y}_\perp) \Big\}$$

Gluon Saturation

Unitarity problem

- The mapping $\mathbf{T} \to \alpha_s N_c \int_z \cdots \mathbf{T}$ has a positive eigenvalue ω
- Solutions of the BFKL equation grow exponentially as $\exp(\omega Y)$ when $Y \to +\infty$ $\quad \triangleright$ violation of unitarity...
- In perturbation theory, the forward scattering amplitude between a small dipole and a target made of gluons reads :

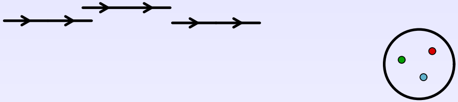
$$\mathbf{T}(\vec{\mathbf{x}}_\perp, \vec{\mathbf{y}}_\perp) \propto |\vec{\mathbf{x}}_\perp - \vec{\mathbf{y}}_\perp|^2 \;\; xG(x, |\vec{\mathbf{x}}_\perp - \vec{\mathbf{y}}_\perp|^{-2})$$

where $\quad Y \equiv \ln(1/x)$

- Therefore, the exponential behavior of $\mathbf{T}$ is related to the increase of the gluon distribution at small x

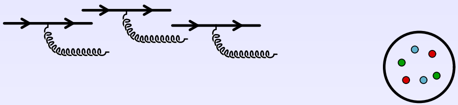
$$\mathbf{T} \sim e^{\omega Y} \qquad \longleftrightarrow \qquad xG(x, Q^2) \sim \frac{1}{x^\omega}$$

Parton evolution under boosts



▷ at low energy, only valence quarks are present in the hadron wave function

Parton evolution under boosts

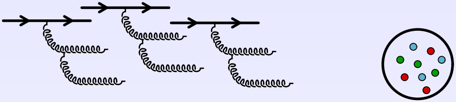


▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with x the longitudinal momentum fraction of the gluon

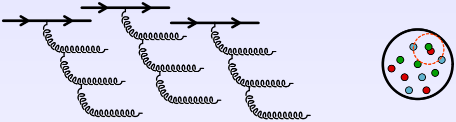
▷ at small-x (i.e. high energy), these logs need to be resummed

Parton evolution under boosts



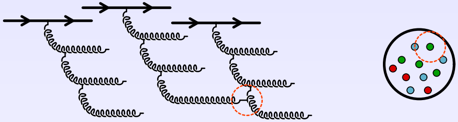
▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step

Parton evolution under boosts



▷ eventually, the partons start overlapping in phase-space

Parton evolution under boosts



▷ parton recombination becomes favorable

▷ after this point, the evolution is non-linear:

the number of partons created at a given step depends non-linearly on the number of partons present previously

Balitsky (1996), Kovchegov (1996,2000)

Jalilian-Marian, Kovner, Leonidov, Weigert (1997,1999)

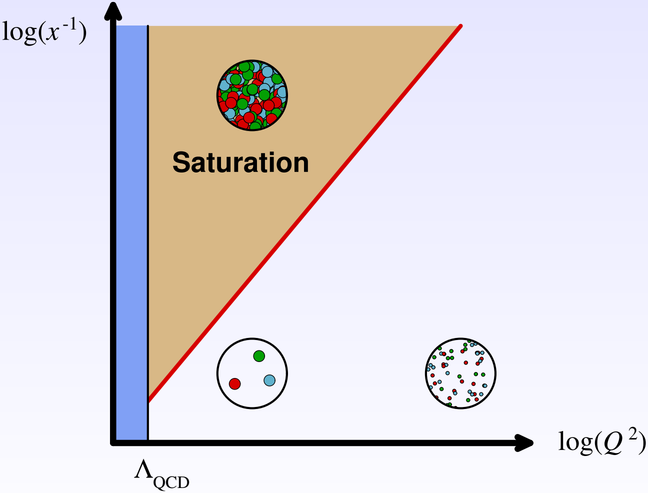
Iancu, Leonidov, McLerran (2001)

Saturation criterion [Gribov, Levin, Ryskin (1983)]

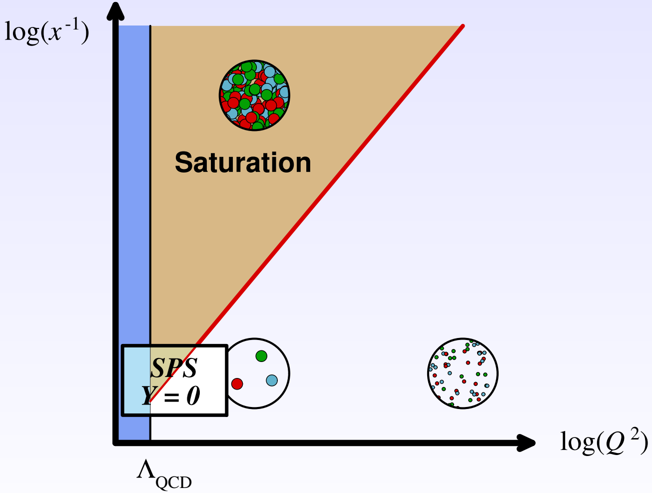
$$\underbrace{\alpha_s Q^{-2}}_{\sigma_{gg\to g}} \times \underbrace{A^{-2/3} xG(x, Q^2)}_{\text{surface density}} \geq 1$$

$$Q^2 \leq \underbrace{Q_s^2 \equiv \frac{\alpha_s xG(x, Q_s^2)}{A^{2/3}}}_{\text{saturation momentum}} \sim A^{1/3} x^{-0.3}$$

Saturation domain



Saturation domain



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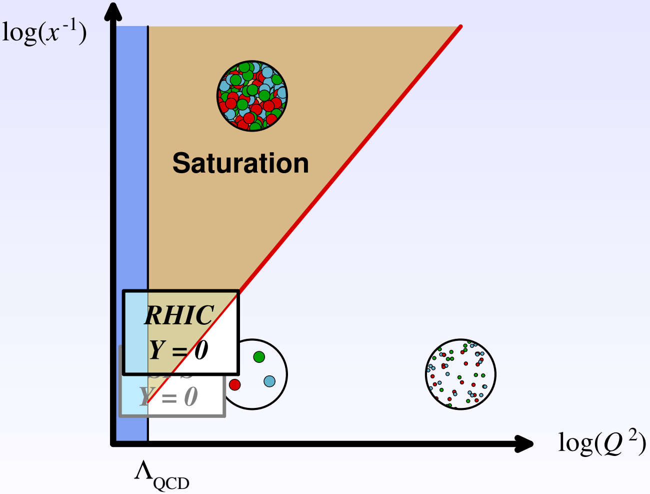

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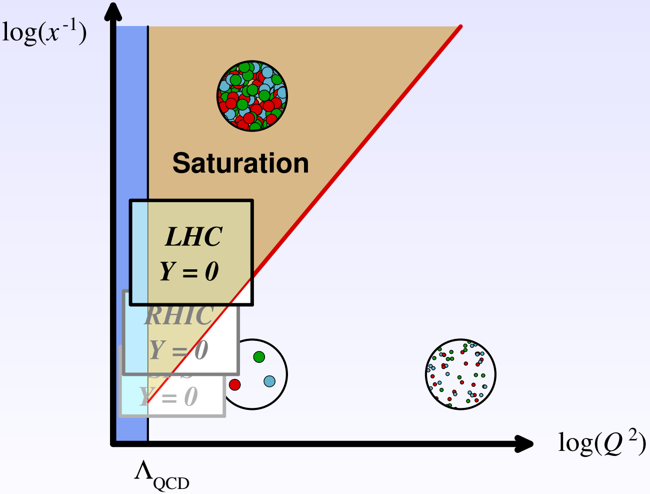

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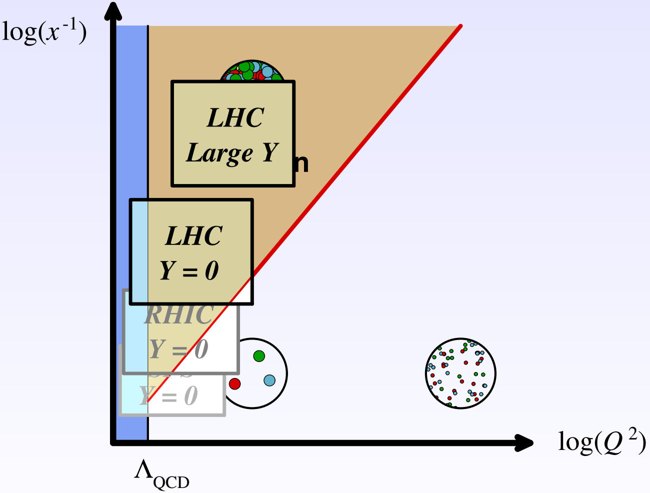
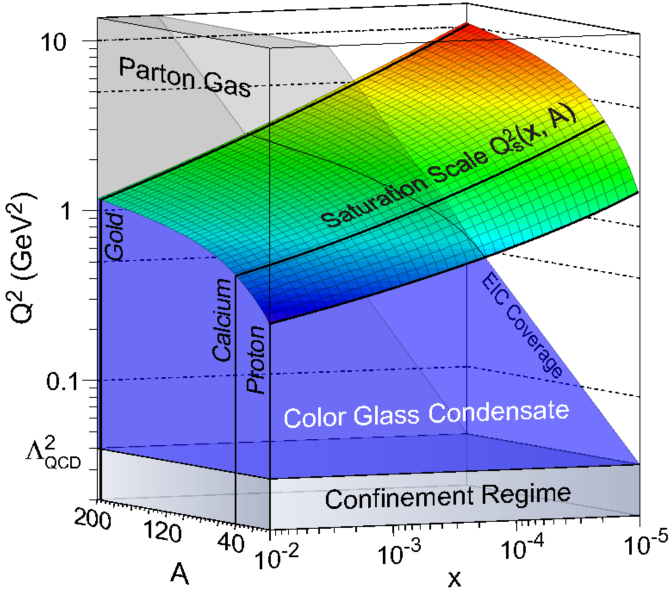


Saturation domain



Balitsky-Kovchegov equation

Non-linear evolution equation

- The first evolution equation we derived has the non-linear effects due to recombination :

$$\frac{\partial\,\mathbf{T}(\vec{x}_\perp,\vec{y}_\perp)}{\partial Y}=\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp\;\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\Big\{\mathbf{T}(\vec{x}_\perp,\vec{z}_\perp)+\mathbf{T}(\vec{z}_\perp,\vec{y}_\perp)-\mathbf{T}(\vec{x}_\perp,\vec{y}_\perp)-\mathbf{T}(\vec{x}_\perp,\vec{z}_\perp)\mathbf{T}(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

(Balitsky-Kovchegov equation)

- The r.h.s. vanishes when $\mathbf{T}$ reaches 1, and the growth stops. The non-linear term lets both dipoles interact after the splitting of the original dipole

- Both $\mathbf{T}=0$ and $\mathbf{T}=1$ are fixed points of this equation
  - $\mathbf{T}=\epsilon$: r.h.s.>0 $\Rightarrow$ $\mathbf{T}=0$ is unstable
  - $\mathbf{T}=1-\epsilon$: r.h.s.>0 $\Rightarrow$ $\mathbf{T}=1$ is stable

Caveats

- So far, we have studied the scattering amplitude between a color dipole and a “god given” patch of color field. This is too crude to describe any realistic situation
- One can describe Deep Inelastic Scattering as an interaction between a dipole and the target, but for that we need to improve the treatment of the target
- An experimentally measured cross-section is an average over many collisions, and the target fields fluctuate event-by-event :

$$\mathbf{T} \quad \to \quad \langle \mathbf{T} \rangle$$

Balitsky hierarchy

- Because of this average over the target configurations, the evolution equation we have derived should be written as :

$$\frac{\partial \left\langle \mathbf{T}(\vec{x}_\perp,\vec{y}_\perp)\right\rangle}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp \; \frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\Big\{\left\langle \mathbf{T}(\vec{x}_\perp,\vec{z}_\perp)\right\rangle+\left\langle \mathbf{T}(\vec{z}_\perp,\vec{y}_\perp)\right\rangle-\left\langle \mathbf{T}(\vec{x}_\perp,\vec{y}_\perp)\right\rangle-\left\langle \mathbf{T}(\vec{x}_\perp,\vec{z}_\perp)\mathbf{T}(\vec{z}_\perp,\vec{y}_\perp)\right\rangle\Big\}$$

- As one can see, the equation is no longer a closed equation, since the equation for $\langle \mathbf{T}\rangle$ depends on a new object, $\langle \mathbf{T}\,\mathbf{T}\rangle$
- One can derive an evolution equation for $\langle \mathbf{T}\,\mathbf{T}\rangle$. Its right hand side contains objects with six Wilson lines
  - There is in fact an infinite hierarchy of nested evolution equations, whose generic structure is

$$\frac{\partial \left\langle (UU^\dagger)^n\right\rangle}{\partial Y} = \int\cdots\left\langle (UU^\dagger)^n\right\rangle \oplus \left\langle (UU^\dagger)^{n+1}\right\rangle$$

BK equation as a mean-field approximation

- In the large N_c approximation, the equations of the Balitsky hierarchy can be rewritten in terms of the dipole operator $\mathbf{T} \equiv \mathrm{tr}(UU^\dagger)$ only. But they still contain averages like $\langle \mathbf{T}^n \rangle$
- In order to truncate the hierarchy of equations, one may assume a mean field approximation

$$\langle \mathbf{T}\,\mathbf{T} \rangle \approx \langle \mathbf{T} \rangle\ \langle \mathbf{T} \rangle$$

- This approximation gives for $\langle \mathbf{T} \rangle$ the same evolution equation as the one we had for a fixed configuration of the target (Balitsky-Kovchegov equation)

Geometrical Scaling from BK evolution

Analogy with reaction-diffusion processes

Munier, Peschanski (2003,2004)

- Assume translation and rotation invariance, and define :

$$N(Y, k_\perp) \equiv 2\pi \int d^2\vec{x}_\perp \; e^{i\vec{k}_\perp \cdot \vec{x}_\perp} \; \frac{\langle \mathbf{T}(0, \vec{x}_\perp) \rangle_Y}{x_\perp^2}$$

- From the Balitsky-Kovchegov equation for $\langle \mathbf{T} \rangle_Y$, we obtain the following equation for N :

$$\frac{\partial N(Y, k_\perp)}{\partial Y} = \frac{\alpha_s N_c}{\pi} \left[\chi(-\partial_L) N(Y, k_\perp) - N^2(Y, k_\perp) \right]$$

with

$$L \equiv \ln(k^2/k_0^2)$$

$$\chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

$$\psi(z) \equiv \frac{d \ln \Gamma(z)}{dz}$$

Analogy with reaction-diffusion processes

- Expand the function $\chi(\gamma)$ to second order (diffusion approximation) around its minimum $\gamma = 1/2$
- Introduce new variables :

$$
\begin{aligned}
t &\sim Y \\
z &\sim L + \frac{\alpha_s N_c}{2\pi} \chi''(1/2)\, Y
\end{aligned}
$$

- The equation for N becomes :

$$
\partial_t N = \partial_z^2 N + N - N^2
$$

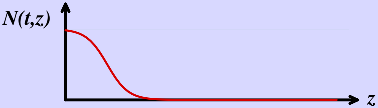
(known as the Fisher-Kolmogorov-Petrov-Piscounov equation)

Analogy with reaction-diffusion processes

- Interpretation : this equation is typical for all the diffusive systems subject to a reaction $A \longleftrightarrow A + A$
  - $\partial_z^2 N$: diffusion term (the quantity under consideration can hop from a site to the neighboring sites)
  - $+N$: gain term corresponding to $A \to A + A$
  - $-N^2$: loss term corresponding to $A + A \to A$
- Note : this equation has two fixed points :
  - $N = 0$: unstable
  - $N = 1$: stable
- The stable fixed point at $N = 1$ exists only if one keeps the loss term. One would not have it from the BFKL equation

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times :

$$N(t,z) \underset{t\to+\infty}{\sim} N(z - v(t))$$

with $v(t) \approx 2t - 3\ln(t)/2$

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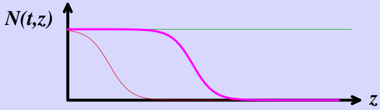

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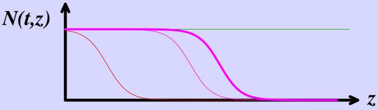

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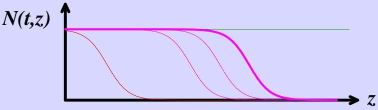

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Geometrical scaling in DIS

Iancu, Itakura, McLerran (2002)
Mueller, Triantafyllopoulos (2002)
Munier, Peschanski (2003)

- Going back to the original variables, one gets :

$$N(Y, k_\perp) = N\left(k_\perp / Q_s(Y)\right)$$

with

$$Q_s^2(Y) = k_0^2 \; Y^{-\frac{3}{2(1-\bar{\gamma})}} \; e^{\overline{\alpha}_s \chi''(\frac{1}{2})(\frac{1}{2}-\bar{\gamma}) Y}$$

- Going from $N(Y, k_\perp)$ to $\left\langle T(0, \vec{x}_\perp) \right\rangle_Y$, we obtain :

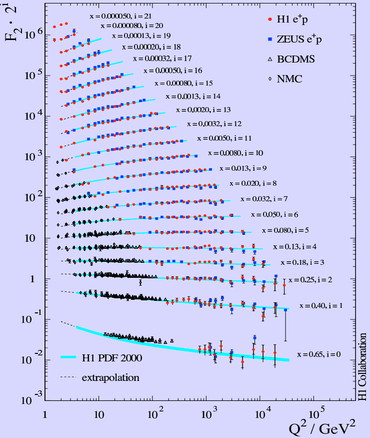
$$\left\langle T(0, \vec{x}_\perp) \right\rangle_Y = T(Q_s(Y) x_\perp)$$

- Reminder : $\gamma^* p$ cross-section expressed in terms of $\mathbf{T}$:

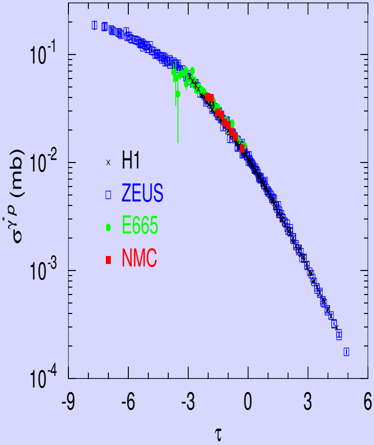
$$\sigma_{\gamma^* p}(Y, Q^2) = 2\,\sigma_0 \int d^2\vec{x}_\perp \int_0^1 dz \left|\psi(z, x_\perp, Q^2)\right|^2 \left\langle \mathbf{T}(0, \vec{x}_\perp)\right\rangle_Y$$

- If one neglects the quark masses in ψ, the scaling property of $\left\langle \mathbf{T}\right\rangle_Y$ imply that $\sigma_{\gamma^* p}$ depends only on the ratio $Q^2/Q_s^2(Y)$, rather than on Q^2 and Y separately

$F_2(x, Q^2)$



$\sigma_{\gamma^* p}(\log(Q^2/Q_s^2(x)))$

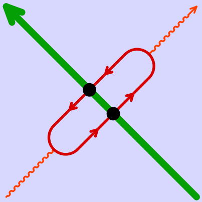


Color Glass Condensate

DIS and other elementary reactions

- Reactions involving a hadron or nucleus and an “elementary” projectile (γ^*, q or g) are fairly straightforward to study

Deep Inelastic Scattering is the archetype of all these processes



- These processes play a role in the study of proton-nucleus collisions, where the proton is described as a dilute beam of quarks and gluons

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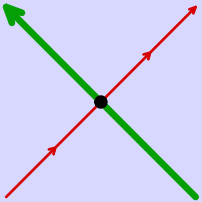
$qA \to qX$



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DIS and other elementary reactions

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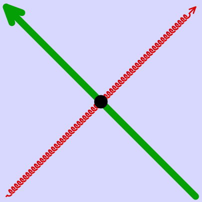
$\mathrm{g}A \to \mathrm{g}X$



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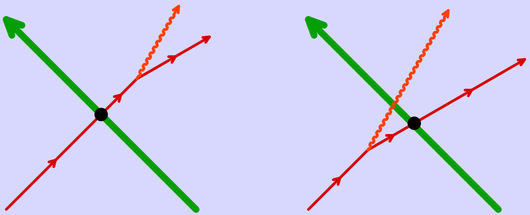
$\mathrm{q}A \to \mathrm{q}\gamma X$



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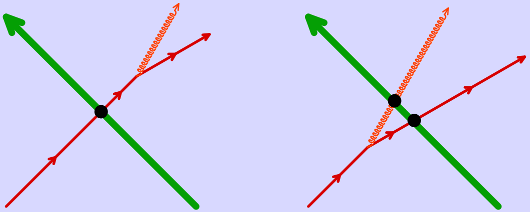

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DIS and other elementary reactions

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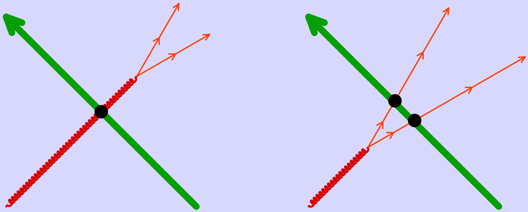
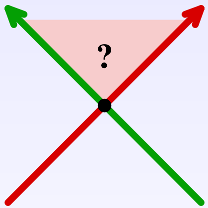
$gA \to q\overline{q}X$



- These processes play a role in the study of proton-nucleus collisions, where the proton is described as a dilute beam of quarks and gluons

Nucleus-nucleus collisions ?

- What about collisions where the two projectiles are equally dense?
- It would be nice to have a formalism that allows one to treat the two projectiles on the same footing :



- Conjecture (for the Leading Order):
  - Before the collision, the two projectiles are described by their own color field, like in DIS
  - After the collision (i.e. in the forward light-cone), there is a color field that obeys Yang-Mills equations, and whose boundary condition on the light-cone is given by the fields of the incoming nuclei

Nucleus-nucleus collisions ?

- Can we set up a framework where this can be justified ?

  Note : this new formalism should lead to the same results for DIS, not something completely different...

- Can we include all the multiple scattering corrections ?

- How do we compute observables for two saturated objects ?

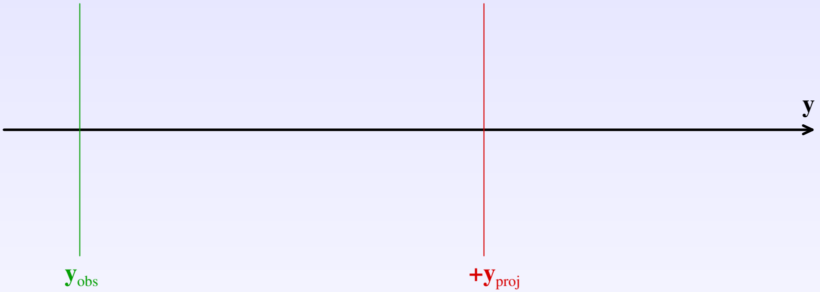
- Can we compute and resum all the large logs of $1/x_{1,2}$?

- The BK equation, can be viewed as a projectile-centric description of a collision process. The rapidity evolution comes from the dressing of the projectile as it is boosted

- One may see the Color Glass Condensate as a description of the same physics from the point of view of the target

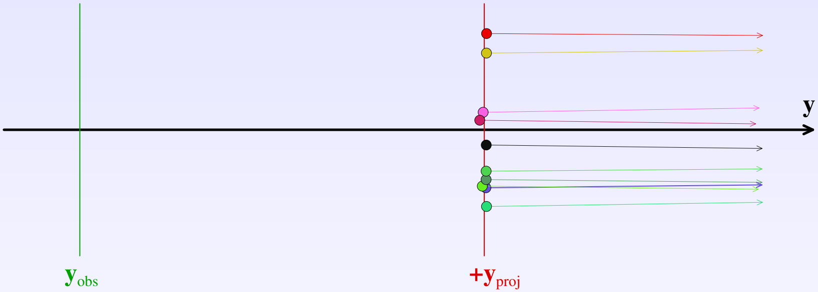
  In this target-centric description, the projectile does not change, but the color fields of the target depend on the rapidity

Degrees of freedom and their interplay



- $p_\perp^2 \sim Q_s^2 \sim \Lambda_{QCD}\, e^{\lambda(y_{proj}-y)}$, $p_z \sim Q_s\, e^{y-y_{obs}}$
- Fast partons : frozen dynamics, negligible $p_\perp$ $\Rightarrow$ classical current
- Slow partons : evolve with time $\Rightarrow$ gauge fields

Degrees of freedom and their interplay



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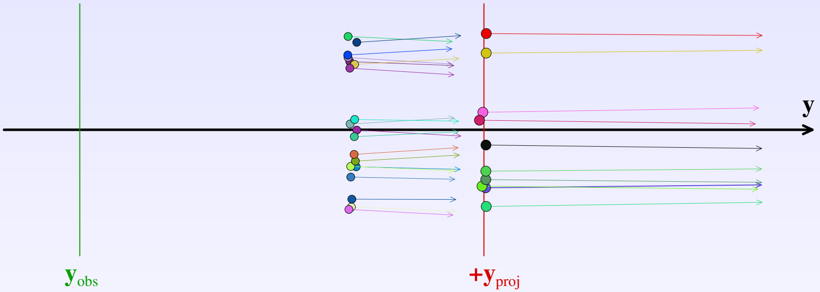

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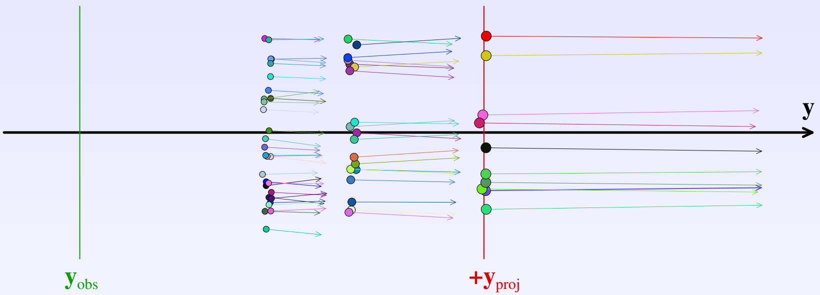

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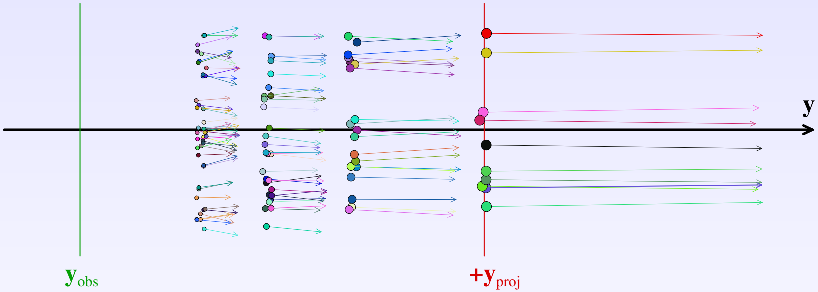

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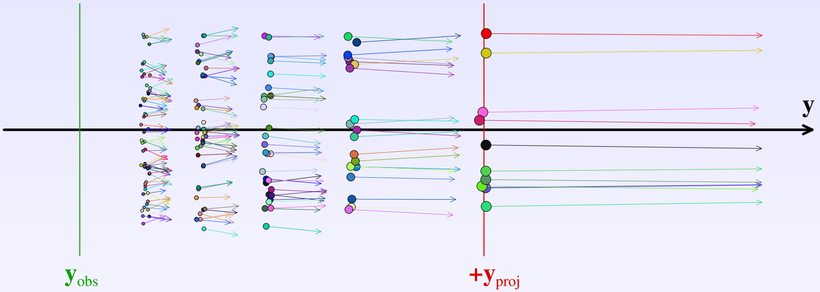

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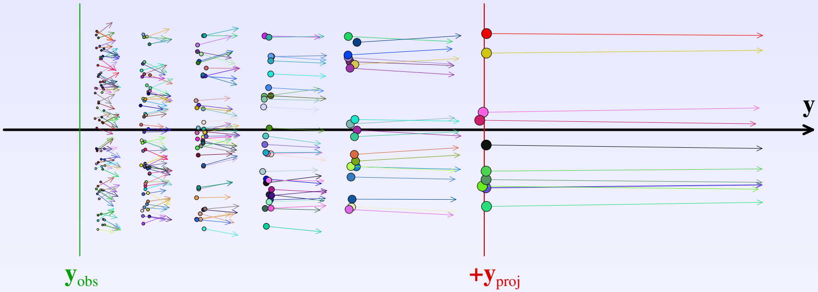

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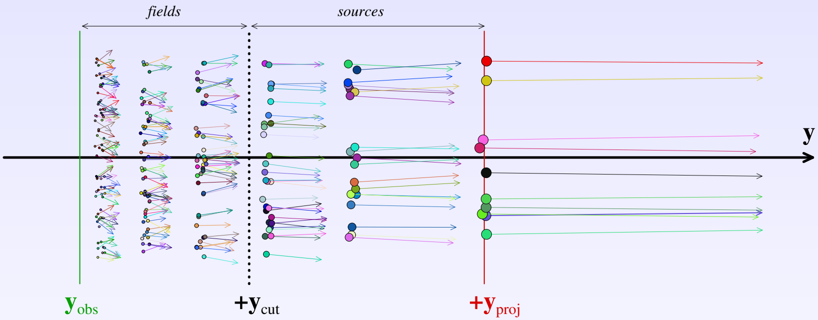

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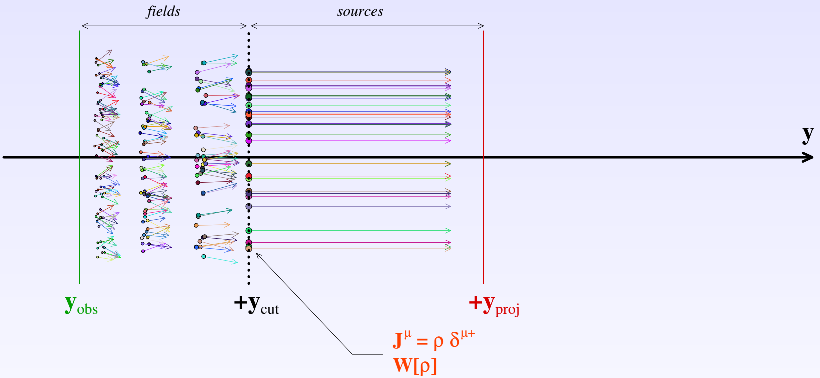

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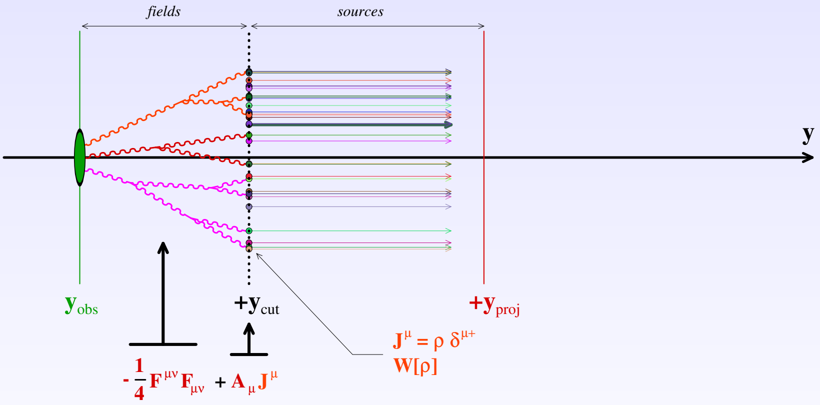

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McLerran (2000)

- Color : quarks and gluons are colored
- Glass : the system has degrees of freedom whose timescale is much larger than the typical timescales for interaction processes. Moreover, these degrees of freedom are stochastic variables, like in “spin glasses” for instance
- Condensate : the soft degrees of freedom are as densely packed as they can (the density remains finite, of order α_s^{-1}, due to the interactions between gluons)

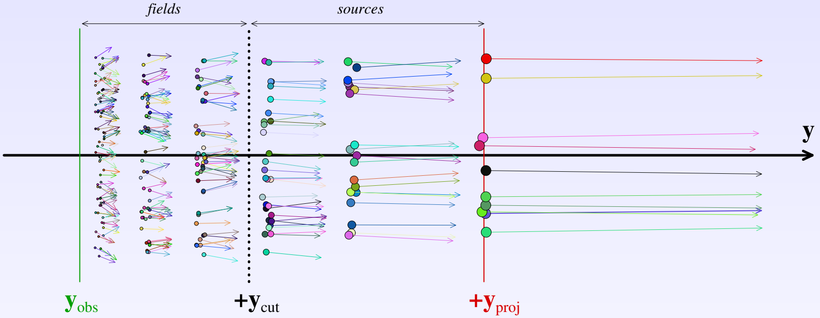
Target average

- The averaged dipole amplitude $\langle \mathbf{T} \rangle$ studied in the Balitsky-Kovchegov approach can be written as :

$$\langle \mathbf{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \rangle = \int [D\rho]\;\; W[\rho]\;\; \left[1 - \frac{1}{N_c} \mathrm{tr}(U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp))\right]$$

- In this formalism, the Y dependence of the expectation value $\langle \mathbf{T} \rangle$ must come from the probability density $W[\rho]$

Cancellation of the cutoff dependence



- The cutoff y_{cut} is arbitrary and should not affect the result

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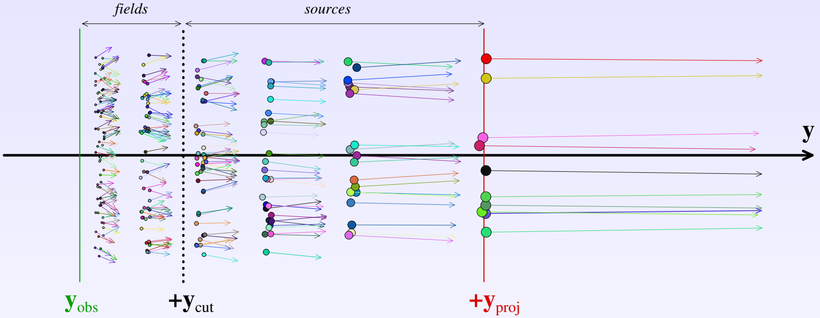

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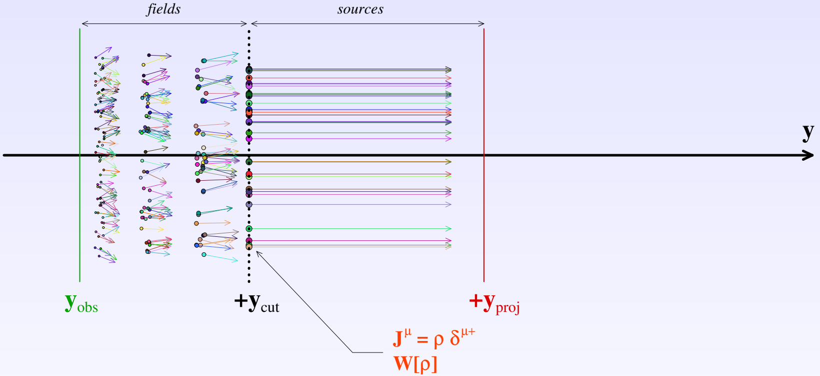

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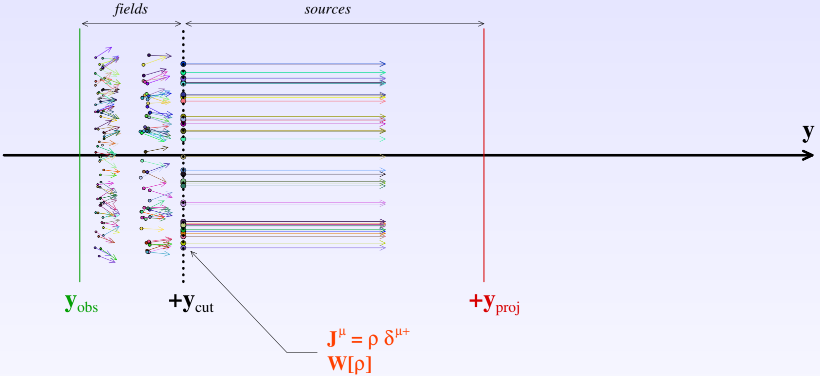

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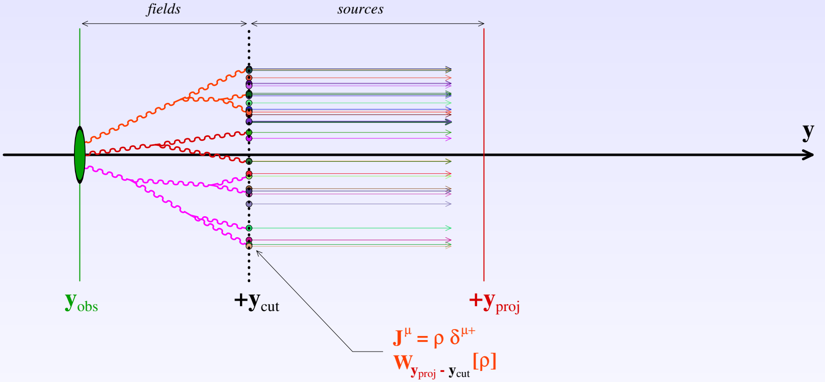
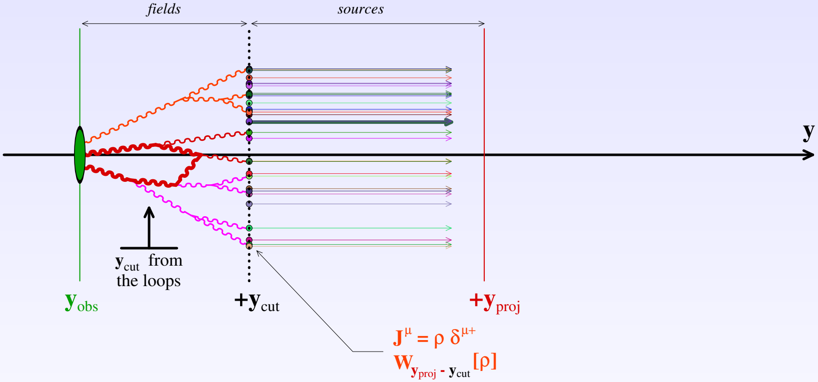


- The cutoff $y_{\rm cut}$ is arbitrary and should not affect the result
- The probability distribution $W[\rho]$ changes with the cutoff

Cancellation of the cutoff dependence



- The cutoff y_{cut} is arbitrary and should not affect the result
- The probability distribution $W[\rho]$ changes with the cutoff
- Loop corrections are also y_{cut}-dependent and cancel the cutoff dependence coming from $W[\rho]$, to all orders $(\alpha_s y_{cut})^n$ (Leading Log)

JIMWLK evolution equation

Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

$$\frac{\partial W_Y[\rho]}{\partial Y} = \underbrace{\frac{1}{2}\int_{\vec{x}_\perp,\vec{y}_\perp} \frac{\delta}{\delta\rho_a(\vec{x}_\perp)}\chi_{ab}(\vec{x}_\perp,\vec{y}_\perp)\frac{\delta}{\delta\rho_b(\vec{y}_\perp)}}_{\mathcal{H}\quad\text{(JIMWLK Hamiltonian)}} \; W_Y[\rho]$$

with

$$\chi_{ab}(\vec{x}_\perp,\vec{y}_\perp) \equiv \frac{\alpha_s}{4\pi^3}\int d^2\vec{z}_\perp \frac{(\vec{x}_\perp-\vec{z}_\perp)\cdot(\vec{y}_\perp-\vec{z}_\perp)}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2} \times\Big[\Big(1-\widetilde{U}^\dagger(\vec{x}_\perp)\widetilde{U}(\vec{z}_\perp)\Big)\Big(1-\widetilde{U}^\dagger(\vec{z}_\perp)\widetilde{U}(\vec{y}_\perp)\Big)\Big]_{ab}$$

- $\widetilde{U}$ is a Wilson line in the adjoint representation, that exponentiates the gauge field A^+ such that $\nabla_\perp^2 A^+ = -\rho$

JIMWLK evolution equation

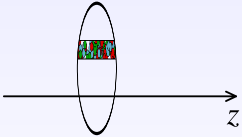
- Sketch of a derivation : exploit the frame independence in order to write :

$$\langle \mathcal{O} \rangle_Y = \underbrace{\int [D\rho] \; W_0[\rho] \; \mathcal{O}_Y[\rho]}_{\text{Balitsky-Kovchegov description}} = \underbrace{\int [D\rho] \; W_Y[\rho] \; \mathcal{O}_0[\rho]}_{\text{CGC description}}$$

- Universality : the evolution of $W_Y[\rho]$ does not depend on the observable one is considering

Initial condition : McLerran–Venugopalan model

- The JIMWLK equation must be completed by an initial condition, given at some moderate x_0
- As with DGLAP, the initial condition is non-perturbative
- The McLerran-Venugopalan model is often used as an initial condition at moderate x_0 for a large nucleus :



- partons distributed randomly
- many partons in a small tube
- no correlations at different $\vec{x}_\perp$

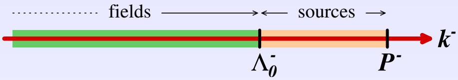
- The MV model assumes that the density of color charges $\rho(\vec{x}_\perp)$ has a Gaussian distribution :

$$W_{x_0}[\rho] = \exp\left[-\int d^2\vec{x}_\perp \frac{\rho_a(\vec{x}_\perp)\rho_a(\vec{x}_\perp)}{2\mu^2(\vec{x}_\perp)}\right]$$

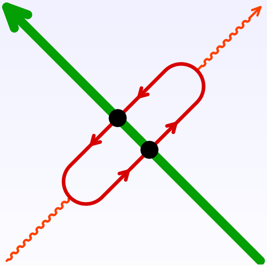
CGC applied to DIS

Inclusive DIS at Leading Order

- CGC effective theory with cutoff at the scale Λ_0^- :



- At Leading Order, DIS can be seen as the interaction between the target and a $q\bar{q}$ fluctuation of the virtual photon :



Inclusive DIS at Leading Order

- Forward dipole amplitude at leading order:

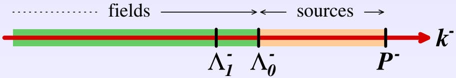
$$\begin{aligned} T_{\rm LO}(\vec{x}_\perp,\vec{y}_\perp) &= 1-\frac{1}{N_c}\,{\rm tr}\,(\underbrace{U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)}_{\text{Wilson lines}}) \\ U(\vec{x}_\perp) &= P\,\exp i g\int^{1/xP^-} dz^+\,\mathcal{A}^-(z^+,\vec{x}_\perp) \\ [\mathcal{D}_\mu,\mathcal{F}^{\mu\nu}] &= \delta^{\nu-}\,\rho(x^+,\vec{x}_\perp) \end{aligned}$$

$\triangleright$ at LO, the scattering amplitude on a saturated target is entirely given by classical fields

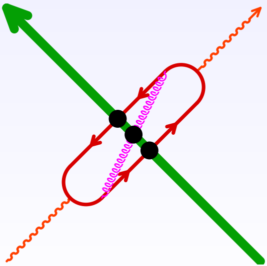
- Note: the $q\bar{q}$ pair couples only to the sources up to the longitudinal coordinate $z^+ \lesssim (xP^-)^{-1}$. The other sources are too slow to be seen by the probe

Inclusive DIS at NLO

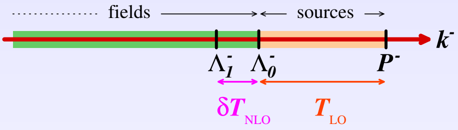
- Consider now quantum corrections to the previous result, restricted to modes with $\Lambda_1^- < k^- < \Lambda_0^-$ (the upper bound prevents double-counting with the sources):



- At NLO, the $q\bar{q}$ dipole must be corrected by a gluon, e.g. :



Inclusive DIS at NLO



- At leading log accuracy, the contribution of the quantum modes in that strip is :

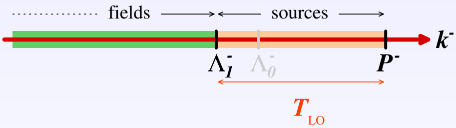
$$\delta \mathbf{T}_{\rm NLO}(\vec{x}_\perp, \vec{y}_\perp) = \log\left(\frac{\Lambda_0^-}{\Lambda_1^-}\right) \mathcal{H}\, \mathbf{T}_{\rm LO}(\vec{x}_\perp, \vec{y}_\perp)$$

Inclusive DIS at NLO

- These NLO corrections can be absorbed in the LO result,

$$\left\langle \mathbf{T}_{\rm LO} + \delta\mathbf{T}_{\rm NLO} \right\rangle_{\Lambda_0^-} = \left\langle \mathbf{T}_{\rm LO} \right\rangle_{\Lambda_1^-}$$

provided one defines a new effective theory with a lower cutoff Λ_1^- and an extended distribution of sources $W_{\Lambda_1^-}[\rho]$:



$$W_{\Lambda_1^-} \equiv \left[1 + \log\left(\frac{\Lambda_0^-}{\Lambda_1^-}\right) \mathcal{H}\right] W_{\Lambda_0^-}$$

(JIMWLK equation for a small change in the cutoff)

Inclusive DIS at Leading Log

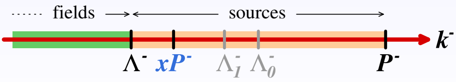
- Iterate the previous process to integrate out all the slow field modes at leading log accuracy:

Inclusive DIS at Leading Log accuracy

$$\sigma_{\gamma^* T} = \int_0^1 dz \int d^2\vec{r}_\perp |\psi(q|z,\vec{r}_\perp)|^2 \sigma_{\rm dipole}(x,\vec{r}_\perp)$$

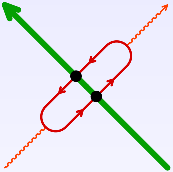
$$\sigma_{\rm dipole}(x,\vec{r}_\perp) \equiv 2\int d^2\vec{X}_\perp \int [D\rho]\, W_{xP^-}[\rho]\, T_{LO}(\vec{x}_\perp,\vec{y}_\perp)$$

- One does not need to evolve down to $\Lambda^- \to 0$: the DIS amplitude becomes independent of Λ^- when $\Lambda^- \lesssim xP^-$



Summary of Lecture II

Projectile-centric description : Balitsky-Kovchegov equation



$$T(x_\perp, y_\perp) \equiv 1 - \frac{1}{N_c}\,\mathrm{tr}\,(U(x_\perp)U^\dagger(y_\perp))$$

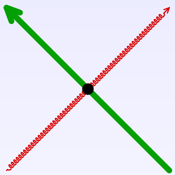
$$\frac{\partial\,\langle T\rangle}{\partial Y} \sim \alpha_s \int \cdots \Big[\ \langle T\rangle - \underbrace{\langle TT\rangle}_{\approx\langle T\rangle\langle T\rangle}\ \Big]$$

- preserves unitarity
- dynamical geometrical scaling
- input: model for $\langle T\rangle$ at the initial Y_0 : Golec-Biernat-Wusthoff, McLerran-Venugopalan,...

Projectile-centric description : Balitsky-Kovchegov equation

$$T(x_\perp, y_\perp) \equiv 1 - \frac{1}{N_c}\,\mathrm{tr}\left(U(x_\perp)U^\dagger(y_\perp)\right)$$

$$\frac{\partial\langle T\rangle}{\partial Y} \sim \alpha_s \int \cdots \left[\,\langle T\rangle - \underbrace{\langle TT\rangle}_{\approx\langle T\rangle\langle T\rangle}\,\right]$$



- preserves unitarity
- dynamical geometrical scaling
- input: model for $\langle T\rangle$ at the initial Y_0 :
  Golec-Biernat-Wusthoff,
  McLerran-Venugopalan,...
- basis of the “hybrid” description in hadron-hadron reactions :
  - projectile 1 : dilute parton beam
  - projectile 2 : saturated

Target-centric description : Color Glass Condensate

- Color source distribution $\rho(x_\perp)$ in the target

$$\Downarrow$$

- Color field $\mathcal{A}^\mu$ given by Yang-Mills equations : $[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}] = \delta^{\nu-}\rho$

$$\Downarrow$$

- Observable $\mathcal{O}$ evaluated on this field configuration

$$\Downarrow$$

- Expectation value obtained by averaging over ρ :

$$\langle \mathcal{O} \rangle = \int [D\rho(x_\perp)] \; W_Y[\rho] \; \mathcal{O}[\rho]$$

- Rapidity evolution : $\frac{\partial W_Y}{\partial Y} = \mathcal{H}\, W_Y$ (JIMWLK equation)

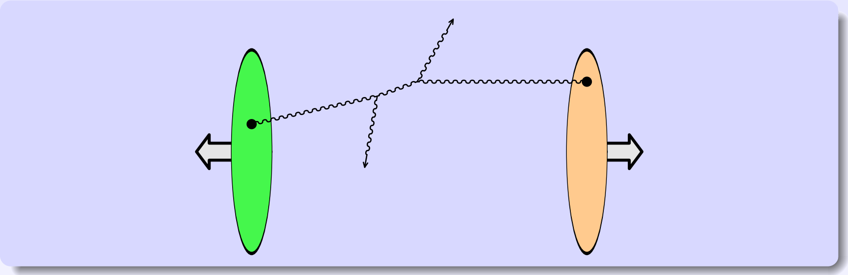
CGC description of A-A collisions at LO

CGC and Nucleus-Nucleus collisions

$$\mathcal{L} = -\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \underbrace{(J_1^\mu + J_2^\mu)}_{J^\mu}A_\mu$$

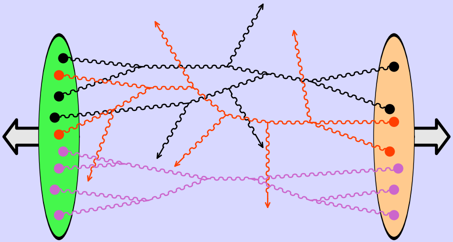
- Given the sources $\rho_{1,2}$ in each projectile, how do we calculate observables? Is there some kind of perturbative expansion?
- Loop corrections and factorization?

Main difficulty : bookkeeping



- Dilute regime : one parton in each projectile interact

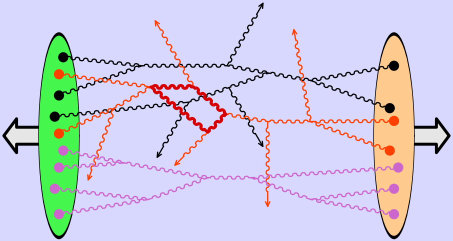
Main difficulty : bookkeeping



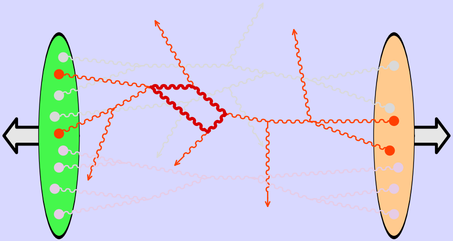
- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial

  (+ pileup of many partonic scatterings in each AA collision)

Power counting



Power counting



- In the saturated regime, the sources are of order $1/g$ (because $\langle\rho\rho\rangle\sim$ occupation number $\sim 1/\alpha_s$)

The order of each connected subdiagram is

$$\frac{1}{g^2}\quad g^{\#\ \text{produced gluons}}\quad g^{2(\#\ \text{loops})}$$

Power counting

- Example : gluon spectrum :

$$\frac{dN_1}{d^3\vec{p}} = \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

- The coefficients $c_0, c_1, \cdots$ are themselves series that resum all orders in $(g\rho_{1,2})^n$. For instance,

$$c_0 = \sum_{n=0}^{\infty} c_{0,n}\,(g\rho_{1,2})^n$$

- At Leading Order, we want to calculate the full c_0/g^2 contribution

Inclusive gluon spectra at LO

- The gluon spectrum at LO is given by :

$$\left.\frac{dN_1}{dY d^2\vec{\mathbf{p}}_\perp}\right|_{LO} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y\sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu\ \mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

where $\mathcal{A}_\mu(x)$ is the classical solution such that $\lim_{x^0\to-\infty}\mathcal{A}_\mu(x) = 0$

Classical Yang-Mills equations

$$[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}] = J_1^\nu + J_2^\nu$$

Inclusive multigluon spectra at Leading Order

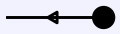
$$\left.\frac{dN_n}{d^3\mathbf{p}_1\cdots d^3\mathbf{p}_n}\right|_{LO} = \left.\frac{dN_1}{d^3\mathbf{p}_1}\right|_{LO}\times\cdots\times\left.\frac{dN_1}{d^3\mathbf{p}_n}\right|_{LO}$$

Retarded classical fields

This sum of trees obeys :

$$\square\mathcal{A} + U'(\mathcal{A}) = J \quad , \quad \lim_{x_0 \to -\infty} \mathcal{A}(x) = 0$$

- Perturbative expansion (illustrated here for $U(\mathcal{A}) \propto \mathcal{A}^3$) :



- Built with retarded propagators

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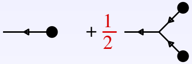

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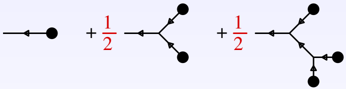

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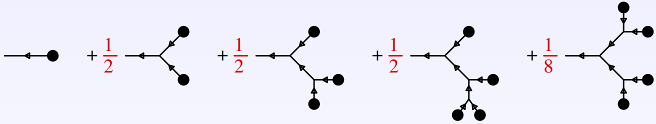


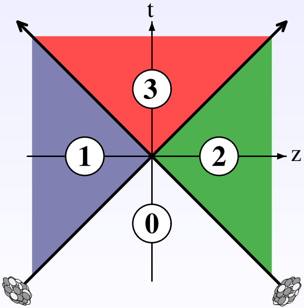
- Built with retarded propagators

- Classical fields resum the full series of tree diagrams

Space-time evolution of the classical field in AA collisions

- Sources located on the light-cone:

$$J^\mu = \delta^{\mu+} \underbrace{\rho_1(x^-, \boldsymbol{x}_\perp)}_{\sim\delta(x^-)} + \delta^{\mu-} \underbrace{\rho_2(x^+, \boldsymbol{x}_\perp)}_{\sim\delta(x^+)}$$



- Region 0 : $\mathcal{A}^\mu = 0$
- Regions 1,2 : $\mathcal{A}^\mu$ depends only on ρ_1 or ρ_2 (known analytically)
- Region 3 : $\mathcal{A}^\mu$ = radiated field after the collision, only known numerically

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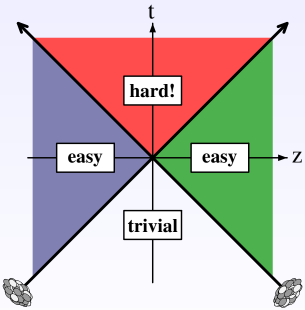

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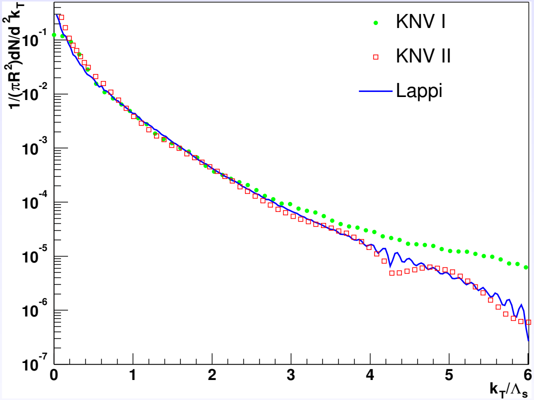
Single gluon spectrum at LO



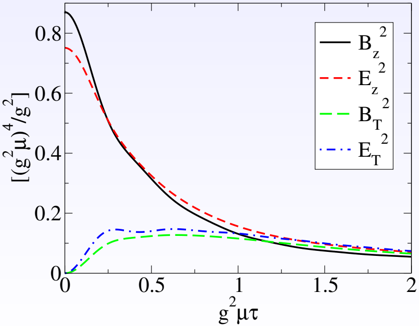
- Lattice artifacts at large momentum
  (they do not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD (saturation)

Initial color fields

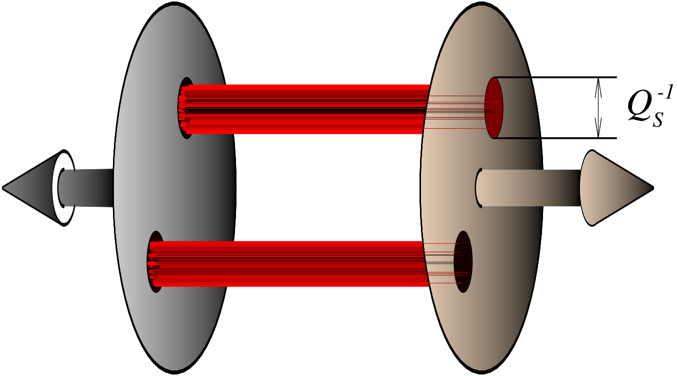
Lappi, McLerran (2006) (Semantics : Glasma $\equiv$ Glas[s - plas]ma)

- Before the collision, the chromo-$\vec{E}$ and $\vec{B}$ fields are localized in two sheets transverse to the beam axis
- Immediately after the collision, the chromo-$\vec{E}$ and $\vec{B}$ fields have become longitudinal :

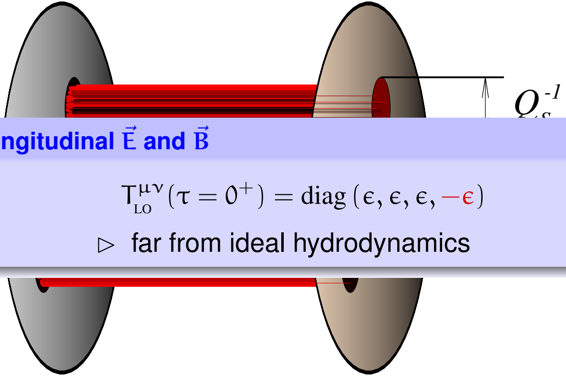
$$E^z = ig\left[\mathcal{A}_1^i, \mathcal{A}_2^i\right] \quad , \qquad B^z = ig\epsilon^{ij}\left[\mathcal{A}_1^i, \mathcal{A}_2^j\right]$$



Energy momentum tensor at LO



Energy momentum tensor at LO



$T^{\mu\nu}$ for longitudinal $\vec{E}$ and $\vec{B}$

$$T^{\mu\nu}_{\mathrm{LO}}(\tau = 0^+) = \mathrm{diag}\,(\epsilon, \epsilon, \epsilon, -\epsilon)$$

▷ far from ideal hydrodynamics

Next–to–Leading Order

Why is the LO insufficient ?

- Naive perturbative expansion :

$$\frac{dN}{d^3\vec{p}} = \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

Note : so far, we have seen how to compute c_0

- Problem : $c_{1,2,\ldots}$ contain powers of the cutoff y_{cut} :

$$\begin{aligned} c_1 &= && & c_{10} &+ \; c_{11}\, y_{cut} \\ c_2 &= c_{20} &+ & & c_{21}\, y_{cut} &+ \underbrace{c_{22}\, y_{cut}^2}_{\text{Leading Log terms}} \end{aligned}$$

- These terms are unphysical. However, they are universal and can be absorbed into the distributions $W[\rho_{1,2}]$

Leading Log corrections to the gluon spectrum

- By keeping only the terms that contain the cutoff, the NLO result can be written as :

$$\left.\frac{dN}{d^3\vec{p}}\right|_{\rm NLO} \underset{\text{Leading Log}}{=} \Big[y^+_{\rm cut}\,\mathcal{H}_1 + y^-_{\rm cut}\,\mathcal{H}_2 \Big] \left.\frac{dN}{d^3\vec{p}}\right|_{\rm LO}$$

$\mathcal{H}_{1,2}$: JIMWLK Hamiltonians for the two nuclei

- Note : the $y_{\rm cut}$ terms do not mix the two nuclei $\Rightarrow$ Factorization

Factorization

- By integrating over $\rho_{1,2}$'s, one can absorb the y_{cut}-dependent terms into universal distributions $W_{1,2}[\rho_{1,2}]$
- $\mathcal{H}$ is a self-adjoint operator :

$$\int [D\rho]\, W\, \big(\mathcal{H}\,\mathcal{O}\big) = \int [D\rho]\, \big(\mathcal{H}\,W\big)\, \mathcal{O}$$

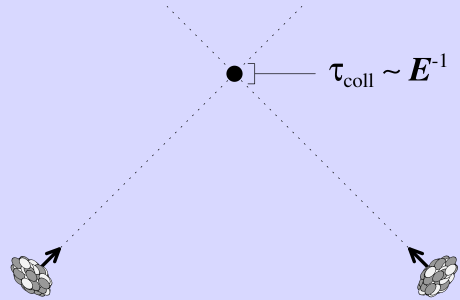
Single inclusive gluon spectrum at Leading Log accuracy

$$\frac{dN_1}{d^3\vec{\mathbf{p}}} \underset{\text{Leading Log}}{=} \int \big[D\rho_1\; D\rho_2\big]\; W_1\big[\rho_1\big]\; W_2\big[\rho_2\big]\; \underbrace{\frac{dN_1}{d^3\vec{\mathbf{p}}}\bigg|_{LO}}_{\text{fixed }\ \rho_{1,2}}$$

- Cutoff absorbed into the evolution of $W_{1,2}$ with rapidity

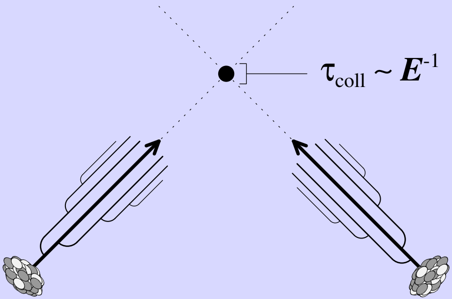
$$\frac{\partial W}{\partial y} = \mathcal{H}\,W \qquad \text{(JIMWLK equation)}$$

Handwaving argument for factorization



- The duration of the collision is very short: $\tau_{\rm coll} \sim E^{-1}$

Handwaving argument for factorization



- The duration of the collision is very short: $\tau_{\rm coll} \sim {\rm E}^{-1}$
- The terms we want to resum are due to the radiation of soft gluons, which takes a long time
  $\triangleright$ it must happen (long) before the collision

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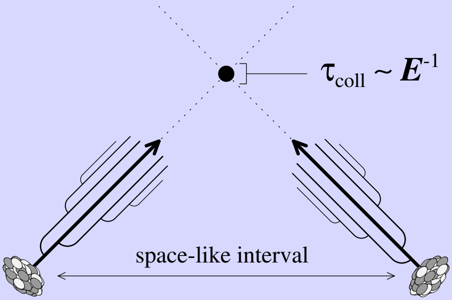


- The duration of the collision is very short: $\tau_{\rm coll} \sim E^{-1}$
- The terms we want to resum are due to the radiation of soft gluons, which takes a long time
  ▷ it must happen (long) before the collision
- The projectiles are not in causal contact before the impact
  ▷ the $y_{\rm cut}$-dependent terms are intrinsic properties of the projectiles, independent of the measured observable

Multi-gluon correlations at Leading Log

- The previous factorization can be extended to multi-particle inclusive spectra :

$$\frac{dN_n}{d^3\vec{p}_1\cdots d^3\vec{p}_n} \underset{\text{Leading Log}}{=} \int \big[D\rho_1\, D\rho_2\big]\, W_1\big[\rho_1\big]\, W_2\big[\rho_2\big]\; \frac{dN_1}{d^3\vec{p}_1}\cdots\frac{dN_1}{d^3\vec{p}_n}\bigg|_{\text{LO}}$$

- At Leading Log accuracy, all the rapidity correlations come from the evolution of the distributions $W[\rho_{1,2}]$

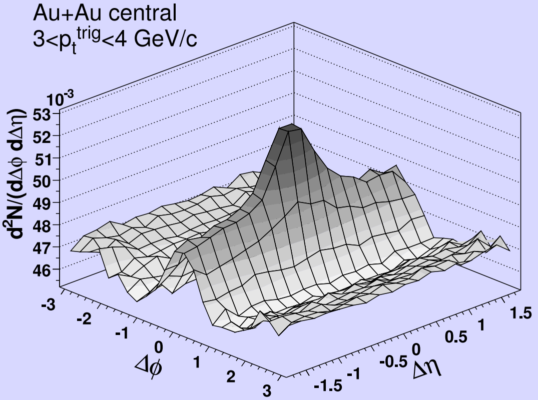
  ▷ they are a property of the pre-collision initial state

- Predicts long range ($\Delta y \sim \alpha_s^{-1}$) correlations in rapidity

Ridge correlations

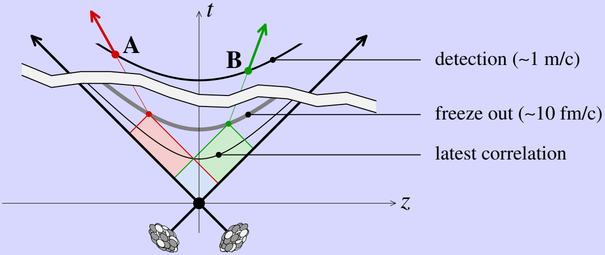
2-particle correlations in AA collisions

[STAR Collaboration, RHIC]



- Long range rapidity correlation
- Narrow correlation in azimuthal angle
- Narrow jet-like correlation near $\Delta y = \Delta \varphi = 0$

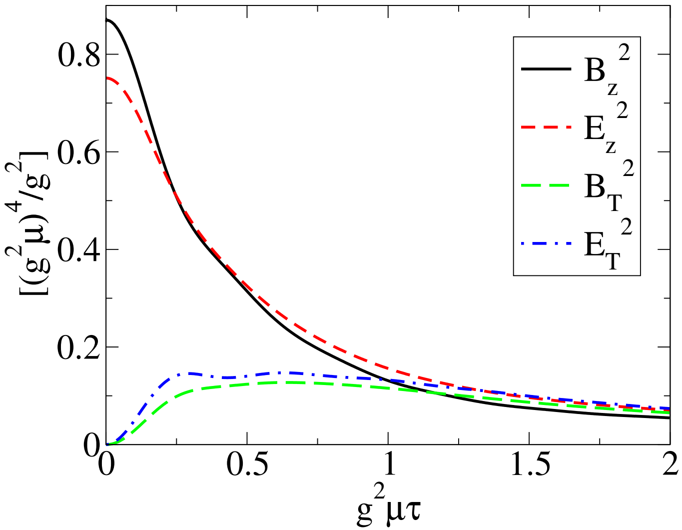
Probing early times with rapidity correlations



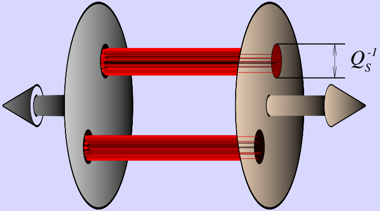
- By causality, long range rapidity correlations are sensitive to the dynamics of the system at early times :

$$\tau_{\text{correlation}} \;\leq\; \tau_{\text{freeze out}}\; e^{-|\Delta y|/2}$$

Color field at early time



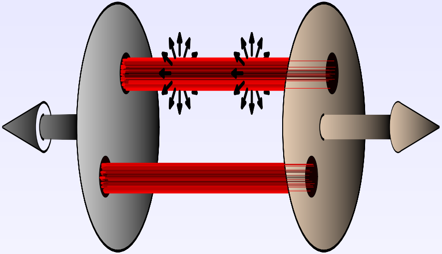
Color field at early time



- The field lines form tubes of transverse size $\sim Q_s^{-1}$
- Rapidity correlation length : $\Delta\eta \sim \alpha_s^{-1}$

2-hadron correlations from color flux tubes

- η-independent fields lead to long range correlations :



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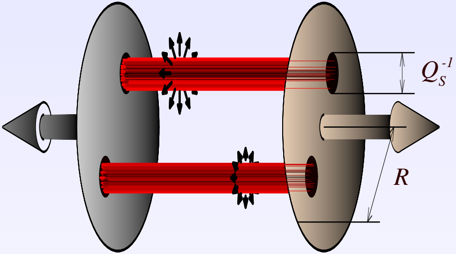

- Particles emitted by different flux tubes are not correlated
  ▷ $(RQ_s)^{-2}$ sets the strength of the correlation

2-hadron correlations from color flux tubes

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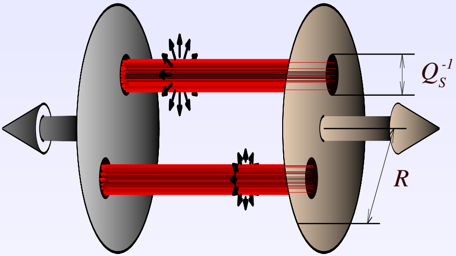

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2-hadron correlations from color flux tubes

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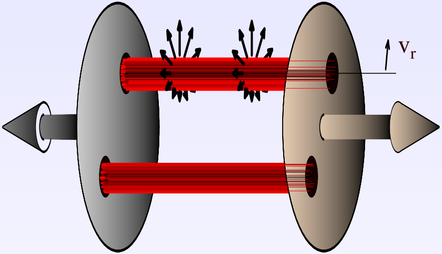
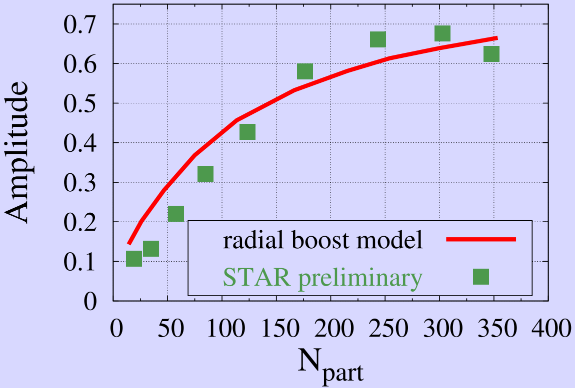


- Particles emitted by different flux tubes are not correlated
  $\triangleright$ $(RQ_s)^{-2}$ sets the strength of the correlation
- At early times, the correlation is flat in $\Delta\varphi$

  The collimation in $\Delta\varphi$ is produced later by radial flow

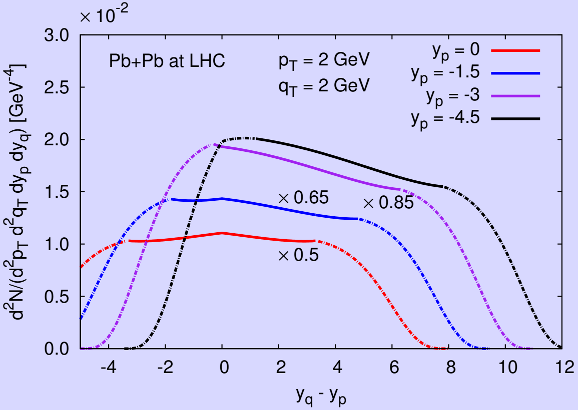
Centrality dependence



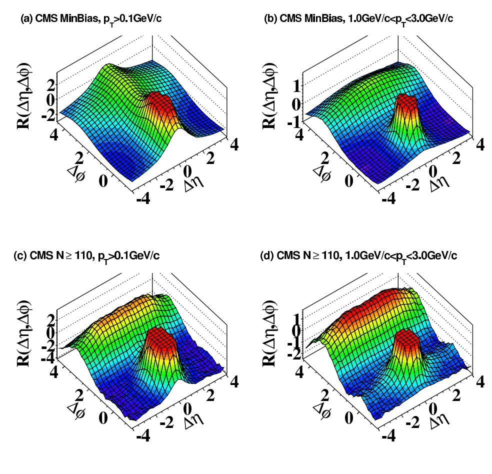
- Main effect : increase of the radial flow velocity with the centrality of the collision

Rapidity dependence

Estimate at LHC energy



High multiplicity proton-proton collisions at the LHC



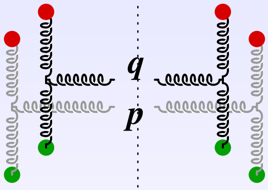
- Similar effect visible for high multiplicity p-p collisions, in an intermediate $p_\perp$ window
- Much weaker than in AA collisions

Possible origin of the angular correlation

- The long range rapidity correlations invoked in A-A collisions are also present in p-p collisions
- Whether there is a sufficient amount of radial flow to induce the azimuthal collimation is unknown
  - less particles are produced
  - the system freezes out much earlier
- There is however an “intrinsic” angular correlation, that exists in the absence of flow (it was there in A-A collisions as well, but neglected because it is a small effect)

Intrinsic angular correlations

- 2-gluon inclusive spectrum before the average over $\rho_{1,2}$:

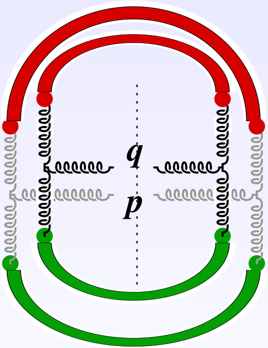


▷ this contribution dominates the 2-gluon spectrum in the regime where the parton densities are large

▷ the average over $\rho_{1,2}$ amounts to connecting the red and green lines in all the possible ways (pairwise if the sources have Gaussian distributions)

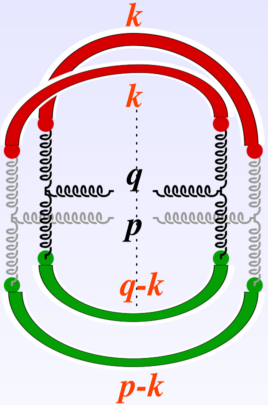
Intrinsic angular correlations

- Trivial connection (no correlation) :



Intrinsic angular correlations

- Non-trivial connection with correlations at $\Delta\varphi < \frac{\pi}{2}$:

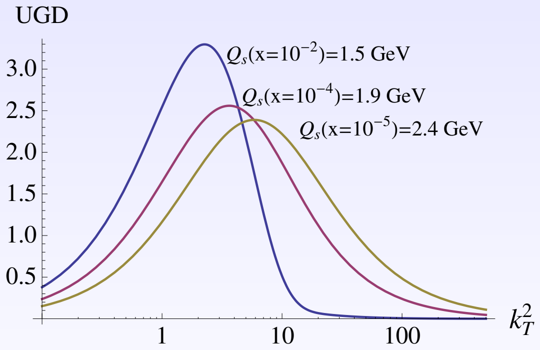


▷ Momentum assignment of the unintegrated gluon distributions:

$$[\phi_1(\mathbf{k}_\perp)]^2 \; \phi_2(|\mathbf{p}_\perp - \mathbf{k}_\perp|) \; \phi_2(|\mathbf{q}_\perp - \mathbf{k}_\perp|)$$

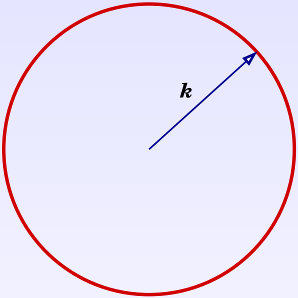
Intrinsic angular correlations

- In the saturation regime, unintegrated gluon distributions are peaked near Q_s:



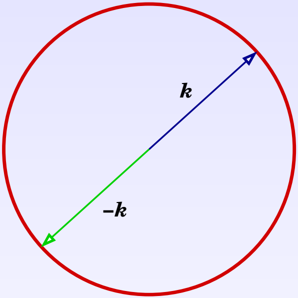
- The presence of this peak is what correlates the directions of $\vec{p}_\perp$ and $\vec{q}_\perp$ around $\Delta\phi = 0$ when we perform the integration over $\vec{k}_\perp$

Intrinsic angular correlations



- $|\vec{k}_\perp| \sim Q_s$

Intrinsic angular correlations



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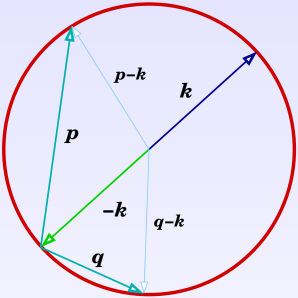
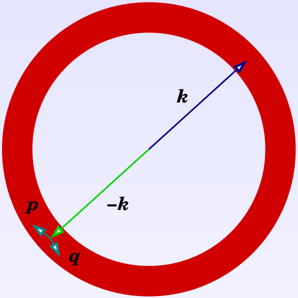
Intrinsic angular correlations



- $|\vec{k}_\perp| \sim Q_s$
- $|\vec{p}_\perp - \vec{k}_\perp| \sim |\vec{q}_\perp - \vec{k}_\perp| \sim Q_s$

Intrinsic angular correlations

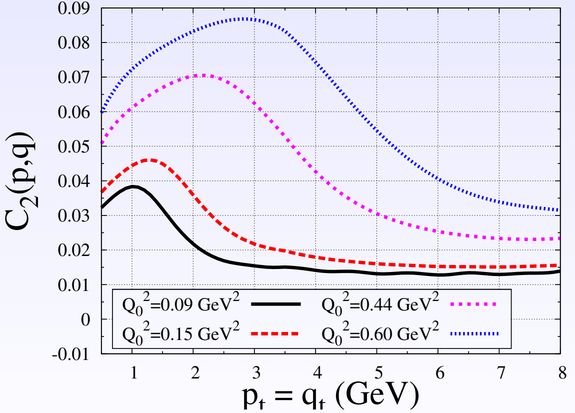


- $|\vec{\mathrm{k}}_\perp| \sim \mathrm{Q}_s$
- $|\vec{\mathrm{p}}_\perp - \vec{\mathrm{k}}_\perp| \sim |\vec{\mathrm{q}}_\perp - \vec{\mathrm{k}}_\perp| \sim \mathrm{Q}_s$
- If the momenta are smaller than the width of the distributions, there is no significant angular correlation

  Similarly, for large momenta there is no correlation because the main contribution does not come from the peak of the distributions anymore

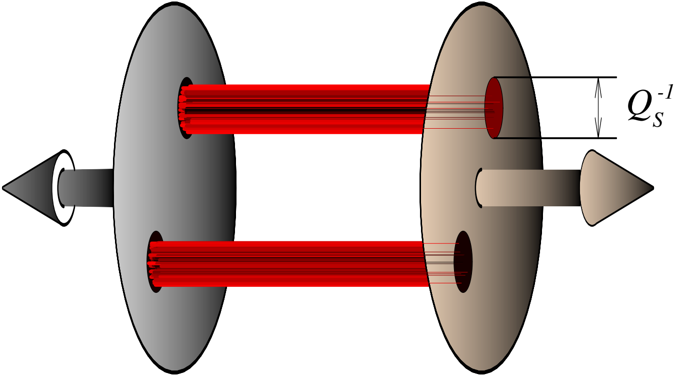
Intrinsic angular correlations

- The effect is maximal for intermediate $p_\perp, q_\perp \sim Q_s$:

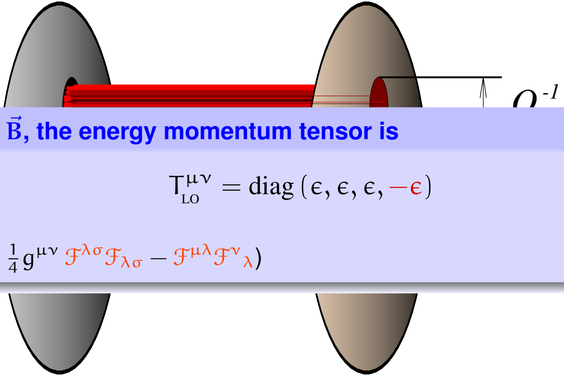


Towards thermalization...

Energy momentum tensor at LO



Energy momentum tensor at LO

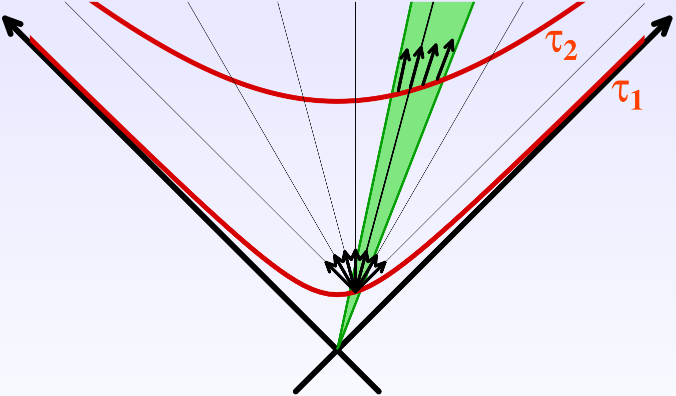


When $\vec{E} \parallel \vec{B}$, the energy momentum tensor is

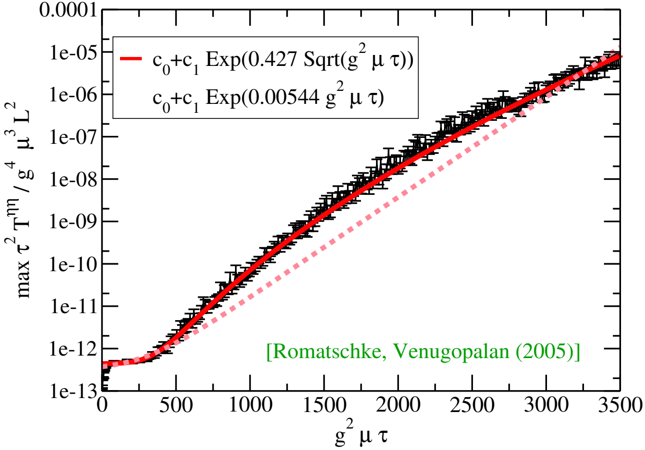
$$T^{\mu\nu}_{\scriptscriptstyle LO} = \mathrm{diag}\,(\epsilon, \epsilon, \epsilon, -\epsilon)$$

$(T^{\mu\nu} = \frac{1}{4} g^{\mu\nu}\, \mathcal{F}^{\lambda\sigma}\mathcal{F}_{\lambda\sigma} - \mathcal{F}^{\mu\lambda}\mathcal{F}^{\nu}{}_{\lambda})$

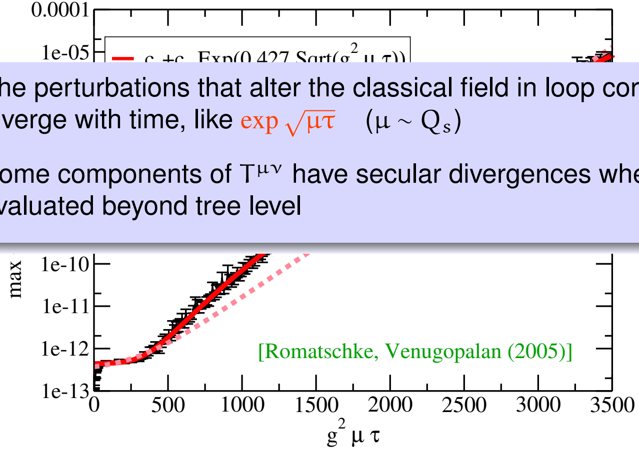
Competition between Expansion and Isotropization



Weibel instabilities for small perturbations



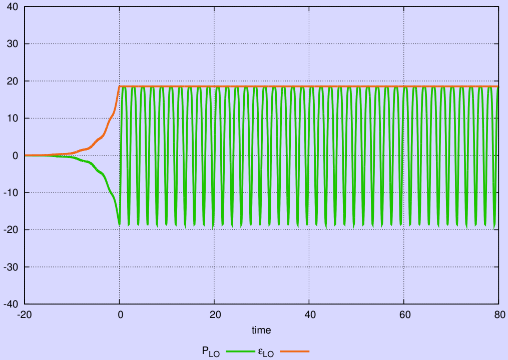
Weibel instabilities for small perturbations



- The perturbations that alter the classical field in loop corrections diverge with time, like $\exp\sqrt{\mu\tau}$ $\quad(\mu \sim Q_s)$
- Some components of $T^{\mu\nu}$ have secular divergences when evaluated beyond tree level

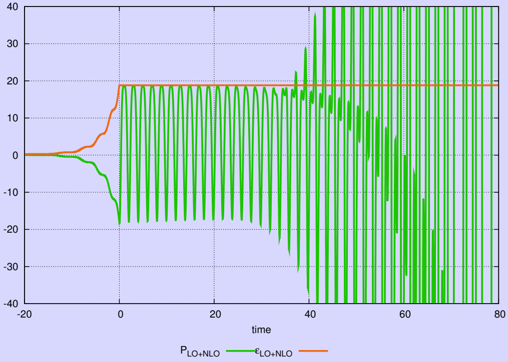
Example of pathologies in fixed order calculations (scalar theory)

LO



Example of pathologies in fixed order calculations (scalar theory)

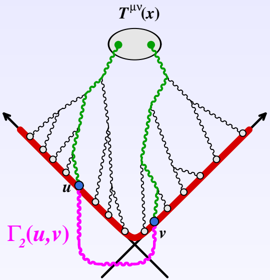
LO + NLO



- Small correction to the energy density
  (protected by energy conservation)
- Secular divergence in the pressure

Improved power counting

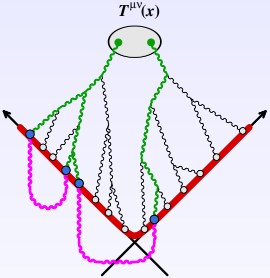
Loop $\sim g^2$, $e^{\sqrt{\mu\tau}}$ for each field perturbation



- 1 loop : $(g e^{\sqrt{\mu\tau}})^2$

Improved power counting

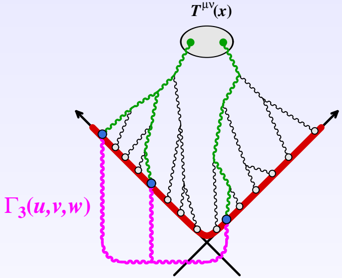
Loop $\sim g^2$, $e^{\sqrt{\mu\tau}}$ for each field perturbation



- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(ge^{\sqrt{\mu\tau}})^4$

Improved power counting

Loop $\sim g^2$, $e^{\sqrt{\mu\tau}}$ for each field perturbation



- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(ge^{\sqrt{\mu\tau}})^4$
- 2 nested loops : $g(ge^{\sqrt{\mu\tau}})^3$
  ▷ subleading

Leading terms at τ_{max}

- All disconnected loops to all orders
  ▷ exponentiation of the 1-loop result

Resummation of the leading secular terms

$$T^{\mu\nu}_{\text{resummed}} = \int [Da]\, \exp\left[-\frac{1}{2} \int_{u,v} a(u) \Gamma_2^{-1}(u,v) a(v) \right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}} + a]$$

- There is a unique choice of the variance Γ_2 such that

$$T^{\mu\nu}_{\text{resummed}} = T^{\mu\nu}_{\text{LO}} + T^{\mu\nu}_{\text{NLO}} + \cdots$$

- This resummation collects all the terms with the worst time behavior

Resummation of the leading secular terms

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$$T^{\mu\nu}_{\text{resummed}} = T^{\mu\nu}_{\text{LO}} + T^{\mu\nu}_{\text{NLO}} + \cdots$$

- This resummation collects all the terms with the worst time behavior
- Equivalent to Gaussian fluctuations of the initial field + classical time evolution
- At $Q_s\tau_0 \ll 1$: $\mathcal{A}_{\text{init}} \sim Q_s/g \quad , \quad a \sim Q_s$

Main steps

1. Determine the 2-point function $\Gamma_2(u,v)$ that defines the Gaussian fluctuations, for the initial time $Q_s\tau_0$ of interest

   Note : this is an initial value problem, whose outcome is uniquely determined by the state of the system at $x^0 = -\infty$, and depends on the history of the system from $x^0 = -\infty$ to $\tau = \tau_0$

   Problem solvable only if the fluctuations are weak, $a^\mu \ll Q_s/g$

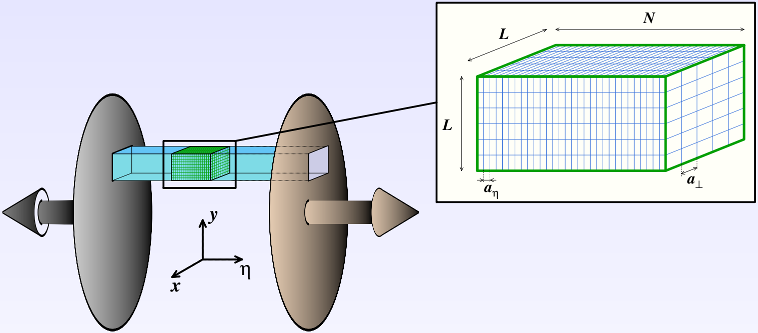
   $Q_s\tau_0 \ll 1$ necessary for the fluctuations to be Gaussian

2. Solve the classical Yang-Mills equations from τ_0 to τ_f

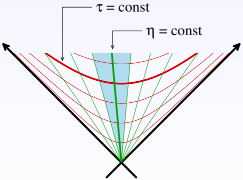
   Note : the problem as a whole is boost invariant, but individual field configurations are not $\Longrightarrow$ 3+1 dimensions necessary

3. Do a Monte-Carlo sampling of the fluctuating initial conditions

Discretization of the expanding volume



- Comoving coordinates : $\tau, \eta, \boldsymbol{x}_\perp$
- Only a sub-volume is simulated + periodic boundary conditions
- $L^2 \times N$ lattice

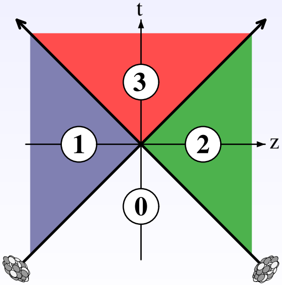


Gaussian spectrum of fluctuations

Expression of the variance (from 1-loop considerations)

$$\Gamma_2(u,v) = \int_{\text{modes } \mathbf{k}} a_k(u)\, a_k^*(v)$$

$$\Big[\mathcal{D}_\rho \mathcal{D}^\rho \delta^\nu_\mu - \mathcal{D}_\mu \mathcal{D}^\nu + ig\,\mathcal{F}_\mu{}^\nu\Big] a_k^\mu = 0 \quad , \quad \lim_{x^0 \to -\infty} a_k(x) \sim e^{ik\cdot x}$$



0. $\mathcal{A}^\mu = 0$, trivial

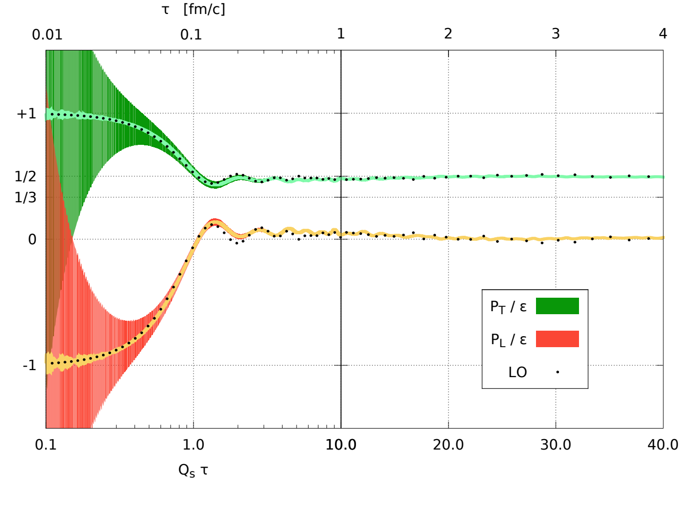
1,2. $\mathcal{A}^\mu =$ pure gauge, analytical solution

3. $\mathcal{A}^\mu$ non-perturbative
$\Rightarrow$ expansion in $Q_s\tau$

- We need the fluctuations in Fock-Schwinger gauge
$x^+ a^- + x^- a^+ = 0$

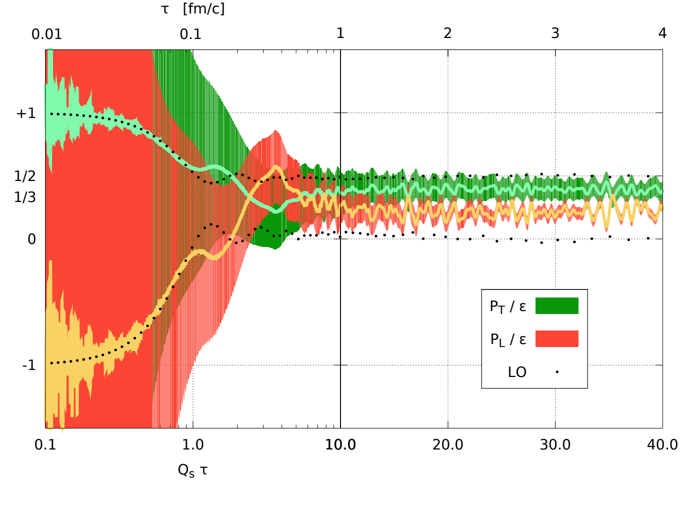
Time evolution of P_T/ϵ and P_L/ϵ $(64 \times 64 \times 128$ lattice)

$g = 0.1$ $(N_{\text{confs}} = 200)$



Time evolution of P_T/ϵ and P_L/ϵ $(64 \times 64 \times 128$ lattice)

$g = 0.5$ $(N_{\text{confs}} = 2000)$



Bose-Einstein condensation

Overpopulated CGC initial conditions

CGC initial conditions

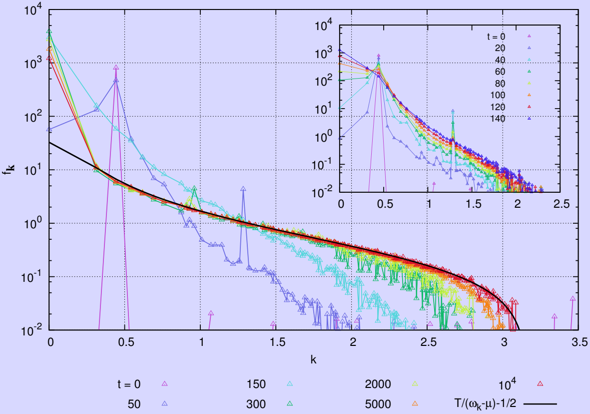
$$\epsilon_0 \sim \frac{Q_s^4}{\alpha_s} \qquad n_0 \sim \frac{Q_s^3}{\alpha_s} \qquad (n\epsilon^{-3/4})_0 \sim \alpha_s^{-1/4}$$

Equilibrium state

$$\epsilon \sim T^4 \qquad n \sim T^3 \qquad n\epsilon^{-3/4} \sim 1$$

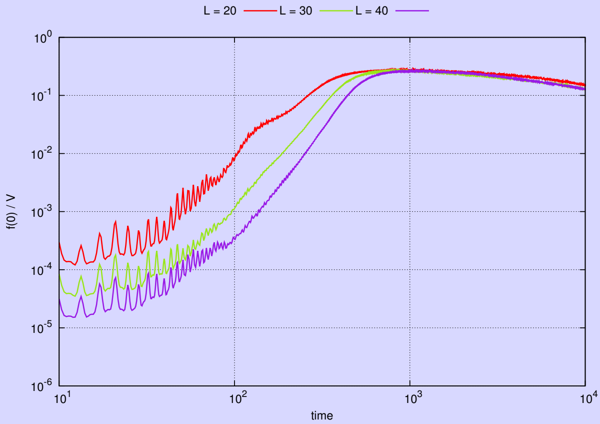
- The excess of gluons can be eliminated in two ways :
  - via inelastic processes $3 \to 2$ (rather slow at weak coupling)
  - by condensation on the zero mode

Bose-Einstein condensation (in a scalar field theory)



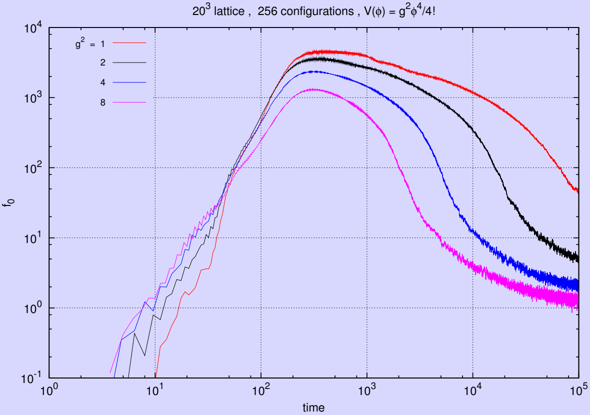
- Start with an overpopulated initial condition, with an empty zero mode
- Very quickly, the zero mode becomes highly occupied

Volume dependence



$$f(\mathbf{k}) = \frac{1}{e^{\beta(\omega_k - \mu)} - 1} + n_0 \delta(\mathbf{k}) \quad \Longrightarrow f(0) \propto V = L^3$$

Evolution of the condensate



- Formation time almost independent of the coupling
- Condensate lifetime much longer than its formation time
- Smaller amplitude and faster decay at large coupling

Summary
and Outlook

Summary

- Gluon saturation and recombination
  - prevents the gluon occupation number to go above $1/\alpha_s$
  - prevents violations of unitarity in scattering amplitudes
- Two equivalent descriptions
  - Balitsky-Kovchegov :

    Non-linear evolution equation for specific matrix elements

    The non-linear terms lead to the dynamical generation of geometrical scaling

    Applicable to collisions between a saturated and a dilute projectile
  - Color Glass Condensate :

    The color fields of the target evolve with rapidity

    More suitable to collisions of two saturated projectiles
- Isotropization, Thermalization
  - Instabilities require the resummation of additional contributions
  - Possibility of the formation of a Bose-Einstein condensate

Semantics

- Weakly coupled : $g \ll 1$

- Weakly interacting : $g\mathcal{A} \ll 1 \qquad g^2 f(\mathbf{p}) \ll 1$

  $$(2 \to 2) \gg (2 \to 3), (3 \to 2), \cdots$$

- Strongly interacting : $g\mathcal{A} \sim 1 \qquad g^2 f(\mathbf{p}) \sim 1$

  $$(2 \to 2) \sim (2 \to 3) \sim (3 \to 2) \sim \cdots$$

  No well defined quasi-particles

CGC = weakly coupled, but strongly interacting effective theory